Electro-Optics:

Electromagnetic Propagation in Anisotropic Media (1)

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Dielectric Tensor of an Anisotropic Medium

Induced polarization:

$$P_x = \varepsilon_0(\chi_{11}E_x + \chi_{12}E_y + \chi_{13}E_z),$$
$$P_y = \varepsilon_0(\chi_{21}E_x + \chi_{22}E_y + \chi_{23}E_z), \quad \leftarrow \ \chi_{ij}: \text{Electric susceptibility tensor}$$
$$P_z = \varepsilon_0(\chi_{31}E_x + \chi_{32}E_y + \chi_{33}E_z).$$

$$\rightarrow \quad \begin{aligned} P_x &= \varepsilon_0\chi_{11}E_x, \\ P_y &= \varepsilon_0\chi_{22}E_y, \\ P_z &= \varepsilon_0\chi_{33}E_z. \end{aligned} \qquad \leftarrow \text{Principal dielectric axes}$$

Constitutive relation:

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_0\mathbf{E} + \mathbf{P}$$

Tensor notation:

$$\rightarrow \quad \varepsilon_{ij} = \varepsilon_0(1+\chi_{ij}) \qquad \leftarrow \text{Dielectric permittivity tensor}$$

$$\rightarrow \quad D_i = \varepsilon_{ij}E_j$$

Dielectric Permittivity Tensor

Energy density:

$$U_e = \frac{1}{2}\mathbf{E}\cdot\mathbf{D} = \frac{1}{2}E_i\varepsilon_{ij}E_j \quad \rightarrow \dot{U}_e = \frac{1}{2}\varepsilon_{ij}\left(\dot{E}_iE_j + E_i\dot{E}_j\right)$$

$\leftarrow$ homogenous, non-absorbing,
and magnetically isotropic

Recall:

$$\cancel{\mathbf{J}\cdot\mathbf{E}} = -\nabla\cdot(\mathbf{E}\times\mathbf{H}) - \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} - \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}$$

$$\rightarrow\ -\nabla\cdot(\mathbf{E}\times\mathbf{H}) = \mathbf{H}\cdot\dot{\mathbf{B}} + \mathbf{E}\cdot\dot{\mathbf{D}}$$

$$\rightarrow\ \frac{1}{2}\varepsilon_{ij}\left(\dot{E}_iE_j + E_i\dot{E}_j\right) = \varepsilon_{ij}E_i\dot{E}_j$$

$$\rightarrow\ \varepsilon_{ij} = \varepsilon_{ji}$$ $\leftarrow$ Symmetric

In a complex dielectric tensor notation:

$$\rightarrow\ \varepsilon_{ij} = \varepsilon_{ji}^{*}$$ $\leftarrow$ Hermitian

Plane-Wave Propagation in Anisotropic Media (1)

Field representation for a monochromatic plane wave:

$$\mathbf{E}\exp[i(\omega t-\mathbf{k}\cdot\mathbf{r})]$$

$$\mathbf{H}\exp[i(\omega t-\mathbf{k}\cdot\mathbf{r})]$$

Recall Maxwell's equations:

$$\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0 \quad \rightarrow \quad \mathbf{k}\times\mathbf{E}=\omega\mu\mathbf{H}$$

$$\leftarrow \quad \nabla\times(\psi\mathbf{U})=\cancel{\psi\nabla\times\mathbf{U}}+\nabla\psi\times\mathbf{U}$$

$$\nabla\times\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t}=\cancel{\mathbf{J}} \quad \rightarrow \quad \mathbf{k}\times\mathbf{H}=-\omega\varepsilon\mathbf{E}$$

Wave equation:

$$\rightarrow \quad \mathbf{k}\times(\mathbf{k}\times\mathbf{E})+\omega^2\mu\varepsilon\mathbf{E}=0$$

Dielectric tensor in the principal coordinate system:

$$\varepsilon=\begin{pmatrix}\varepsilon_x & 0 & 0\\ 0 & \varepsilon_y & 0\\ 0 & 0 & \varepsilon_z\end{pmatrix}$$

Plane-Wave Propagation in Anisotropic Media (2)

Wave equation:

$$\begin{pmatrix} \omega^2\mu\varepsilon_x - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_y k_x & \omega^2\mu\varepsilon_y - k_x^2 - k_z^2 & k_y k_z \\ k_z k_x & k_z k_y & \omega^2\mu\varepsilon_z - k_x^2 - k_y^2 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

