Electro-Optics:

Electromagnetic Fields (2)

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Complex-Function Formalism

Field component:

$$a(t) = |A|\cos(\omega t + \alpha)$$

Angular frequency (ω), *Amplitude* ($|A|$), *Phase* (α)

Consider a complex amplitude:

$$A = |A|e^{i\alpha}$$

$$\rightarrow \quad a(t) = \mathrm{Re}[Ae^{i\omega t}]$$

Product of two sinusoidal functions:

$$a(t) = |A|\cos(\omega t + \alpha) \qquad b(t) = |B|\cos(\omega t + \beta)$$

$$= \mathrm{Re}[Ae^{i\omega t}] \qquad\qquad = \mathrm{Re}[Be^{i\omega t}]$$

$$\rightarrow \quad a(t)b(t) = \frac{1}{2}|AB|[\cos(2\omega t + \alpha + \beta) + \cos(\alpha - \beta)]$$

$$\rightarrow \quad Ae^{i\omega t}Be^{i\omega t} = |AB|e^{i(2\omega t + \alpha + \beta)}$$

$$\rightarrow \quad \boxed{\mathrm{Re}[x]\mathrm{Re}[y] \neq \mathrm{Re}[xy]}$$

Time-Averaging Sinusoidal Products

Time average of the product of two sinusoidal functions of the same freq.:

$$\langle a(t)b(t)\rangle = \frac{1}{T}\int_0^T |A|\cos(\omega t+\alpha)|B|\cos(\omega t+\beta)dt$$

$$= \frac{1}{2}|AB|\cos(\alpha-\beta)$$

$$\rightarrow \langle a(t)b(t)\rangle = \frac{1}{2}\mathrm{Re}[AB^*]$$

$$\rightarrow \langle \mathrm{Re}[x(t)]\mathrm{Re}[y(t)]\rangle = \frac{1}{2}\mathrm{Re}[x(t)y(t)^*]$$

Time-averaged Poynting vector and energy density:

$$\mathbf{S} = \frac{1}{2}\mathrm{Re}[\mathbf{E}\times\mathbf{H}^*]$$

$$U = \frac{1}{4}\mathrm{Re}[\mathbf{E}\cdot\mathbf{D}^* + \mathbf{B}\cdot\mathbf{H}^*]$$

Wave Equations & Monochromatic Plane Waves

For an isotropic medium:

No charge density & no current density:

$$\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0 \quad \rightarrow \nabla\times\left(\frac{1}{\mu}\nabla\times\mathbf{E}\right)+\frac{\partial}{\partial t}\nabla\times\mathbf{H}=0 \quad \leftarrow \boxed{\rho=0,\ \mathbf{J}=0}$$

$$\rightarrow \nabla\times\left(\frac{1}{\mu}\nabla\times\mathbf{E}\right)+\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}=0$$

$$\leftarrow \nabla\times\left(\frac{1}{\mu}\nabla\times\mathbf{E}\right)=\frac{1}{\mu}\nabla\times(\nabla\times\mathbf{E})+\left(\nabla\frac{1}{\mu}\right)\times\nabla\times\mathbf{E}$$

$$\leftarrow \nabla\times(\nabla\times\mathbf{E})=\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}$$

$$\rightarrow \nabla^2\mathbf{E}-\mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}+(\nabla\log\mu)\times(\nabla\times\mathbf{E})-\nabla(\nabla\cdot\mathbf{E})=0$$

$$\rightarrow \nabla\cdot(\varepsilon\mathbf{E})=\varepsilon(\nabla\cdot\mathbf{E})+\mathbf{E}\cdot\nabla\varepsilon=\rho=0$$

Wave equation:

$$\rightarrow \nabla^2\mathbf{E}-\mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}+(\nabla\log\mu)\times(\nabla\times\mathbf{E})+\nabla(\mathbf{E}\cdot\nabla\log\varepsilon)=0$$

$$\rightarrow \nabla^2\mathbf{H}-\mu\varepsilon\frac{\partial^2\mathbf{H}}{\partial t^2}+(\nabla\log\varepsilon)\times(\nabla\times\mathbf{H})+\nabla(\mathbf{H}\cdot\nabla\log\mu)=0$$

Monochromatic Plane Waves

Wave equations: *Homogeneous, isotropic & no source*

$$\nabla^2\mathbf{E}-\mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}=0,\ \ \nabla^2\mathbf{H}-\mu\varepsilon\frac{\partial^2\mathbf{H}}{\partial t^2}=0$$

Plane waves:

$$\psi=e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})},\quad |\mathbf{k}|=\omega\sqrt{\mu\varepsilon}$$

Phase velocity:

$$\omega t-\mathbf{k}\cdot\mathbf{r}=\text{constant},\quad v_p=\frac{\omega}{k}=\frac{1}{\sqrt{\mu\varepsilon}},$$

$$c=\frac{1}{\sqrt{\mu_o\varepsilon_o}}=2.997930\ \times 10^8\ \ (\text{m/s})$$

Field vectors:

$$\mathbf{E}=\mathbf{u}_1E_0e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}$$

$$\mathbf{H}=\mathbf{u}_2H_0e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}$$

$$\leftarrow\ \mathbf{u}_1\cdot\mathbf{k}=\mathbf{u}_2\cdot\mathbf{k}=0$$

$$\leftarrow\ \mathbf{u}_2=\frac{\mathbf{k}\times\mathbf{u}_1}{|\mathbf{k}|}$$

Impedance of space:

$$\eta=\frac{E_0}{H_0}=\sqrt{\frac{\mu}{\varepsilon}}$$

Time-averaged power flow & energy density:

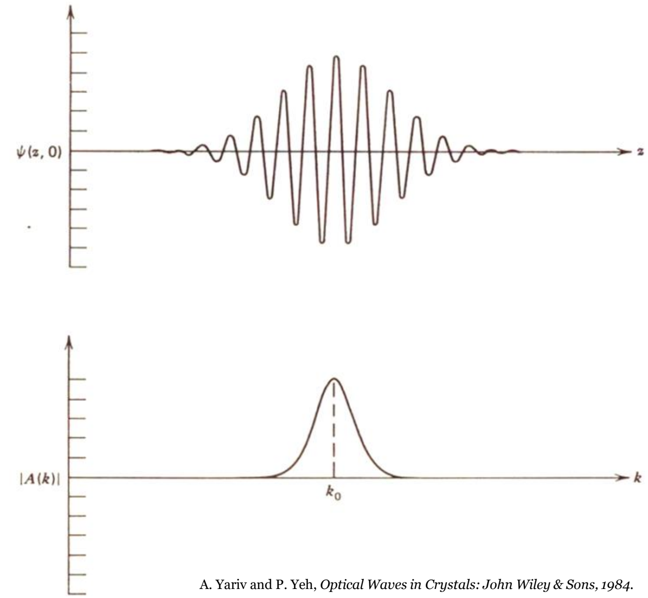
$$\mathbf{S}=\frac{|E_0|^2\mathbf{u}_3}{2\eta}=\frac{\mathbf{k}}{2\omega\mu}|E_0|^2$$

$$U=\frac{1}{2}\varepsilon|E_0|^2$$

Propagation of a Laser Pulse (1)

Wave packet:

$$\psi(z,t)=\int_{-\infty}^{\infty}A(k)e^{i[\omega(k)t-kz]}dk$$



A. Yariv and P. Yeh, *Optical Waves in Crystals: John Wiley & Sons, 1984.*

Dispersion relation: $\omega=f(k)$

$\rightarrow$ Taylor series expansion

$$\omega(k)=\omega_0+\left(\frac{d\omega}{dk}\right)_0(k-k_0)+\cdots$$

$$\rightarrow\ \psi(z,t)=\int_{-\infty}^{\infty}A(k)e^{i[\omega(k)t-kz]}dk$$

$$\cong\int_{-\infty}^{\infty}A(k)e^{i\{[\omega_0+\left(\frac{d\omega}{dk}\right)_0(k-k_0)]t-kz\}}dk$$

$\leftarrow$ First-order perturbation

$$=e^{i(\omega_0t-k_0z)}\int_{-\infty}^{\infty}A(k)\exp\left\{i\left[\left(\frac{d\omega}{dk}\right)_0t-z\right](k-k_0)\right\}dk$$

Carrier

$$=e^{i(\omega_0t-k_0z)}F\left[z-\left(\frac{d\omega}{dk}\right)_0t\right]$$

Envelop

Propagation of a Laser Pulse (2)

Wave packet:

$$\psi(z,t) = \int_{-\infty}^{\infty} A(k) e^{i[\omega(k)t - kz]} dk \quad \cong e^{i(\omega_0 t - k_0 z)} F\left[z - \left(\frac{d\omega}{dk} \right)_0 t \right]$$

Group velocity: Velocity of the envelop

$$v_g = \left(\frac{d\omega}{dk} \right)_0 = \frac{c}{n + \omega(dn/d\omega)}$$

Dispersion relation:

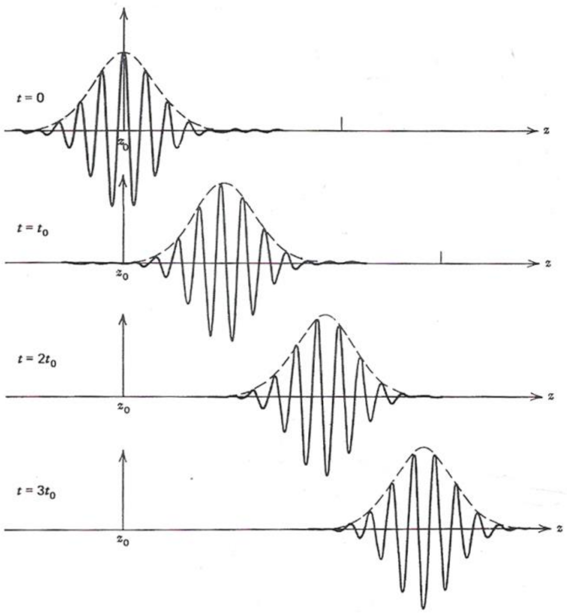
$$k = \frac{\omega}{c} n(\omega)$$

Phase velocity:

$$v_p = \frac{c}{n(\omega)}$$

Group velocity dispersion:

$$\Delta v_g \sim \left(\frac{d^2\omega}{dk^2} \right)_0 \Delta k$$



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