Four-wave mixing II

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Tensor notation for the third-order susceptibility

Constitutive relations for E-field:

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_o\mathbf{E} + \mathbf{P},$$

$$\mathbf{P} = \varepsilon_o\chi\mathbf{E} = \varepsilon_o\left(\chi^{(1)}\mathbf{E} + \chi^{(2)}\mathbf{EE} + \underline{\chi^{(3)}}\mathbf{EEE} + \cdots\right)$$

Tensor notation for $\chi^{(3)}$:

$$\chi^{(3)} \rightarrow \chi^{(3)}_{ijkl}(\omega_4 = \omega_1 + \omega_2 + \omega_3)$$

$$P_{i,NL} \leftarrow \sum_{j,k,l} \chi^{(3)}_{ijkl} E_j E_k E_l$$

Note that we normally omit summation notation.

Tensor nature of the third-order susceptibility

How many elements for $\chi_{ijkl}^{(3)}$? *81 elements*

For isotropic media:

$$\chi_{1111}^{(3)} = \chi_{2222}^{(3)} = \chi_{3333}^{(3)},$$
$$\chi_{1122}^{(3)} = \chi_{1133}^{(3)} = \chi_{2211}^{(3)} = \chi_{2233}^{(3)} = \chi_{3311}^{(3)} = \chi_{3322}^{(3)},$$
$$\chi_{1212}^{(3)} = \chi_{1313}^{(3)} = \chi_{2323}^{(3)} = \chi_{2121}^{(3)} = \chi_{3131}^{(3)} = \chi_{3232}^{(3)},$$
$$\chi_{1221}^{(3)} = \chi_{1331}^{(3)} = \chi_{2112}^{(3)} = \chi_{2332}^{(3)} = \chi_{3113}^{(3)} = \chi_{3223}^{(3)}.$$

$\rightarrow$ *21 nonzero elements*

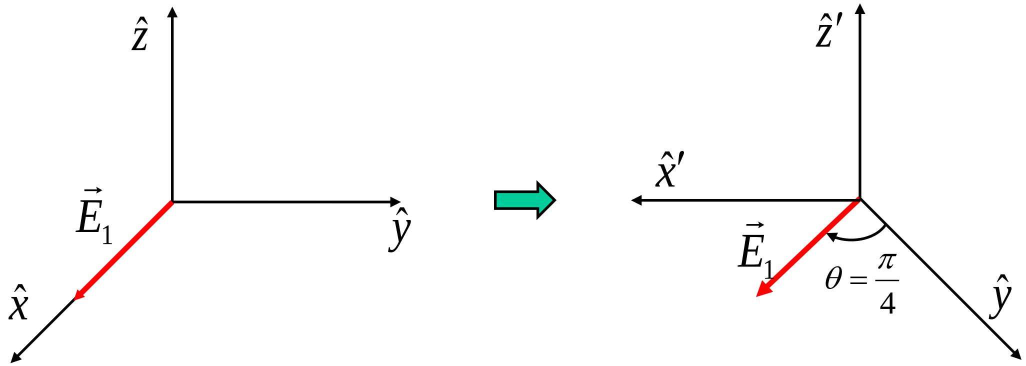
In addition:

$$\chi_{1111}^{(3)} = \chi_{1122}^{(3)} + \chi_{1212}^{(3)} + \chi_{1221}^{(3)}$$

$\rightarrow$ *Why?*

$\rightarrow$ In the compact form: $\chi_{ijkl}^{(3)} = \chi_{1122}^{(3)}\delta_{ij}\delta_{kl} + \chi_{1212}^{(3)}\delta_{ik}\delta_{jl} + \chi_{1221}^{(3)}\delta_{il}\delta_{jk}$

Proof



$$\vec{P}_{NL} = \hat{x}\frac{3}{4}\varepsilon_0\chi^{(3)}_{xxxx}|E_1^2|E_1$$

$$\vec{E}_1' = \hat{x}'\frac{1}{\sqrt{2}}E_1 + \hat{y}'\frac{1}{\sqrt{2}}E_1 = \hat{x}'E'_{1,x} + \hat{y}'E'_{1,y}$$

$$P'_{NL,x'} = \frac{3}{4}\varepsilon_0(\chi^{(3)}_{xxxx}|E'^2_{1,x}|E_{1,x} + \chi^{(3)}_{xxyy}|E'^2_{1,y}|E_{1,x} + \chi^{(3)}_{xyxy}|E'^2_{1,y}|E_{1,x} + \chi^{(3)}_{xyyx}|E'^2_{1,y}|E_{1,x})$$

$$= \frac{3}{4}\varepsilon_0|E_1|^2E_1\frac{1}{2\sqrt{2}}(\chi^{(3)}_{xxxx} + \chi^{(3)}_{xxyy} + \chi^{(3)}_{xyxy} + \chi^{(3)}_{xyyx})$$

