

# Electro-Optics:

## Electromagnetic Propagation in Anisotropic Media (2)

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# Index Ellipsoid (1)

Surface of constant energy density:

$$U_e = \frac{1}{2} \mathbf{E} \cdot \mathbf{D} \rightarrow \frac{D_x^2}{\epsilon_x^2} + \frac{D_y^2}{\epsilon_y^2} + \frac{D_z^2}{\epsilon_z^2} = 2U_e \leftarrow n_i^2 = \epsilon_i / \epsilon_0$$

Index ellipsoid:

$$\rightarrow \frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1$$

Impermeability tensor:

$$\eta_{ij} = \epsilon_0 (\epsilon^{-1})_{ij} \rightarrow \mathbf{E} = \frac{1}{\epsilon_0} \boldsymbol{\eta} \mathbf{D}$$

Wave equation:

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) + \omega^2 \mu \epsilon \mathbf{E} = 0 \rightarrow \mathbf{s} \times (\mathbf{s} \times \boldsymbol{\eta} \mathbf{D}) + \frac{1}{n^2} \mathbf{D} = 0$$

Let:  $\rightarrow \mathbf{s} = (0 \ 0 \ 1)^T$

$$\rightarrow \begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} \mathbf{D} = \frac{1}{n^2} \mathbf{D}$$

# Index Ellipsoid (2)

Transverse impermeability tensor:

$$\eta_t = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix}$$

Wave equation:

$$\rightarrow \left( \eta_t - \frac{1}{n^2} \right) \mathbf{D} = 0 \quad \leftarrow \text{Eigenvalue problem}$$

Index ellipsoid:

$$\rightarrow \eta_{11} \xi_1^2 + \eta_{22} \xi_2^2 + 2\eta_{12} \xi_1 \xi_2 = 1$$

$$\rightarrow \eta_{\alpha\beta} \xi_\alpha \xi_\beta = 1 \quad (\alpha, \beta = 1, 2, 3)$$

$$\rightarrow \frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1$$

# Phase Velocity, Group Velocity, and Energy Velocity

Phase velocity:  $\mathbf{v}_p = \frac{\omega}{k} \mathbf{s}$

Group velocity:  $\mathbf{v}_g = \nabla_{\mathbf{k}} \omega(\mathbf{k})$

Velocity of energy flow:  $\mathbf{v}_e = \frac{\mathbf{S}}{U}$  ← Poynting vector  
 ← Energy density

Monochromatic plane waves:  $\rightarrow \delta \mathbf{k} \rightarrow \delta \omega, \delta \mathbf{E}, \& \delta \mathbf{H}$

$$\begin{aligned} \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 &\rightarrow \mathbf{k} \times \mathbf{E} = \omega \mu \mathbf{H} \rightarrow \delta \mathbf{k} \times \mathbf{E} + \mathbf{k} \times \delta \mathbf{E} = \delta \omega \mu \mathbf{H} + \omega \mu \delta \mathbf{H} \\ &\rightarrow \mathbf{H} \cdot (\delta \mathbf{k} \times \mathbf{E} + \mathbf{k} \times \delta \mathbf{E}) = \mathbf{H} \cdot (\delta \omega \mu \mathbf{H} + \omega \mu \delta \mathbf{H}) \\ &\rightarrow \delta \mathbf{k} \cdot (\mathbf{E} \times \mathbf{H}) - \delta \mathbf{E} \cdot (\mathbf{k} \times \mathbf{H}) = \delta \omega \mathbf{H} \cdot \mu \mathbf{H} + \omega (\mathbf{H} \cdot \mu \delta \mathbf{H}) \end{aligned}$$

