Electro-Optics:

Electromagnetic Propagation in Anisotropic Media (2)

Yoonchan Jeong

School of Electrical Engineering, Seoul National University

Tel: +82 (0)2 880 1623, Fax: +82 (0)2 873 9953

Email: yoonchan@snu.ac.kr

Index Ellipsoid (1)

Surface of constant energy density:

$$U_e = \frac{1}{2}\mathbf{E}\cdot\mathbf{D} \rightarrow \frac{D_x^2}{\varepsilon_x} + \frac{D_y^2}{\varepsilon_y} + \frac{D_z^2}{\varepsilon_z} = 2U_e \leftarrow n_i^2 = \varepsilon_i / \varepsilon_0$$

Index ellipsoid:

$$\leftarrow x_i \equiv \frac{D_i}{\sqrt{2U_e\varepsilon_0}}$$

$$\rightarrow \frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1$$

Impermeability tensor:

$$\eta_{ij} = \varepsilon_0 (\varepsilon^{-1})_{ij} \rightarrow \mathbf{E} = \frac{1}{\varepsilon_0}\eta\mathbf{D}$$

"back-cab" rule

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

Wave equation:

$$\mathbf{k}\times(\mathbf{k}\times\mathbf{E}) + \omega^2\mu\varepsilon\mathbf{E} = 0 \rightarrow \mathbf{s}\times(\mathbf{s}\times\eta\mathbf{D}) + \frac{1}{n^2}\mathbf{D} = 0$$

Let: $\rightarrow \mathbf{s} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$

$$\rightarrow \begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix}\mathbf{D} - \mathbf{s}(\mathbf{s}\cdot\eta\mathbf{D}) = \frac{1}{n^2}\mathbf{D}$$

Index Ellipsoid (2)

Transverse impermeability tensor:

$$\eta_t = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix}$$

Wave equation:

$$\rightarrow \left(\eta_t - \frac{1}{n^2} \right) \mathbf{D} = 0 \qquad \leftarrow \text{Eigenvalue problem}$$

Index ellipsoid:

$$\rightarrow \eta_{\alpha\beta}\xi_\alpha\xi_\beta = 1 \quad (\alpha, \beta = 1, 2, 3)$$

$$\rightarrow \eta_{11}\xi_1^2 + \eta_{22}\xi_2^2 + 2\eta_{12}\xi_1\xi_2 = 1$$

$$\rightarrow \frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1 \qquad \leftarrow \text{Principal coordinates}$$

Phase Velocity, Group Velocity, and Energy Velocity

Phase velocity: $\mathbf{v}_p = \dfrac{\omega}{k}\mathbf{s}$

Group velocity: $\mathbf{v}_g = \nabla_{\mathbf{k}}\omega(\mathbf{k})$

Velocity of energy flow: $\mathbf{v}_e = \dfrac{\mathbf{S}}{U}$ ← Poynting vector ← Energy density

Monochromatic plane waves: $\rightarrow \delta\mathbf{k} \rightarrow \delta\omega, \delta\mathbf{E}, \& \ \delta\mathbf{H}$

$$\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0 \rightarrow \mathbf{k}\times\mathbf{E}=\omega\mu\mathbf{H} \rightarrow \delta\mathbf{k}\times\mathbf{E}+\mathbf{k}\times\delta\mathbf{E}=\delta\omega\mu\mathbf{H}+\omega\mu\delta\mathbf{H}$$

$$\rightarrow \mathbf{H}\cdot(\delta\mathbf{k}\times\mathbf{E}+\mathbf{k}\times\delta\mathbf{E})=\mathbf{H}\cdot(\delta\omega\mu\mathbf{H}+\omega\mu\delta\mathbf{H})$$

$$\rightarrow \delta\mathbf{k}\cdot(\mathbf{E}\times\mathbf{H})-\delta\mathbf{E}\cdot(\mathbf{k}\times\mathbf{H})=\delta\omega\mathbf{H}\cdot\mu\mathbf{H}+\omega(\mathbf{H}\cdot\mu\delta\mathbf{H})$$

$$\nabla\times\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t}=\mathbf{J} \rightarrow \mathbf{k}\times\mathbf{H}=-\omega\varepsilon\mathbf{E} \rightarrow \delta\mathbf{k}\times\mathbf{H}+\mathbf{k}\times\delta\mathbf{H}=-\delta\omega\varepsilon\mathbf{E}-\omega\varepsilon\delta\mathbf{E}$$

$$\rightarrow \mathbf{E}\cdot(\delta\mathbf{k}\times\mathbf{H}+\mathbf{k}\times\delta\mathbf{H})=\mathbf{E}\cdot(-\delta\omega\varepsilon\mathbf{E}-\omega\varepsilon\delta\mathbf{E})$$

$$\rightarrow -\delta\mathbf{k}\cdot(\mathbf{E}\times\mathbf{H})-\delta\mathbf{H}\cdot(\mathbf{k}\times\mathbf{E})=-\delta\omega(\mathbf{E}\cdot\varepsilon\mathbf{E})-\omega(\mathbf{E}\cdot\varepsilon\delta\mathbf{E})$$