Rotation invariant:

$$P_{NL} = P'_{NL}$$

$$\rightarrow \chi^{(3)}_{xxxx} = \chi^{(3)}_{xxyy} + \chi^{(3)}_{xyxy} + \chi^{(3)}_{xyyx}$$

$$P'_{NL,y'} = \frac{3}{4}\varepsilon_0(\chi^{(3)}_{yyyy}|E'^2_{1,y}|E_{1,y} + \chi^{(3)}_{yyxx}|E'^2_{1,x}|E_{1,y} + \chi^{(3)}_{yxyx}|E'^2_{1,x}|E_{1,y} + \chi^{(3)}_{yxxy}|E'^2_{1,x}|E_{1,y})$$

$$= \frac{3}{4}\varepsilon_0|E_1|^2E_1\frac{1}{2\sqrt{2}}(\chi^{(3)}_{xxxx} + \chi^{(3)}_{xxyy} + \chi^{(3)}_{xyxy} + \chi^{(3)}_{xyyx})$$

$$\vec{P}'_{NL} = \hat{x}'P'_{NL,x'} + \hat{y}'P'_{NL,y'}$$

$$P'_{NL} = \sqrt{2}P'_{NL,x'} = \frac{3}{4}\varepsilon_0|E_1|^2E_1\frac{1}{2}(\chi^{(3)}_{xxxx} + \chi^{(3)}_{xxyy} + \chi^{(3)}_{xyxy} + \chi^{(3)}_{xyyx})$$

Four-wave mixing (FWM)

Consider four optical waves oscillating at $\omega_1, \omega_2, \omega_3$ and ω_4 :

$$\mathbf{E}(r,t) = \frac{1}{2}\hat{x}\sum_{j=1}^{4} E_j \exp[i(k_j z - \omega_j t)] + c.c.$$

Induced nonlinear polarisation:

$$\mathbf{P}_{NL} = \varepsilon_0 \chi^{(3)} \mathbf{EEE} = \frac{1}{2}\hat{x}\sum_{j=1}^{4} P_j \exp[i(k_j z - i\omega_j t)] + c.c$$

$$+\cdots$$

For ω_4 :

$$P_4 = \frac{3\varepsilon_0 \chi^{(3)}_{xxxx}}{4}[|E_4|^2 E_4 + 2(|E_1|^2 + |E_2|^2 + |E_3|^2)E_4$$

$$+ 2E_1E_2E_3 \exp(i\theta_+) + 2E_1E_2E_3^* \exp(i\theta_-) + \cdots],$$

where

$$\theta_+ = (k_1 + k_2 + k_3 - k_4)z - (\omega_1 + \omega_2 + \omega_3 - \omega_4)t,$$

$$\theta_- = (k_1 + k_2 - k_3 - k_4)z - (\omega_1 + \omega_2 - \omega_3 - \omega_4)t.$$

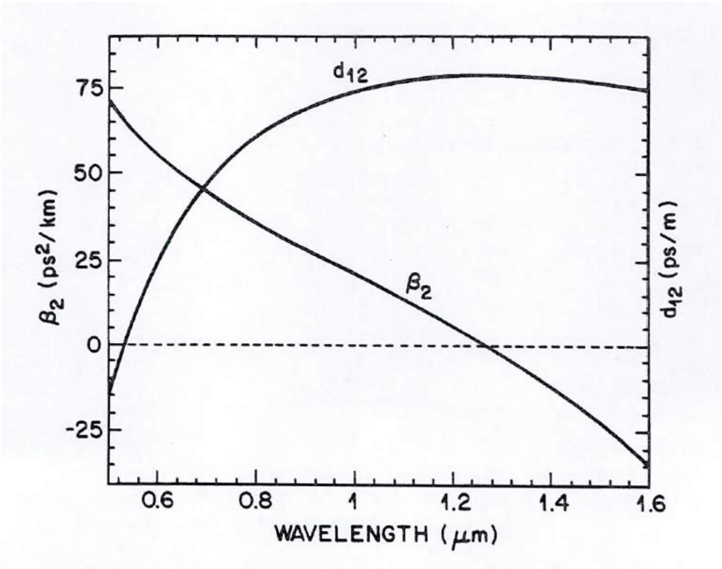
Easy for phase-matching

Phase-matching condition

For energy conservation:

$$\omega_3 + \omega_4 - \omega_1 - \omega_2 = 0$$

$$\omega_3 + \omega_4 - 2\omega_1 = 0, \quad if\ \omega_1 = \omega_2$$



Source: Nonlinear Fiber Optics, G. P. Agrawal

For momentum conservation:

$$\Delta k = k_3 + k_4 - k_1 - k_2$$
$$= (n_3\omega_3 + n_4\omega_4 - n_1\omega_1 - n_2\omega_2)/c = 0$$

$$\Delta k = k_3 + k_4 - 2k_1$$
$$= (n_3\omega_3 + n_4\omega_4 - 2n_1\omega_1)/c = 0, \quad if\ \omega_1 = \omega_2$$

Is this it?

Recall: $\beta(\omega) = n(\omega)\dfrac{\omega}{c} = \beta_0 + \beta_1(\omega - \omega_0) + \dfrac{1}{2}\beta_2(\omega - \omega_0)^2 + \cdots,$

$$where \quad \beta_m = \left(\frac{d^m\beta}{d\omega_m}\right)_{\omega=\omega_0} \quad (m = 1, 2, 3, \ldots)$$

Coupled wave equations

Assuming: $E_j(\mathbf{r}) = F_j(x,y)A_j(z),$

$$\frac{dA_1}{dz} = \frac{in_2'\omega_1}{c}[(f_{11}|A_1|^2 + 2\sum_{j\neq 1}|A_j|^2)A_1 + 2f_{1234}A_2^*A_3A_4e^{i\Delta kz},$$

$$\frac{dA_2}{dz} = \frac{in_2'\omega_2}{c}[(f_{22}|A_2|^2 + 2\sum_{j\neq 2}|A_j|^2)A_2 + 2f_{2134}A_1^*A_3A_4e^{i\Delta kz},$$

$$\frac{dA_3}{dz} = \frac{in_2'\omega_3}{c}[(f_{33}|A_3|^2 + 2\sum_{j\neq 3}|A_j|^2)A_3 + 2f_{3412}A_1A_2A_4^*e^{-i\Delta kz},$$

$$\frac{dA_4}{dz} = \frac{in_2'\omega_4}{c}[(f_{44}|A_4|^2 + 2\sum_{j\neq 4}|A_j|^2)A_4 + 2f_{4312}A_1A_2A_3^*e^{-i\Delta kz},$$

where $$f_{jk} = \frac{\langle |F_j|^2|F_k|^2\rangle}{\langle |F_j|^2\rangle\langle |F_k|^2\rangle},$$

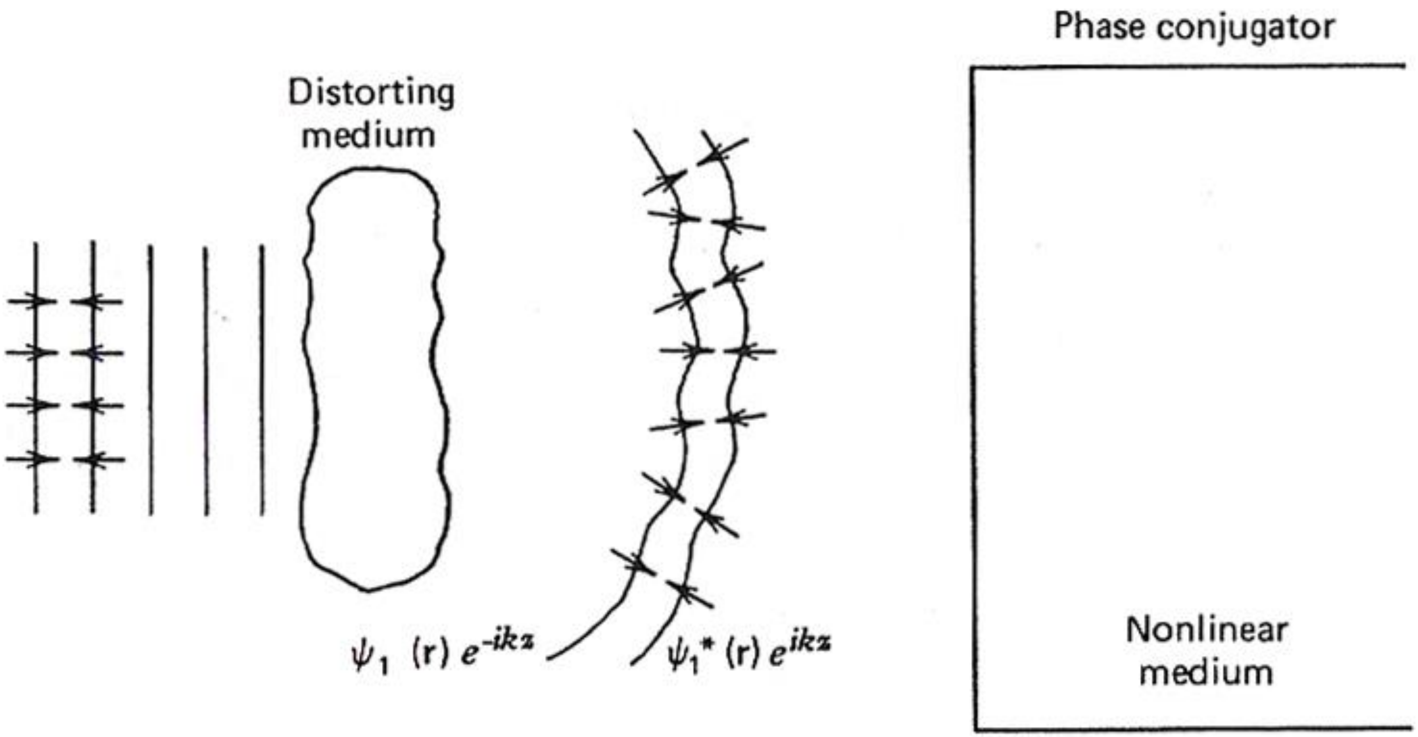
$$f_{ijkl} = \frac{\langle F_i^*F_j^*F_kF_l\rangle}{[\langle |F_i|^2\rangle\langle |F_j|^2\rangle\langle |F_k|^2\rangle\langle |F_l|^2\rangle]^{1/2}}.$$

Effective phase-matching condition:

$$\kappa = \Delta k + 2\gamma P_0 = 0$$

where $P_0 = P_1 + P_2$

Phase conjugation

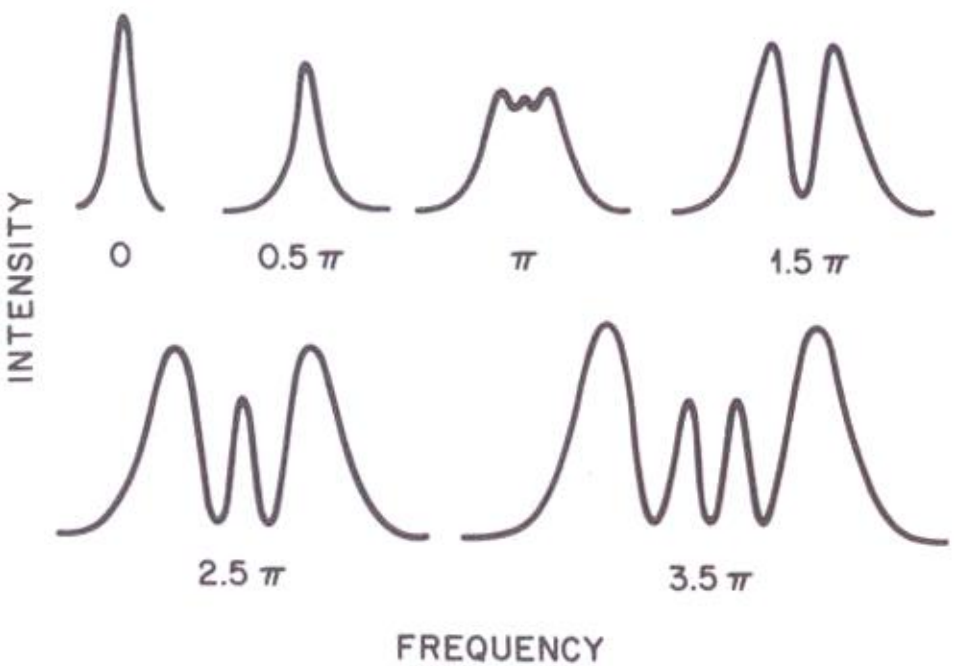


Source: *Optical Waves in Crystals*, A. Yariv and P. Yeh

Complex conjugation of an input field via FWM

Self-phase modulation (SPM)?

For Gaussian pulses:



Source: Nonlinear Fiber Optics, G. P. Agrawal

SPM or XPM?

$$P_1 = \frac{3\varepsilon_0 \chi^{(3)}_{xxxx}}{4}[|E_1|^2 E_1 + 2(|E_2|^2 + |E_3|^2 + |E_4|^2)E_1 \,?$$