

Electro-Optics:

Electromagnetic Propagation in Periodic Media (1)

Yoonchan Jeong

School of Electrical Engineering, Seoul National University

Tel: +82 (0)2 880 1623, Fax: +82 (0)2 873 9953

Email: yunchan@snu.ac.kr

Periodic Media

Translational symmetry:

$$\varepsilon(\mathbf{x}) = \varepsilon(\mathbf{x} + \mathbf{a}), \quad \mu(\mathbf{x}) = \mu(\mathbf{x} + \mathbf{a})$$

Maxwell's equations:

$$\nabla \times \mathbf{H} = i\omega\varepsilon\mathbf{E}$$

$$\nabla \times \mathbf{E} = -i\omega\mu\mathbf{H}$$

Normal modes:

$$\rightarrow \mathbf{E} = \mathbf{E}_{\mathbf{K}}(\mathbf{x})e^{-i\mathbf{K}\cdot\mathbf{x}} \quad \leftarrow \mathbf{E}_{\mathbf{K}}(\mathbf{x}) = \mathbf{E}_{\mathbf{K}}(\mathbf{x} + \mathbf{a})$$

$$\rightarrow \mathbf{H} = \mathbf{H}_{\mathbf{K}}(\mathbf{x})e^{-i\mathbf{K}\cdot\mathbf{x}} \quad \leftarrow \mathbf{H}_{\mathbf{K}}(\mathbf{x}) = \mathbf{H}_{\mathbf{K}}(\mathbf{x} + \mathbf{a})$$

\leftarrow Floquet (or Bloch) theorem (to be proved)

\leftarrow \mathbf{K} : Bloch wave vector

Dispersion relation:

$$\rightarrow \omega = \omega(\mathbf{K})$$

One-Dimensional Periodic Media (1)

Permittivity:

$$\varepsilon(z) = \varepsilon(z + l\Lambda)$$

Wave equation:

$$\nabla \times (\nabla \times \mathbf{E}) - \omega^2 \mu \varepsilon \mathbf{E} = 0$$

Dielectric tensor in a Fourier series:

$$\rightarrow \varepsilon(\mathbf{x}) = \sum_{\mathbf{G}} \varepsilon_{\mathbf{G}} e^{-i\mathbf{G} \cdot \mathbf{x}} \quad \leftarrow \quad \mathbf{G} = l\mathbf{g} = l \frac{2\pi}{\Lambda} \hat{\mathbf{z}}$$

\leftarrow Reciprocal-lattice vector

$$\rightarrow \varepsilon(z) = \sum_l \varepsilon_l e^{-il(2\pi/\Lambda)z}$$

Electric-field vector in a Fourier integral:

$$\mathbf{E} = \int d^3k \mathbf{A}(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{x}}$$

$$\rightarrow \int d^3k \mathbf{k} \times [\mathbf{k} \times \mathbf{A}(\mathbf{k})] e^{-i\mathbf{k} \cdot \mathbf{x}} + \omega^2 \mu \sum_{\mathbf{G}} \int d^3k \varepsilon_{\mathbf{G}} \mathbf{A}(\mathbf{k} - \mathbf{G}) e^{-i\mathbf{k} \cdot \mathbf{x}} = 0$$

$$\rightarrow \mathbf{k} \times [\mathbf{k} \times \mathbf{A}(\mathbf{k})] + \omega^2 \mu \sum_{\mathbf{G}} \varepsilon_{\mathbf{G}} \mathbf{A}(\mathbf{k} - \mathbf{G}) = 0 \quad \text{for all } \mathbf{k}$$

One-Dimensional Periodic Media (2)

Electric-field vector:

$$\rightarrow \mathbf{k} \times [\mathbf{k} \times \mathbf{A}(\mathbf{k})] + \omega^2 \mu \sum_{\mathbf{G}} \varepsilon_{\mathbf{G}} \mathbf{A}(\mathbf{k} - \mathbf{G}) = 0 \quad \text{for all } \mathbf{k}$$