$$\rightarrow 2\delta\mathbf{k}\cdot(\mathbf{E}\times\mathbf{H})-\delta\omega(\mathbf{E}\cdot\varepsilon\mathbf{E}+\mathbf{H}\cdot\mu\mathbf{H})=\delta\mathbf{H}\cdot(\omega\mu\mathbf{H}-\mathbf{k}\times\mathbf{E})+\delta\mathbf{E}\cdot(\omega\varepsilon\mathbf{E}+\mathbf{k}\times\mathbf{H})$$

$$\rightarrow \delta\mathbf{k}\cdot(\mathbf{E}\times\mathbf{H})=\delta\omega\frac{1}{2}(\mathbf{E}\cdot\varepsilon\mathbf{E}+\mathbf{H}\cdot\mu\mathbf{H})$$

Group Velocity and Energy Velocity

Continued:

$$\rightarrow \; \delta\mathbf{k}\cdot(\mathbf{E}\times\mathbf{H}) = \delta\omega\frac{1}{2}(\mathbf{E}\cdot\varepsilon\mathbf{E}+\mathbf{H}\cdot\mu\mathbf{H})$$

$$\rightarrow \; \delta\mathbf{k}\cdot\mathbf{S} = \delta\omega U$$

$$\rightarrow \; \delta\omega = \delta\mathbf{k}\cdot\frac{\mathbf{S}}{U}$$

$$\rightarrow \; \delta\omega = \delta\mathbf{k}\cdot\mathbf{v}_e \;\; \leftarrow \;\; \delta\omega = (\nabla_{\mathbf{k}}\omega\cdot\delta\mathbf{k}) = \mathbf{v}_g\cdot\delta\mathbf{k}$$

Equivalence of group velocity and energy velocity:

$$\rightarrow \; \mathbf{v}_g = \mathbf{v}_e$$

Relationship between the Poynting vector and the normal surface:

$$\rightarrow \; \delta\mathbf{k}' \, // \, \textit{normal surface}$$

$$\rightarrow \; \delta\mathbf{k}'\cdot(\mathbf{E}\times\mathbf{H})$$

$$= 0 \;\; \leftarrow \;\; \delta\omega = 0$$

$$\rightarrow \; \mathbf{E}\times\mathbf{H} \perp \textit{normal surface}$$

Energy flow

Classification of Anisotropic Media

Anisotropic media:

$$\varepsilon = \varepsilon_o \begin{pmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{pmatrix} \quad \begin{cases} n_x = n_y = n_z \rightarrow \mathit{Isotropic} \\ n_x = n_y \neq n_z \rightarrow \mathit{Uniaxial} \\ n_x \neq n_y \neq n_z \rightarrow \mathit{Biaxial} \end{cases}$$

Shape of normal surface: $k_x^2 YZ + k_y^2 ZX + k_z^2 XY - XYZ = 0$

k_z-k_x plane: $\rightarrow \; k_y = 0 \quad \rightarrow \; k_x^2 YZ + k_z^2 XY - XYZ = 0$

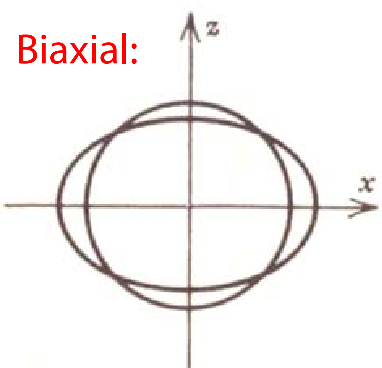
$$\rightarrow Y = 0 \qquad \rightarrow k_z^2 + k_x^2 = \left(\frac{\omega}{c} n_y\right)^2$$

or

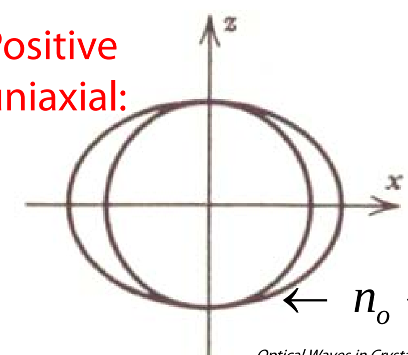
$$\rightarrow k_x^2 Z + k_z^2 X - XZ = 0 \quad \rightarrow \frac{k_z^2}{\left(\frac{\omega}{c} n_x\right)^2} + \frac{k_x^2}{\left(\frac{\omega}{c} n_z\right)^2} = 1$$

Classification:

Biaxial:

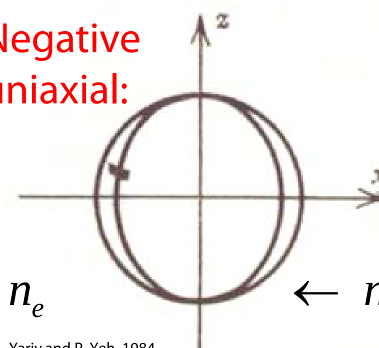


Positive uniaxial:



$\leftarrow n_o < n_e$

Negative uniaxial:



$\leftarrow n_o > n_e$

Optical Waves in Crystals, A. Yariv and P. Yeh, 1984.