$$\begin{aligned} \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} &\rightarrow \mathbf{k} \times \mathbf{H} = -\omega \epsilon \mathbf{E} \rightarrow \delta \mathbf{k} \times \mathbf{H} + \mathbf{k} \times \delta \mathbf{H} = -\delta \omega \epsilon \mathbf{E} - \omega \epsilon \delta \mathbf{E} \\ &\rightarrow \mathbf{E} \cdot (\delta \mathbf{k} \times \mathbf{H} + \mathbf{k} \times \delta \mathbf{H}) = \mathbf{E} \cdot (-\delta \omega \epsilon \mathbf{E} - \omega \epsilon \delta \mathbf{E}) \\ &\rightarrow -\delta \mathbf{k} \cdot (\mathbf{E} \times \mathbf{H}) - \delta \mathbf{H} \cdot (\mathbf{k} \times \mathbf{E}) = -\delta \omega (\mathbf{E} \cdot \epsilon \mathbf{E}) - \omega (\mathbf{E} \cdot \epsilon \delta \mathbf{E}) \end{aligned}$$

$$\begin{aligned} \rightarrow 2\delta \mathbf{k} \cdot (\mathbf{E} \times \mathbf{H}) - \delta \omega (\mathbf{E} \cdot \epsilon \mathbf{E} + \mathbf{H} \cdot \mu \mathbf{H}) &= \delta \mathbf{H} \cdot (\omega \mu \mathbf{H} - \mathbf{k} \times \mathbf{E}) + \delta \mathbf{E} \cdot (\omega \epsilon \mathbf{E} + \mathbf{k} \times \mathbf{H}) \\ &\rightarrow \delta \mathbf{k} \cdot (\mathbf{E} \times \mathbf{H}) = \delta \omega \frac{1}{2} (\mathbf{E} \cdot \epsilon \mathbf{E} + \mathbf{H} \cdot \mu \mathbf{H}) \end{aligned}$$

# Group Velocity and Energy Velocity

Continued:

$$\rightarrow \delta \mathbf{k} \cdot (\mathbf{E} \times \mathbf{H}) = \delta \omega \frac{1}{2} (\mathbf{E} \cdot \epsilon \mathbf{E} + \mathbf{H} \cdot \mu \mathbf{H})$$

$$\rightarrow \delta \mathbf{k} \cdot \mathbf{S} = \delta \omega U$$

$$\rightarrow \delta \omega = \delta \mathbf{k} \cdot \frac{\mathbf{S}}{U}$$

$$\rightarrow \delta \omega = \delta \mathbf{k} \cdot \mathbf{v}_e \quad \leftarrow \quad \delta \omega = (\nabla_{\mathbf{k}} \omega \cdot \delta \mathbf{k}) = \mathbf{v}_g \cdot \delta \mathbf{k}$$

Equivalence of group velocity and energy velocity:

$$\rightarrow \mathbf{v}_g = \mathbf{v}_e$$

Relationship between the Poynting vector and the normal surface:

$$\rightarrow \delta \mathbf{k}' \parallel \text{normal surface}$$

$$\rightarrow \delta \mathbf{k}' \cdot (\mathbf{E} \times \mathbf{H}) = 0 \quad \leftarrow \quad \delta \omega = 0$$

$$\rightarrow \mathbf{E} \times \mathbf{H} \perp \text{normal surface}$$

# Classification of Anisotropic Media

Anisotropic media:

$$\epsilon = \epsilon_0 \begin{pmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{pmatrix} \begin{cases} n_x = n_y = n_z \rightarrow \textit{Isotropic} \\ n_x = n_y \neq n_z \rightarrow \textit{Uniaxial} \\ n_x \neq n_y \neq n_z \rightarrow \textit{Biaxial} \end{cases}$$

Shape of normal surface:  $k_x^2 YZ + k_y^2 ZX + k_z^2 XY - XYZ = 0$

$k_z$ - $k_x$  plane:  $\rightarrow k_y = 0 \rightarrow k_x^2 YZ + k_z^2 XY - XYZ = 0$

$$\rightarrow Y = 0 \rightarrow k_z^2 + k_x^2 = \left( \frac{\omega}{c} n_y \right)^2$$

or

$$\rightarrow k_x^2 Z + k_z^2 X - XZ = 0 \rightarrow \frac{k_z^2}{\left( \frac{\omega}{c} n_x \right)^2} + \frac{k_x^2}{\left( \frac{\omega}{c} n_z \right)^2} = 1$$

Classification:

