

Electro-Optics:

Electromagnetic Propagation in Anisotropic Media (1)

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Dielectric Tensor of an Anisotropic Medium

Induced polarization:

$$P_x = \varepsilon_0(\chi_{11}E_x + \chi_{12}E_y + \chi_{13}E_z),$$

$$P_y = \varepsilon_0(\chi_{21}E_x + \chi_{22}E_y + \chi_{23}E_z), \quad \leftarrow \chi_{ij} : \text{Electric susceptibility tensor}$$

$$P_z = \varepsilon_0(\chi_{31}E_x + \chi_{32}E_y + \chi_{33}E_z).$$

$$P_x = \varepsilon_0\chi_{11}E_x,$$

$$\rightarrow P_y = \varepsilon_0\chi_{22}E_y, \quad \leftarrow \text{Principal dielectric axes}$$

$$P_z = \varepsilon_0\chi_{33}E_z.$$

Constitutive relation:

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_0\mathbf{E} + \mathbf{P}$$

Tensor notation:

$$\rightarrow \varepsilon_{ij} = \varepsilon_0(1 + \chi_{ij}) \quad \leftarrow \text{Dielectric permittivity tensor}$$

$$\rightarrow D_i = \varepsilon_{ij}E_j$$

Dielectric Permittivity Tensor

Energy density:

$$U_e = \frac{1}{2} \mathbf{E} \cdot \mathbf{D} = \frac{1}{2} E_i \varepsilon_{ij} E_j \quad \rightarrow \quad \dot{U}_e = \frac{1}{2} \varepsilon_{ij} (\dot{E}_i E_j + E_i \dot{E}_j)$$

← homogenous, non-absorbing,
and magnetically isotropic

Recall:

$$\begin{aligned} \mathbf{J} \cdot \mathbf{E} &= -\nabla \cdot (\mathbf{E} \times \mathbf{H}) - \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \\ &\rightarrow -\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot \dot{\mathbf{B}} + \mathbf{E} \cdot \dot{\mathbf{D}} \\ &\rightarrow \frac{1}{2} \varepsilon_{ij} (\dot{E}_i E_j + E_i \dot{E}_j) = \varepsilon_{ij} E_i \dot{E}_j \end{aligned}$$

$$\rightarrow \varepsilon_{ij} = \varepsilon_{ji} \quad \leftarrow \text{Symmetric}$$

In a complex dielectric tensor notation:

$$\rightarrow \varepsilon_{ij} = \varepsilon_{ji}^* \quad \leftarrow \text{Hermitian}$$

Plane-Wave Propagation in Anisotropic Media (1)

Field representation for a monochromatic plane wave:

$$\mathbf{E} \exp[i(\omega t - \mathbf{k} \cdot \mathbf{r})]$$

$$\mathbf{H} \exp[i(\omega t - \mathbf{k} \cdot \mathbf{r})]$$

Recall Maxwell's equations:

$$\begin{aligned} \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 & \quad \rightarrow \quad \mathbf{k} \times \mathbf{E} = \omega \mu \mathbf{H} \\ \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} & \quad \rightarrow \quad \mathbf{k} \times \mathbf{H} = -\omega \varepsilon \mathbf{E} \end{aligned} \quad \leftarrow \quad \nabla \times (\psi \mathbf{A}) = \psi \nabla \times \mathbf{A} + \nabla \psi \times \mathbf{A}$$

Wave equation:

$$\rightarrow \mathbf{k} \times (\mathbf{k} \times \mathbf{E}) + \omega^2 \mu \varepsilon \mathbf{E} = 0$$

Dielectric tensor in the principal coordinate system:

$$\varepsilon = \begin{pmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix}$$

Plane-Wave Propagation in Anisotropic Media (2)

Wave equation:

$$\begin{pmatrix} \omega^2 \mu \epsilon_x - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_y k_x & \omega^2 \mu \epsilon_y - k_x^2 - k_z^2 & k_y k_z \\ k_z k_x & k_z k_y & \omega^2 \mu \epsilon_z - k_x^2 - k_y^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

For nontrivial solutions to exist:

$$\rightarrow \begin{vmatrix} \omega^2 \mu \epsilon_x - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_y k_x & \omega^2 \mu \epsilon_y - k_x^2 - k_z^2 & k_y k_z \\ k_z k_x & k_z k_y & \omega^2 \mu \epsilon_z - k_x^2 - k_y^2 \end{vmatrix} = 0$$

