Nonlinear pulse propagation in optical fibres

Dr Yoonchan Jeong
School of Electrical Engineering, Seoul National University
Tel: +82 (0)2 880 1623, Fax: +82 (0)2 873 9953
Email: yoonchan@snu.ac.kr

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Maxwell's equation

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

Constitutive relations for E-field

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_o \mathbf{E} + \mathbf{P},$$

$$\mathbf{P} = \varepsilon_o \chi \mathbf{E} = \varepsilon_o \left(\chi^{(1)} \mathbf{E} + \chi^{(2)} \mathbf{EE} + \chi^{(3)} \mathbf{EEE} + \cdots \right)$$

Let $\mathbf{J} = 0, \ \rho = 0, \ \mu = \mu_0 \ \textit{and} \ \chi^{(2)} = 0.$

Recall the coupled-mode equations derived from Lorentz reciprocity theorem

$$\nabla \cdot (\boldsymbol{E}' \times \boldsymbol{H}_p^* + \boldsymbol{E}_p^* \times \boldsymbol{H}') = -i\omega \boldsymbol{E}_p^* \Delta\varepsilon \boldsymbol{E}' \quad (p = 1, 2, 3, \ldots)$$

Dispersion relation

Mode-propagation constant β in a Taylor series:

$$\beta(\omega) = n(\omega)\frac{\omega}{c} = \beta_0 + \beta_1(\omega - \omega_0) + \frac{1}{2}\beta_2(\omega - \omega_0)^2 + \cdots,$$

$$where \quad \beta_m = \left(\frac{d^m\beta}{d\omega_m}\right)_{\omega=\omega_0} \quad (m = 1, 2, 3, \ldots)$$

Wave-packet representation

Summation of eigen waves:

$$E(\mathbf{r},t)=\int \Phi(\omega)\underline{F(x,y,\omega)\exp[i(\omega t-\beta z)]}d\omega$$

Mode-propagation constant β in a Taylor series:

$$\beta(\omega)=n(\omega)\frac{\omega}{c}=\beta_0+\beta_1(\omega-\omega_0)+\frac{1}{2}\beta_2(\omega-\omega_0)^2+\cdots,$$

$$\text{where}\quad \beta_m=\left(\frac{d^m\beta}{d\omega_m}\right)_{\omega=\omega_0}\qquad (m=1,2,3,\ldots)$$

In result:

$$\begin{aligned}E(\mathbf{r},t)&=\int \Phi(\Omega)F(x,y,\Omega)\exp[i(\Omega t-Bz)]d\Omega\cdot\exp[i(\omega_0 t-\beta_0 z)]\\&\approx\int \Phi(\Omega)\exp(-iBz)\exp(i\Omega t)d\Omega\cdot \underline{F_0(x,y)}\exp[i(\omega_0 t-\beta_0 z)]\\&=\underline{\int \tilde{A}(z,\Omega)\exp(i\Omega t)d\Omega}\cdot F_0(x,y)\exp[i(\omega_0 t-\beta_0 z)]\\&=\underbrace{A(z,t)}_{\text{Envelop}}\cdot\underbrace{F_0(x,y)\exp[i(\omega_0 t-\beta_0 z)]}_{\text{Carrier}},\end{aligned}$$

Envelop *Carrier*

$$\text{where}\quad \Omega=\omega-\omega_0,\quad \mathrm{B}=\beta-\beta_0=\beta_1\Omega+\frac{\beta_2}{2}\Omega^2+\cdots.$$

Envelop evolution

For linear & lossless dispersive media:

$$A(z,t) = \int \tilde{A}(z,\Omega)\exp(i\Omega t)d\Omega$$

$$= \int \Phi(\Omega)\exp[i(\Omega t - Bz)]d\Omega$$

$$= \int \Phi(\Omega)\exp[i\Omega t - i(\beta_1\Omega + \frac{\beta_2}{2}\Omega^2 + \cdots)z]d\Omega$$

$$\rightarrow \frac{\partial A}{\partial z} + \beta_1\frac{\partial A}{\partial t} - \frac{i\beta_2}{2}\frac{\partial^2 A}{\partial t^2} + \cdots = 0$$

