Electro-Optics:

Electromagnetic Propagation in Periodic Media (3)

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Coupled-Mode Theory (1)

Dielectric tensor:

$$\varepsilon(x,y,z) = \varepsilon_0(x,y) + \Delta\varepsilon(x,y,z)$$

Unperturbed part   Perturbed (periodic) part

Normal modes:

$$\mathbf{E}_m(x,y)e^{i(\omega t-\beta_m z)}$$

$$\rightarrow \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \omega^2\mu\varepsilon_0(x,y) - \beta_m^2\right]\mathbf{E}_m(x,y) = 0$$

Electric-field for the "unperturbed" medium:

$$\mathbf{E} = \sum_m A_m \mathbf{E}_m(x,y)e^{i(\omega t-\beta_m z)}$$

Orthogonal relation of the normal modes:

TM modes? $\rightarrow$ H.W.

$$\frac{1}{2}\int(\mathbf{E}_k \times \mathbf{H}_l^*)_z\,dxdy = \delta_{lk} \quad \leftarrow \quad \mathbf{a}_z \cdot \mathbf{E} = 0$$ (TE modes)

$$\rightarrow \int \mathbf{E}_k^*(x,y)\cdot\mathbf{E}_l(x,y)dxdy = \frac{2\omega\mu}{|\beta_k|}\delta_{lk}$$

Coupled-Mode Theory (2)

In the presence of the dielectric perturbation:

$$\Delta\mathbf{P} = \Delta\varepsilon(x,y,z)\mathbf{E}_l(x,y)e^{i(\omega t-\beta_l z)}$$

$$\rightarrow \mathbf{E} = \sum_m A_m(z)\mathbf{E}_m(x,y)e^{i(\omega t-\beta_m z)}$$ ← Variation of constants

Wave equation:

$$\left\{\nabla^2 + \omega^2\mu[\varepsilon_0(x,y) + \Delta\varepsilon(x,y,z)]\right\}\mathbf{E} = 0$$

"Slowly varying amplitude"

$$\rightarrow \sum_k \left[\frac{d^2}{dz^2}A_k - 2i\beta_k\frac{d}{dz}A_k\right]\mathbf{E}_k(x,y)e^{-i\beta_k z} = -\omega^2\mu\sum_l \Delta\varepsilon(x,y,z)A_l\mathbf{E}_l(x,y)e^{-i\beta_l z}$$

$$\leftarrow \left|\frac{d^2}{dz^2}A_k\right| << \left|\beta_k\frac{d}{dz}A_k\right|$$

$$\rightarrow -2i\sum_k \beta_k\left(\frac{d}{dz}A_k\right)\mathbf{E}_k(x,y)e^{-i\beta_k z} = -\omega^2\mu\sum_l \Delta\varepsilon(x,y,z)A_l\mathbf{E}_l(x,y)e^{-i\beta_l z}$$

Coupled-Mode Theory (3)

Wave equation with the orthogonal relation:

$$\rightarrow -2i\sum_k \beta_k \left(\frac{d}{dz}A_k\right)\mathbf{E}_k(x,y)e^{-i\beta_k z} = -\omega^2\mu\sum_l \Delta\varepsilon(x,y,z)A_l\mathbf{E}_l(x,y)e^{-i\beta_l z}$$

$$\rightarrow <k\,|\,k> \frac{d}{dz}A_k(z) = \frac{\omega^2\mu}{2i\beta_k}\sum_l <k\,|\,\Delta\varepsilon\,|\,l> A_l(z)e^{i(\beta_k-\beta_l)z}$$

$$\leftarrow\ <k\,|\,k>=\int \mathbf{E}_k^*(x,y)\cdot\mathbf{E}_k(x,y)dxdy = \frac{2\omega\mu}{|\beta_k|}$$ (TE modes)

$$\leftarrow\ <k\,|\,\Delta\varepsilon\,|\,l>=\int \mathbf{E}_k^*(x,y)\cdot\Delta\varepsilon(x,y,z)\mathbf{E}_l(x,y)dxdy$$

Dielectric perturbation in a Fourier series:

$$\Delta\varepsilon(x,y,z) = \sum_{m\neq 0}\varepsilon_m(x,y)\exp\left[-im\frac{2\pi}{\Lambda}z\right]$$

$$\rightarrow \frac{d}{dz}A_k(z) = -i\frac{\beta_k}{|\beta_k|}\sum_l\sum_m C_{kl}^{(m)}A_l(z)e^{i(\beta_k-\beta_l-m2\pi/\Lambda)z}$$

$$\leftarrow\ C_{kl}^{(m)} = \frac{\omega}{4}<k\,|\,\varepsilon_m\,|\,l>$$

Coupled-Mode Theory (4)

Resonant coupling:

$$\frac{d}{dz}A_k(z) = -i\frac{\beta_k}{|\beta_k|}\sum_l \sum_m C_{kl}^{(m)} A_l(z) e^{i(\beta_k - \beta_l - m2\pi/\Lambda)z}$$

$$\rightarrow \ \beta_k - \beta_l - m\frac{2\pi}{\Lambda} = 0$$

Coupled-mode equations:

$$\frac{d}{dz}A_1 = -i\frac{\beta_1}{|\beta_1|}C_{12}^{(m)} A_2 e^{i\Delta\beta z}$$

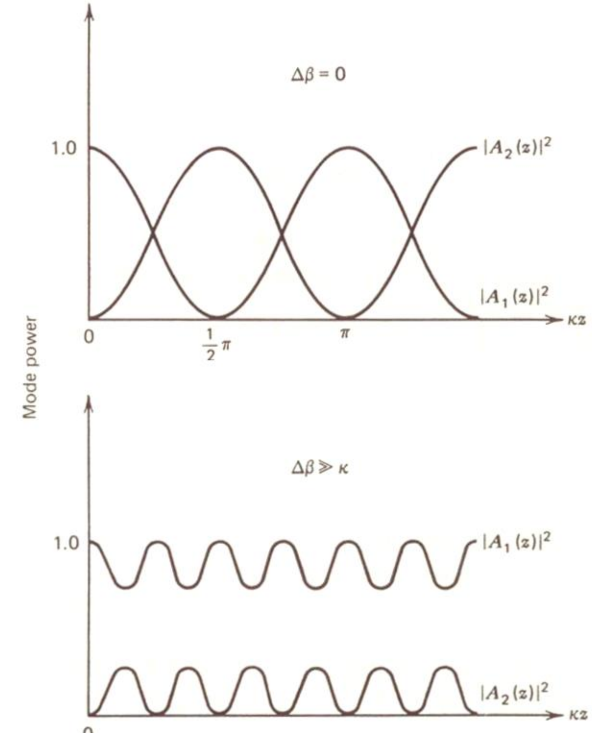
$$\frac{d}{dz}A_2 = -i\frac{\beta_2}{|\beta_2|}C_{21}^{(-m)} A_1 e^{-i\Delta\beta z} \quad \leftarrow \ \Delta\beta = \beta_1 - \beta_2 - m\frac{2\pi}{\Lambda}$$

$$\leftarrow \ C_{12}^{(m)} = [C_{21}^{(-m)}]^*$$

For plane waves:

$$\rightarrow \ C_{kl}^{(m)} = \frac{\omega^2 \mu}{2\sqrt{|\beta_k \beta_l|}} \mathbf{p}_k^* \cdot \varepsilon_m \mathbf{p}_l$$ $\leftarrow$ $\mathbf{p}_k$, $\mathbf{p}_l$: Unit pol. vectors

Codirectional Coupling



Optical Waves in Crystals, A. Yariv and P. Yeh, 1984.