For nontrivial solutions to exist:

$$\rightarrow \begin{vmatrix} \omega^2\mu\varepsilon_x - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_y k_x & \omega^2\mu\varepsilon_y - k_x^2 - k_z^2 & k_y k_z \\ k_z k_x & k_z k_y & \omega^2\mu\varepsilon_z - k_x^2 - k_y^2 \end{vmatrix} = 0$$

Let:

$$k^2 - \omega^2\mu\varepsilon_x \equiv X, \qquad \rightarrow \det = k_x^2 YZ + k_y^2 ZX + k_z^2 XY - XYZ = 0$$

$$k^2 - \omega^2\mu\varepsilon_y \equiv Y, \qquad \rightarrow\ XYZ\left(\frac{k_x^2}{X} + \frac{k_y^2}{Y} + \frac{k_z^2}{Z} - 1\right) = 0$$

$$k^2 - \omega^2\mu\varepsilon_z \equiv Z.$$

In case:

$$\rightarrow\ XYZ \neq 0 \qquad \rightarrow\ \frac{k_x^2}{X} + \frac{k_y^2}{Y} + \frac{k_z^2}{Z} = 1$$

Plane-Wave Propagation in Anisotropic Media (3)

Normal surface and electric-field vector in case $XYZ \neq 0$:

$$\rightarrow \frac{k_x^2}{k^2-\omega^2\mu\varepsilon_x}+\frac{k_y^2}{k^2-\omega^2\mu\varepsilon_y}+\frac{k_z^2}{k^2-\omega^2\mu\varepsilon_z}=1$$

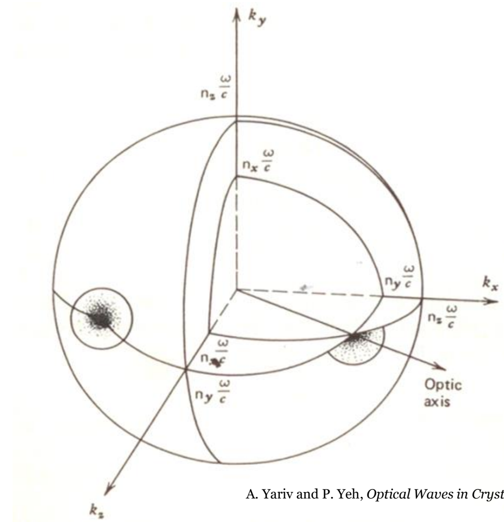
$$\rightarrow \mathbf{E}=\begin{pmatrix} \dfrac{k_x}{k^2-\omega^2\mu\varepsilon_x} \\ \dfrac{k_y}{k^2-\omega^2\mu\varepsilon_y} \\ \dfrac{k_z}{k^2-\omega^2\mu\varepsilon_z} \end{pmatrix}$$

$$\rightarrow \mathbf{k}=\frac{\omega}{c}n\mathbf{s}$$

$$\rightarrow \frac{s_x^2}{n^2-\varepsilon_x/\varepsilon_0}+\frac{s_y^2}{n^2-\varepsilon_y/\varepsilon_0}+\frac{s_z^2}{n^2-\varepsilon_z/\varepsilon_0}=\frac{1}{n^2}$$

$$\rightarrow \mathbf{E}=\begin{pmatrix} \dfrac{s_x}{n^2-\varepsilon_x/\varepsilon_0} \\ \dfrac{s_y}{n^2-\varepsilon_y/\varepsilon_0} \\ \dfrac{s_z}{n^2-\varepsilon_z/\varepsilon_0} \end{pmatrix}$$

Plane-Wave Propagation in Anisotropic Media (4)



A. Yariv and P. Yeh, *Optical Waves in Crystals*, 1984

Shape of normal surface :

$$k_x^2 YZ + k_y^2 ZX + k_z^2 XY - XYZ = 0$$

k_x-k_y plane:

$$\rightarrow k_z = 0 \quad \rightarrow k_x^2 YZ + k_y^2 ZX - XYZ = 0$$

$$\rightarrow Z = 0 \qquad \rightarrow k_x^2 + k_y^2 = \left(\frac{\omega}{c} n_z\right)^2$$

or

$$\rightarrow k_x^2 Y + k_y^2 X - XY = 0 \quad \rightarrow \frac{k_x^2}{\left(\frac{\omega}{c} n_y\right)^2} + \frac{k_y^2}{\left(\frac{\omega}{c} n_x\right)^2} = 1$$

k_y-k_z plane:

$$\rightarrow k_y^2 + k_z^2 = \left(\frac{\omega}{c} n_x\right)^2 \quad \rightarrow \frac{k_y^2}{\left(\frac{\omega}{c} n_z\right)^2} + \frac{k_z^2}{\left(\frac{\omega}{c} n_y\right)^2} = 1$$

k_z-k_x plane:

$$\rightarrow k_z^2 + k_x^2 = \left(\frac{\omega}{c} n_y\right)^2 \quad \rightarrow \frac{k_z^2}{\left(\frac{\omega}{c} n_x\right)^2} + \frac{k_x^2}{\left(\frac{\omega}{c} n_z\right)^2} = 1$$