For nonlinear & lossy dispersive media:

$$A(z,t) = \int \tilde{A}(z,\Omega)\exp(i\Omega t)d\Omega$$

Loss

Nonlinear phase shift due to $\chi^{(3)}$

$$= \int \Phi(\Omega)\exp[i(\Omega t - Bz) - \frac{\alpha}{2}z - i\gamma|A|^2 z]d\Omega$$

$$= \int \Phi(\Omega)\exp[i\Omega t - i(\beta_1\Omega + \frac{\beta_2}{2}\Omega^2 + \cdots)z - \frac{\alpha}{2}z - i\gamma|A|^2 z]d\Omega$$

$$\rightarrow \frac{\partial A}{\partial z} + \beta_1\frac{\partial A}{\partial t} - \frac{i\beta_2}{2}\frac{\partial^2 A}{\partial t^2} + \frac{\alpha}{2}A + \cdots = -i\gamma|A|^2 A$$

Derivation of nonlinear Schrödinger equation (NLSE)

Perturbed field

$$\tilde{E}'(z,\omega) = \underline{\tilde{A}(z,\Omega)} \cdot \underline{F_0(x,y)\exp(-i\beta_0 z)}$$

Slowly varying amplitude — *Eigen mode*

$$\beta(\omega) = n(\omega)\frac{\omega}{c} = \beta_0 + \underline{\beta_1(\omega-\omega_0) + \frac{1}{2}\beta_2(\omega-\omega_0)^2 + \cdots}$$

$$= \beta_0 + \Delta\beta_L, \text{ where } \beta_0 = n(\omega_0)\frac{\omega_0}{c}$$

$$\rightarrow \quad n(\omega) = n(\omega_0) + \Delta n_L$$

$$\Delta n_L \approx \frac{c}{\omega_0}\Delta\beta_L$$

$$\Delta\varepsilon_L = \varepsilon_0 n^2(\omega) - \varepsilon_0 n^2(\omega_0)$$

$$= \varepsilon_0[n(\omega_0)+\Delta n_L]^2 - \varepsilon_0 n^2(\omega_0)$$

$$\approx 2\varepsilon_0 n(\omega_0)\Delta n_L = \frac{2\varepsilon_0 n(\omega_0) c}{\omega_0}\Delta\beta_L$$

If nonlinear perturbation involved:

$$n = n_L + n_2 I$$

$$\Delta\varepsilon_{NL} \approx 2\varepsilon_0 n(\omega_0) n_2 I = 2\varepsilon_0 n(\omega_0) n_2 \cdot \frac{1}{2}\varepsilon_0 c n(\omega_0)|A|^2 |F_0(x,y)|^2$$

$$= \varepsilon_0^2 c n^2(\omega_0) n_2 |A|^2 |F_0(x,y)|^2, \text{ where } n_2 = \frac{3\chi^{(3)}}{4\varepsilon_0 c n^2}$$

$$= \varepsilon_0 \frac{3}{4}\chi^{(3)} |A|^2 |F_0(x,y)|^2$$

Recall the coupled-mode equations!

Without second-order nonlinearity

$$\nabla \cdot (E' \times H_p^* + E_p^* \times H') = -i\omega E_p^* \cdot \Delta\varepsilon\, E', \quad (p = 1,2,3,\ldots)$$

$$\Delta\varepsilon_{NL(q)} = \varepsilon_o \frac{3}{4} \chi^{(3)}(x,y,z) \cdot \sum_s \alpha_{(q,s)} \left|E_s(t,z)\right|^2 \left|\hat{e}_s\right|^2, \; \alpha_{(q,s)} = \begin{cases} 1 & (q = s), \\ 2 & (q \neq s), \end{cases}$$

$$\frac{da_p(z)}{dz} \exp(-i\beta_p z) = -i \frac{\omega}{4} \sum_q a_q(z) \exp(-i\beta_q z) \int \boldsymbol{e}_p^*(x,y) \cdot (\Delta\varepsilon_L + \Delta\varepsilon_{NL}) \boldsymbol{e}_q(x,y)\, dxdy$$

For each Fourier component:

$$\rightarrow \quad \frac{\partial \tilde{A}(z,\Omega)}{\partial z} = -i \frac{\omega_0}{4} \tilde{A} \cdot \int \left|F_0(x,y)\right|^2 (\Delta\varepsilon_L + \Delta\varepsilon_{NL})\, dxdy$$

$$\text{where } \frac{\varepsilon_0 c n(\omega_0)}{2} \int \left|F_0(x,y)\right|^2 dxdy = 1.$$

$$\frac{\partial \tilde{A}}{\partial z} = -i\frac{\omega_0}{4}\tilde{A}\cdot\int\left|F_0(x,y)\right|^2(\Delta\varepsilon_L + \Delta\varepsilon_{NL})dxdy$$