Coupled-mode equations:

$$\frac{d}{dz}A_1 = -i\kappa A_2 e^{i\Delta\beta z}$$

$$\frac{d}{dz}A_2 = -i\kappa^* A_1 e^{-i\Delta\beta z} \leftarrow \quad \kappa = C_{12}^{(m)}$$

$$\rightarrow \frac{d}{dz}\left\{|A_1|^2 + |A_2|^2\right\} = 0$$

"Conservation of energy"

General solution:

$$A_1(z) = e^{i(\Delta\beta/2)z}\left\{\left[\cos sz - i\frac{\Delta\beta}{2s}\sin sz\right]A_1(0) - i\frac{\kappa}{s}\sin sz A_2(0)\right\}$$

$$A_2(z) = e^{-i(\Delta\beta/2)z}\left\{-i\frac{\kappa^*}{s}\sin sz A_1(0) + \left[\cos sz + i\frac{\Delta\beta}{2s}\sin sz\right]A_2(0)\right\}$$

$$\leftarrow \quad s^2 = \kappa\kappa^* + \left(\frac{\Delta\beta}{2}\right)^2$$

Fraction of power exchange:

$$\rightarrow \frac{|\kappa|^2}{|\kappa|^2 + (\Delta\beta/2)^2}\sin^2\sqrt{|\kappa|^2 + \left(\frac{\Delta\beta}{2}\right)^2}\,z \qquad \leftarrow \quad \Delta\beta = 0$$

Contradirectional Coupling

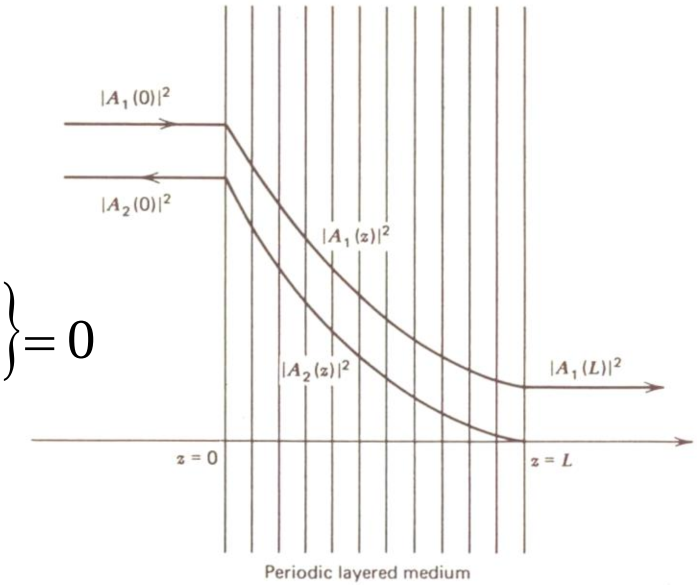
Coupled-mode equations:

$$\frac{d}{dz}A_1 = -i\kappa A_2 e^{i\Delta\beta z}$$

$$\frac{d}{dz}A_2 = i\kappa^* A_1 e^{-i\Delta\beta z}$$

$$\rightarrow \frac{d}{dz}\left\{ |A_1|^2 - |A_2|^2 \right\} = 0$$

"Conservation of energy"



Optical Waves in Crystals, A. Yariv and P. Yeh, 1984.

General solution:

$$A_1(z) = e^{i(\Delta\beta/2)z}\left\{ \frac{s\cosh s(L-z) + i(\Delta\beta/2)\sinh s(L-z)}{s\cosh sL + i(\Delta\beta/2)\sinh sL} A_1(0) + \frac{-i\kappa e^{i(\Delta\beta/2)L}\sinh sz}{s\cosh sL + i(\Delta\beta/2)\sinh sL} A_2(L) \right\}$$

$$A_2(z) = e^{-i(\Delta\beta/2)z}\left\{ \frac{-i\kappa^*\sinh s(L-z)}{s\cosh sL + i(\Delta\beta/2)\sinh sL} A_1(0) + e^{i(\Delta\beta/2)L}\frac{s\cosh sz + i(\Delta\beta/2)\sinh sz}{s\cosh sL + i(\Delta\beta/2)\sinh sL} A_2(L) \right\}$$

$$\leftarrow \; s^2 = \kappa\kappa^* - \left(\frac{\Delta\beta}{2}\right)^2$$

Fraction of power exchange:

$$\rightarrow \frac{|\kappa|^2 \sinh^2 sL}{s^2\cosh^2 sL + (\Delta\beta/2)^2 \sinh^2 sL} \qquad \leftarrow \; \Delta\beta = 0$$