Solution of a subset labeled by \mathbf{K} (normal mode):

$$\begin{aligned} \rightarrow \mathbf{E}_{(\mathbf{K})} &= \sum_{\mathbf{G}} \mathbf{A}(\mathbf{K} - \mathbf{G}) e^{-i(\mathbf{K} - \mathbf{G}) \cdot \mathbf{x}} \\ &= e^{-i\mathbf{K} \cdot \mathbf{x}} \sum_{\mathbf{G}} \mathbf{A}(\mathbf{K} - \mathbf{G}) e^{i\mathbf{G} \cdot \mathbf{x}} \\ &= e^{-i\mathbf{K} \cdot \mathbf{x}} \mathbf{E}_{\mathbf{K}}(\mathbf{x}) \quad \leftarrow \quad \mathbf{E}_{\mathbf{K}}(\mathbf{x}) = \sum_l \mathbf{A}(\mathbf{K} - l\mathbf{g}) e^{il(2\pi/\Lambda)z} \\ &\rightarrow \mathbf{E}_{\mathbf{K}}(\mathbf{x}) = \mathbf{E}_{\mathbf{K}}(\mathbf{x} + \mathbf{a}) \end{aligned}$$

\rightarrow Floquet (or Bloch) theorem proved

Bloch mode or wave:

$$\rightarrow \mathbf{E} = e^{-i(K_x x + K_y y)} e^{-iK_z z} \mathbf{E}_{\mathbf{K}}(z)$$

One-Dimensional Periodic Media (3)

Wave propagating in the z direction (isotropic):

$$\rightarrow k^2 A(k) - \omega^2 \mu \sum_l \varepsilon_l A(k - lg) = 0$$

$$\leftarrow k = K, K \pm g, K \pm 2g, \dots$$

$$\rightarrow K^2 A(K) - \omega^2 \mu \varepsilon_0 A(K) - \omega^2 \mu \varepsilon_1 A(K - g) - \omega^2 \mu \varepsilon_{-1} A(K + g) - \dots = 0$$

$$\rightarrow A(K) = \frac{1}{K^2 - \omega^2 \mu \varepsilon_0} \left[\omega^2 \mu \varepsilon_1 A(K - g) + \omega^2 \mu \varepsilon_{-1} A(K + g) + \dots \right]$$

$$\rightarrow A(K - g) = \frac{1}{(K - g)^2 - \omega^2 \mu \varepsilon_0} \left[\omega^2 \mu \varepsilon_1 A(K - 2g) + \omega^2 \mu \varepsilon_{-1} A(K) + \dots \right]$$

$$\rightarrow A(K + g) = \frac{1}{(K + g)^2 - \omega^2 \mu \varepsilon_0} \left[\omega^2 \mu \varepsilon_1 A(K) + \omega^2 \mu \varepsilon_{-1} A(K + 2g) + \dots \right]$$

If: $|K - g| \cong K$ & $K^2 \cong \omega^2 \mu \varepsilon_0$

$$\rightarrow (K^2 - \omega^2 \mu \varepsilon_0) A(K) - \omega^2 \mu \varepsilon_1 A(K - g) = 0$$

$$\rightarrow -\omega^2 \mu \varepsilon_{-1} A(K) + [(K - g)^2 - \omega^2 \mu \varepsilon_0] A(K - g) = 0$$

Bragg condition

One-Dimensional Periodic Media (4)

Nontrivial solution:

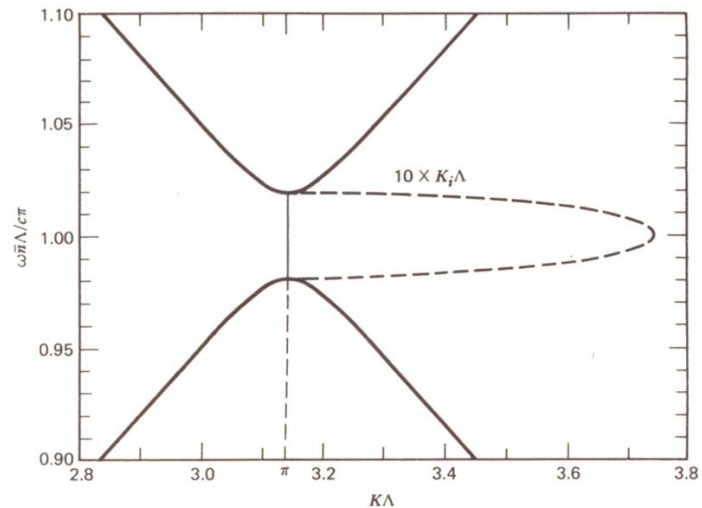
$$\rightarrow \begin{vmatrix} K^2 - \omega^2 \mu \epsilon_0 & -\omega^2 \mu \epsilon_1 \\ -\omega^2 \mu \epsilon_{-1} & (K - g)^2 - \omega^2 \mu \epsilon_0 \end{vmatrix} = 0 \quad \leftarrow \epsilon_{-1} = \epsilon_1^*$$

$$\rightarrow (K^2 - \omega^2 \mu \epsilon_0) [(K - g)^2 - \omega^2 \mu \epsilon_0] - (\omega^2 \mu |\epsilon_1|)^2 = 0$$