Let: $k^2 - \omega^2 \mu \epsilon_x \equiv X,$ $\rightarrow \det = k_x^2 YZ + k_y^2 ZX + k_z^2 XY - XYZ = 0$

$k^2 - \omega^2 \mu \epsilon_y \equiv Y,$ $\rightarrow XYZ \left(\frac{k_x^2}{X} + \frac{k_y^2}{Y} + \frac{k_z^2}{Z} - 1 \right) = 0$

$k^2 - \omega^2 \mu \epsilon_z \equiv Z.$

In case:

$$\rightarrow XYZ \neq 0 \quad \rightarrow \frac{k_x^2}{X} + \frac{k_y^2}{Y} + \frac{k_z^2}{Z} = 1$$

Plane-Wave Propagation in Anisotropic Media (3)

Normal surface and electric-field vector in case $XYZ \neq 0$:

$$\rightarrow \frac{k_x^2}{k^2 - \omega^2 \mu \epsilon_x} + \frac{k_y^2}{k^2 - \omega^2 \mu \epsilon_y} + \frac{k_z^2}{k^2 - \omega^2 \mu \epsilon_z} = 1$$
$$\rightarrow \mathbf{k} = \frac{\omega}{c} n \mathbf{s}$$
$$\rightarrow \frac{s_x^2}{n^2 - \epsilon_x / \epsilon_0} + \frac{s_y^2}{n^2 - \epsilon_y / \epsilon_0} + \frac{s_z^2}{n^2 - \epsilon_z / \epsilon_0} = \frac{1}{n^2}$$
$$\rightarrow \mathbf{E} = \begin{pmatrix} \frac{k_x}{k^2 - \omega^2 \mu \epsilon_x} \\ \frac{k_y}{k^2 - \omega^2 \mu \epsilon_y} \\ \frac{k_z}{k^2 - \omega^2 \mu \epsilon_z} \end{pmatrix}$$

$$\rightarrow \mathbf{E} = \begin{pmatrix} \frac{s_x}{n^2 - \epsilon_x / \epsilon_0} \\ \frac{s_y}{n^2 - \epsilon_x / \epsilon_0} \\ \frac{s_z}{n^2 - \epsilon_x / \epsilon_0} \end{pmatrix}$$

Plane-Wave Propagation in Anisotropic Media (4)

Shape of normal surface :

$$k_x^2 YZ + k_y^2 ZX + k_z^2 XY - XYZ = 0$$

k_x - k_y plane:

$$\rightarrow k_z = 0$$

$$\rightarrow k_x^2 YZ + k_y^2 ZX - XYZ = 0$$

$$\rightarrow Z = 0 \quad \rightarrow k_x^2 + k_y^2 = \left(\frac{\omega}{c} n_z \right)^2$$

or

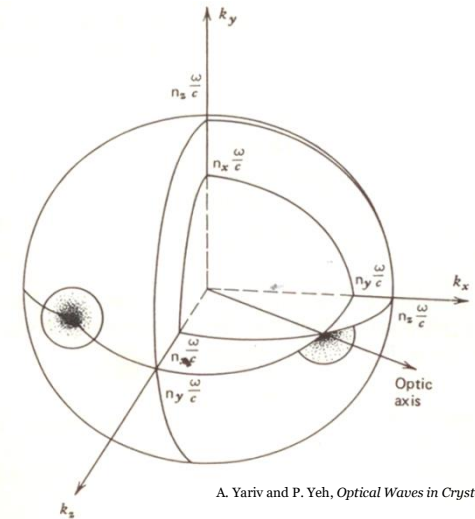
$$\rightarrow k_x^2 Y + k_y^2 X - XY = 0 \quad \rightarrow \frac{k_x^2}{\left(\frac{\omega}{c} n_y \right)^2} + \frac{k_y^2}{\left(\frac{\omega}{c} n_x \right)^2} = 1$$

k_y - k_z plane:

$$\rightarrow k_y^2 + k_z^2 = \left(\frac{\omega}{c} n_x \right)^2 \quad \rightarrow \frac{k_y^2}{\left(\frac{\omega}{c} n_z \right)^2} + \frac{k_z^2}{\left(\frac{\omega}{c} n_y \right)^2} = 1$$

k_z - k_x plane:

$$\rightarrow k_z^2 + k_x^2 = \left(\frac{\omega}{c} n_y \right)^2 \quad \rightarrow \frac{k_z^2}{\left(\frac{\omega}{c} n_x \right)^2} + \frac{k_x^2}{\left(\frac{\omega}{c} n_z \right)^2} = 1$$