Orthogonality Properties of the Eigenmodes (1)

Displacement vectors:

$$\nabla \cdot \mathbf{D} = 0 \quad \rightarrow \mathbf{D} \perp \mathbf{s}$$

$$\mathbf{D} = -\frac{n}{c}\mathbf{s}\times\mathbf{H} \quad \leftarrow \mathbf{H} = \frac{n}{\mu c}\mathbf{s}\times\mathbf{E}$$

$$\rightarrow \mathbf{D} = -\frac{n^2}{c^2\mu}\mathbf{s}\times(\mathbf{s}\times\mathbf{E}) = \frac{n^2}{c^2\mu}[\mathbf{E}-\mathbf{s}(\mathbf{s}\cdot\mathbf{E})]$$

$$= \frac{n^2}{c^2\mu}\mathbf{E}_{transverse} = n^2\varepsilon_0\mathbf{E}_{transverse}$$

Orthogonality relations for eigenmodes:

$$\mathbf{E}_1 = \mathbf{E}_{1t} + \mathbf{E}_{1l}, \quad \mathbf{E}_2 = \mathbf{E}_{2t} + \mathbf{E}_{2l}$$

$$\rightarrow \mathbf{D}_1 = n_1^2\varepsilon_0\mathbf{E}_{1t}, \quad \mathbf{D}_2 = n_2^2\varepsilon_0\mathbf{E}_{2t}$$

$$\rightarrow \mathbf{E}_2 \cdot \mathbf{D}_1 = E_{2i}\varepsilon_{ij}E_{1j}$$

$$\rightarrow \mathbf{E}_1 \cdot \mathbf{D}_2 = E_{1i}\varepsilon_{ij}E_{2j} = E_{1i}\varepsilon_{ji}E_{2j} = E_{1j}\varepsilon_{ij}E_{2i}$$

$$\rightarrow \mathbf{E}_2 \cdot \mathbf{D}_1 = \mathbf{E}_1 \cdot \mathbf{D}_2$$

$$\rightarrow \mathbf{E}_2 \cdot (n_1^2\varepsilon_0\mathbf{E}_{1t}) = \mathbf{E}_1 \cdot (n_2^2\varepsilon_0\mathbf{E}_{2t})$$

$$\rightarrow \varepsilon_0(n_1^2 - n_2^2)\mathbf{E}_{1t}\cdot\mathbf{E}_{2t} = 0$$

$$\leftarrow n_1^2 \neq n_2^2$$

$$\rightarrow \mathbf{E}_{1t}\cdot\mathbf{E}_{2t} = 0$$

$$\rightarrow \mathbf{D}_1\cdot\mathbf{D}_2 = 0$$

$$\rightarrow \mathbf{D}_1\cdot\mathbf{E}_2 = \mathbf{D}_2\cdot\mathbf{E}_1 = 0$$

$$\rightarrow \mathbf{s}\cdot\mathbf{D}_1 = \mathbf{s}\cdot\mathbf{D}_2 = 0$$

Orthogonality Properties of the Eigenmodes (2)

Orthogonality relations for eignemodes:

$$\mathbf{s}\cdot(\mathbf{E}_1\times\mathbf{H}_2) = \frac{n_2}{\mu c}\mathbf{s}\cdot[\mathbf{E}_1\times(\mathbf{s}\times\mathbf{E}_2)] \quad \leftarrow \quad \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

$$= \frac{n_2}{\mu c}[(\mathbf{s}\times\mathbf{E}_2)\cdot(\mathbf{s}\times\mathbf{E}_1)]$$

$$= \frac{n_2}{\mu c}[(\mathbf{s}\times\mathbf{E}_{2t})\cdot(\mathbf{s}\times\mathbf{E}_{1t})]$$

$$= 0$$

$$\rightarrow \quad \mathbf{s}\cdot(\mathbf{E}_1\times\mathbf{H}_2) = \mathbf{s}\cdot(\mathbf{E}_2\times\mathbf{H}_1) = 0$$

$\leftarrow$ No power flow