Bragg condition:

$$\rightarrow K = \frac{1}{2} g = \frac{\pi}{\Lambda}$$

$$\rightarrow \omega_{\pm}^2 = \frac{K^2}{\mu(\epsilon_0 \pm |\epsilon_1|)}$$



Optical Waves in Crystals, A. Yariv and P. Yeh, 1984.

→ Forbidden band between ω_+ and ω_-

One-Dimensional Periodic Media (5)

Forbidden band:

$$\rightarrow \omega^2 = \left(\frac{1}{2}g\right)^2 / \mu\varepsilon_0$$

$$\rightarrow K = \frac{1}{2}g + x, \quad |x| \ll \frac{1}{2}g$$

$$\rightarrow (K^2 - \omega^2 \mu\varepsilon_0)[(K - g)^2 - \omega^2 \mu\varepsilon_0] - (\omega^2 \mu |\varepsilon_1|)^2 = 0$$

$$\rightarrow g^2 x^2 + \left(\frac{|\varepsilon_1|}{\varepsilon_0}\right)^2 \left(\frac{1}{4}g^2\right)^2 = 0$$

$$\rightarrow K = \frac{1}{2}g \left(1 \pm i \frac{|\varepsilon_1|}{2\varepsilon_0}\right) \quad \leftarrow \Delta\omega_{gap} = |\omega_+ - \omega_-|$$

$$\rightarrow K_i = \frac{1}{4}g \frac{\Delta\omega_{gap}}{\omega} \quad = \omega \frac{|\varepsilon_1|}{\varepsilon_0}$$

One-Dimensional Periodic Media (6)

Higher-order forbidden band:

$$|K - lg| \cong K, \quad l = \pm 1, \pm 2, \dots$$

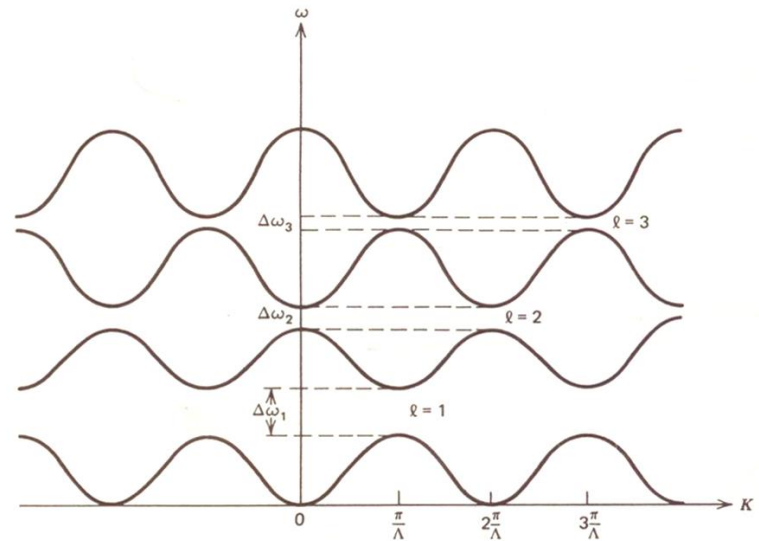
$$\& K^2 \cong \omega^2 \mu \varepsilon_0$$

$$\rightarrow (K^2 - \omega^2 \mu \varepsilon_0)A(K) - \omega^2 \mu \varepsilon_1 A(K - lg) = 0$$

$$\rightarrow -\omega^2 \mu \varepsilon_{-l} A(K) + [(K - lg)^2 - \omega^2 \mu \varepsilon_0]A(K - lg) = 0$$

$$\rightarrow K = l \frac{g}{2} = l \frac{\pi}{\Lambda}$$

$$\rightarrow (\Delta\omega_{gap})_l = \omega \frac{|\varepsilon_l|}{\varepsilon_0}$$



Optical Waves in Crystals, A. Yariv and P. Yeh, 1984.