Orthogonality Properties of the Eigenmodes (1)

Displacement vectors:

$$\nabla \cdot \mathbf{D} = 0 \quad \rightarrow \quad \mathbf{D} \perp \mathbf{s}$$

$$\mathbf{D} = -\frac{n}{c} \mathbf{s} \times \mathbf{H} \quad \leftarrow \quad \mathbf{H} = \frac{n}{\mu c} \mathbf{s} \times \mathbf{E}$$

$$\begin{aligned} \rightarrow \mathbf{D} &= -\frac{n^2}{c^2 \mu} \mathbf{s} \times (\mathbf{s} \times \mathbf{E}) = \frac{n^2}{c^2 \mu} [\mathbf{E} - \mathbf{s}(\mathbf{s} \cdot \mathbf{E})] \\ &= \frac{n^2}{c^2 \mu} \mathbf{E}_{\text{transverse}} = n^2 \varepsilon_0 \mathbf{E}_{\text{transverse}} \end{aligned}$$

Orthogonality relations for eigenmodes:

$$\begin{aligned} \mathbf{E}_1 &= \mathbf{E}_{1t} + \mathbf{E}_{1l}, \quad \mathbf{E}_2 = \mathbf{E}_{2t} + \mathbf{E}_{2l} & \rightarrow \mathbf{E}_2 \cdot (n_1^2 \varepsilon_0 \mathbf{E}_{1t}) &= \mathbf{E}_1 \cdot (n_2^2 \varepsilon_0 \mathbf{E}_{2t}) \\ \rightarrow \mathbf{D}_1 &= n_1^2 \varepsilon_0 \mathbf{E}_{1t}, \quad \mathbf{D}_2 = n_2^2 \varepsilon_0 \mathbf{E}_{2t} & \rightarrow \varepsilon_0 (n_1^2 - n_2^2) \mathbf{E}_{1t} \cdot \mathbf{E}_{2t} &= 0 \quad \leftarrow n_1^2 \neq n_2^2 \\ \rightarrow \mathbf{E}_2 \cdot \mathbf{D}_1 &= E_{2i} \varepsilon_{ij} E_{1j} & \rightarrow \mathbf{E}_{1t} \cdot \mathbf{E}_{2t} &= 0 \\ \rightarrow \mathbf{E}_1 \cdot \mathbf{D}_2 &= E_{1i} \varepsilon_{ij} E_{2j} = E_{1i} \varepsilon_{ji} E_{2j} = E_{1j} \varepsilon_{ij} E_{2i} & \rightarrow \mathbf{D}_1 \cdot \mathbf{D}_2 &= 0 \\ \rightarrow \mathbf{E}_2 \cdot \mathbf{D}_1 &= \mathbf{E}_1 \cdot \mathbf{D}_2 & \rightarrow \mathbf{D}_1 \cdot \mathbf{E}_2 &= \mathbf{D}_2 \cdot \mathbf{E}_1 = 0 \\ & & \rightarrow \mathbf{s} \cdot \mathbf{D}_1 &= \mathbf{s} \cdot \mathbf{D}_2 = 0 \quad 8 \end{aligned}$$

Orthogonality Properties of the Eigenmodes (2)

Orthogonality relations for eignemodes:

$$\begin{aligned}\mathbf{s} \cdot (\mathbf{E}_1 \times \mathbf{H}_2) &= \frac{n_2}{\mu c} \mathbf{s} \cdot [\mathbf{E}_1 \times (\mathbf{s} \times \mathbf{E}_2)] \quad \leftarrow \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \\ &= \frac{n_2}{\mu c} [(\mathbf{s} \times \mathbf{E}_2) \cdot (\mathbf{s} \times \mathbf{E}_1)] \\ &= \frac{n_2}{\mu c} [(\mathbf{s} \times \mathbf{E}_{2t}) \cdot (\mathbf{s} \times \mathbf{E}_{1t})] \\ &= 0\end{aligned}$$

$$\rightarrow \mathbf{s} \cdot (\mathbf{E}_1 \times \mathbf{H}_2) = \mathbf{s} \cdot (\mathbf{E}_2 \times \mathbf{H}_1) = 0$$

← No power flow