$$= -i\frac{\omega_0}{4}\tilde{A}\cdot(\Delta\varepsilon_L\int\left|F_0(x,y)\right|^2 dxdy + \varepsilon_0\frac{3}{4}\chi^{(3)}\left|A\right|^2\int\left|F_0(x,y)\right|^4 dxdy$$

$$= -i\frac{\omega_0}{4}\tilde{A}\cdot(\Delta\varepsilon_L\frac{2}{\varepsilon_0 cn} + \varepsilon_0\frac{3}{4}\chi^{(3)}\left|A\right|^2\frac{4}{\varepsilon_0^2c^2n^2}\frac{1}{A_{eff}}) \quad \leftarrow \quad A_{eff} = \frac{(\int\left|F_0(x,y)\right|^2 dxdy)^2}{\int\left|F_0(x,y)\right|^4 dxdy}$$

$$= -i\frac{\omega_0}{4}\Delta\varepsilon_L\frac{2}{\varepsilon_0 cn}\tilde{A} - i\frac{\omega_0}{4}\varepsilon_0\frac{3}{4}\chi^{(3)}\frac{4}{\varepsilon_0^2c^2n^2}\frac{1}{A_{eff}}\left|A\right|^2\tilde{A}$$

$$= -i\frac{\omega_0}{4}\frac{2\varepsilon_0 n(\omega_0)c}{\omega_0}\frac{2c\mu_0}{n}\Delta\beta_L\tilde{A} - i\frac{\omega_0}{4}\varepsilon_0\frac{3}{4}\chi^{(3)}\frac{4}{\varepsilon_0^2c^2n^2}\frac{1}{A_{eff}}\left|A\right|^2\tilde{A}$$

$$= -i\Delta\beta_L\tilde{A} - i\frac{3\omega_0\chi^{(3)}}{4\varepsilon_0c^2n^2A_{eff}}\left|A\right|^2\tilde{A}$$

$$= -i\Delta\beta_L\tilde{A} - i\frac{n_2\omega_0}{cA_{eff}}\left|A\right|^2\tilde{A} \quad \leftarrow \quad \gamma = \frac{n_2\omega_0}{cA_{eff}}$$

$$= -i\Delta\beta_L\tilde{A} - i\gamma\left|A\right|^2\tilde{A}$$

Transform into the time domain via inverse Fourier transform:

$$\frac{\partial \tilde{A}(z,\Omega)}{\partial z} = -i\Delta\beta_L \tilde{A}(z,\Omega) - i\gamma|A|^2 \tilde{A}(z,\Omega)$$

$$= -i(\beta_1\Omega + \frac{\beta_2}{2}\Omega^2 + \cdots)\tilde{A}(z,\Omega) - i\gamma|A|^2 \tilde{A}(z,\Omega)$$

$$\Omega \to -i\frac{\partial}{\partial t}$$

Nonlinear Schrödinger equation:

$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} - i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial t^2} + \cdots = -i\gamma|A|^2 A$$ ←*for lossless media*

or

$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} - i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial t^2} + \frac{\alpha}{2}A \cdots = -i\gamma|A|^2 A$$ ←*for lossy media*

Nonlinear pulse propagation

NLSE:

$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} - i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial t^2} + \frac{\alpha}{2}A = -i\gamma|A|^2 A$$

Let $$A(z,t) = \int \Phi(z,\Omega)\underline{\exp\{[-i(\beta_1\Omega + \frac{\beta_2}{2}\Omega^2) - \frac{\alpha}{2})]z\}}\exp(i\Omega t)d\Omega$$

$$= \int T(z,\Omega)\cdot\Phi(z,\Omega)\exp(i\Omega t)d\Omega,$$

where $$\Phi(0,\Omega) = \frac{1}{2\pi}\int A(0,t)\exp(-i\Omega t)dt$$

Let $$\gamma|A|^2 A = \int T(z,\Omega)G(z,\Omega)\exp(i\Omega t)d\Omega,$$

where $$G(z,\Omega) = \frac{1}{2\pi}\int T^{-1}(z,\Omega)\gamma|A|^2 A\exp(-i\Omega t)dt$$

In result: $$\frac{\partial}{\partial z}\Phi(z,\Omega) = -iG(z,\Omega),$$

which can be solved by the fast Fourier transform method in a predictor-corrector scheme.

Predictor-corrector scheme

For a first-order differential equation:

$$\frac{dy}{dz} = f(z, y),$$

$$y_p(z + \Delta z) = y(z) + f[z, y(z)]\Delta z,$$

$$y(z + \Delta z) = y(z) + \frac{1}{2}\{f[z, y(z)] + f[z, y_p(z + \Delta z)]\}\Delta z.$$

The value predicted by an initial rough estimation is corrected by iterations.

Split-step Fourier method

Fibre length is divided into a large number of segments of with Δz. Within a segment, the effect of nonlinearity is included at the midplane.

$$\frac{\partial A}{\partial z}+\beta_1\frac{\partial A}{\partial t}-i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial t^2}+\frac{\alpha}{2}A=-i\gamma|A|^2 A$$

$$\frac{\partial A}{\partial z}+\beta_1\frac{\partial A}{\partial t}-i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial t^2}+\frac{\alpha}{2}A=0$$

$$\frac{\partial A}{\partial z}=-i\gamma|A|^2 A$$

Nonlinear phase shift

Recall NLSE:

Loss

Nonlinear phase shift due to $\chi^{(3)}$

$$A(z,t) = \int \tilde{A}(z,\Omega)\exp(i\Omega t)d\Omega$$

$$= \int \Phi(\Omega)\exp[i(\Omega t - Bz) - \frac{\alpha}{2}z - i\gamma|A|^2 z]d\Omega$$

$$= \int \Phi(\Omega)\exp[i\Omega t - i(\beta_1\Omega + \frac{\beta_2}{2}\Omega^2 + \cdots)z - \frac{\alpha}{2}z - i\gamma|A|^2 z]d\Omega$$

$$\rightarrow \frac{\partial A}{\partial z} + \beta_1\frac{\partial A}{\partial t} - \frac{i\beta_2}{2}\frac{\partial^2 A}{\partial t^2} + \frac{\alpha}{2}A + \cdots = -i\gamma|A|^2 A$$

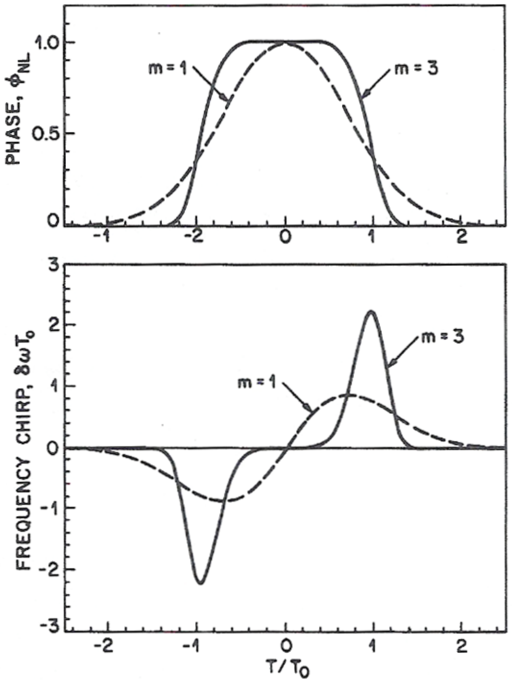
Nonlinear phase shift:

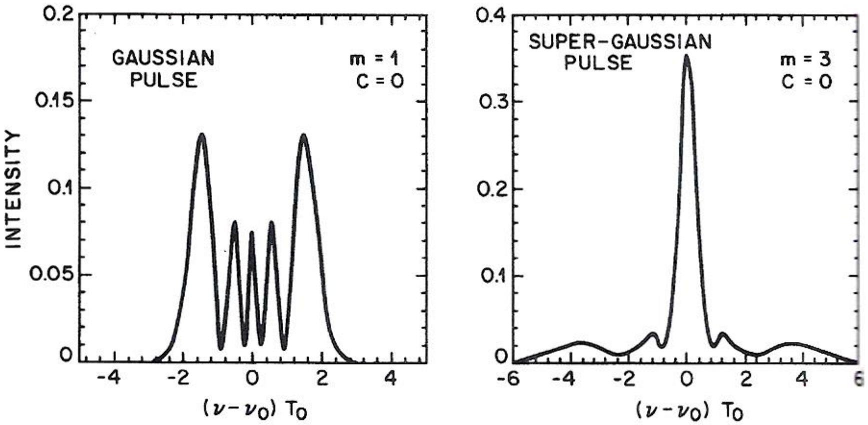
$$\phi_{NL}(z,t) = -\gamma|A(z,t)|^2 z$$

$$\delta\omega(z,t) = \frac{\partial}{\partial t}\left(-\gamma|A(z,t)|^2 z\right)$$

Self-phase modulation (SPM)

For Gaussian and super-Gaussian pulses:





$$\phi_{NL,\max} = 4.5\pi$$

Frequency chirp incurred

Source: Nonlinear Fiber Optics, G. P. Agrawal