Electro-Optics:

Guided Waves

Dr Yoonchan Jeong
School of Electrical Engineering, Seoul National University
Tel: +82 (0)2 880 1623, Fax: +82 (0)2 873 9953
Email: yoonchan@snu.ac.kr

Circular Waveguides

Propagation constant and fields:

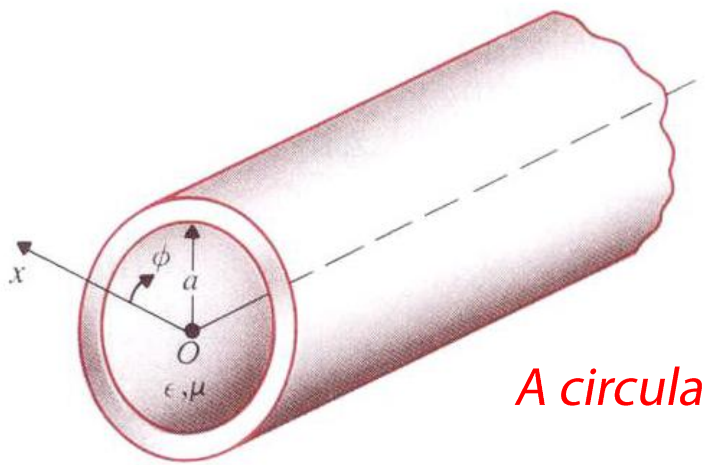
$$\gamma = \alpha + j\beta$$

$$\mathbf{E}(x, y, z; t) = \mathrm{Re}[\mathbf{E}^0(x, y)e^{j\omega t - \gamma z}]$$

$$H_z = 0 \quad \rightarrow \nabla_{xy}^2 E_z^0 + (\gamma^2 + k^2)E_z^0 = 0 \quad \rightarrow \nabla_{xy}^2 E_z^0 + h^2 E_z^0 = 0$$

$$E_z = 0 \quad \rightarrow \nabla_{xy}^2 H_z^0 + (\gamma^2 + k^2)H_z^0 = 0 \quad \rightarrow \nabla_{xy}^2 H_z^0 + h^2 H_z^0 = 0$$

where $h^2 = \gamma^2 + k^2$



A circular waveguide

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

→ Choice of the coordinate system:

$$\rightarrow \nabla_{r\phi}^2 E_z^0 + h^2 E_z^0 = 0$$

$$\rightarrow \nabla_{r\phi}^2 H_z^0 + h^2 H_z^0 = 0$$

Bessel's Differential Equation

Consider:

$$\nabla^2_{r\phi} E_z^0 + h^2 E_z^0 = 0$$

$$\rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial E_z^0}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 E_z^0}{\partial \phi^2} + h^2 E_z^0 = 0$$

Let: $E_z^0 = R(r)\Phi(\phi)$

$$\rightarrow \frac{d^2\Phi(\phi)}{d\phi^2} + n^2\Phi(\phi) = 0$$

$$\rightarrow \frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} + (h^2 - \frac{n^2}{r^2})R(r) = 0$$

→ Bessel's differential equation

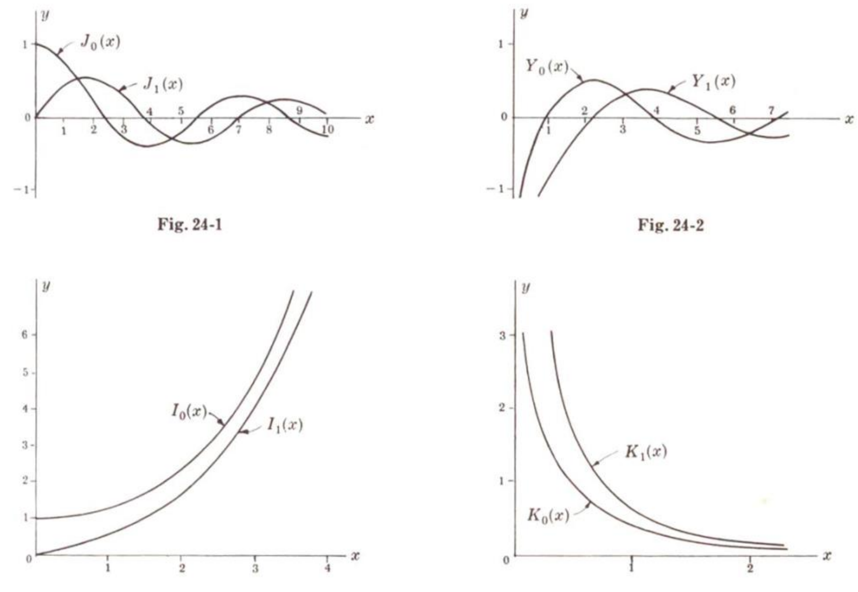


Fig. 24-1 Fig. 24-2 Fig. 24-3 Fig. 24-4

M. R. Spiegel, Mathematical Handbook of Formulas and Tables, McGraw-Hill, 1990.

$$R(r) = \begin{cases} CJ_n(hr) + DN_n(hr) & \text{for } h^2 > 0 \\ CI_n(qr) + DK_n(qr) & \text{for } h^2 < 0 \end{cases}$$

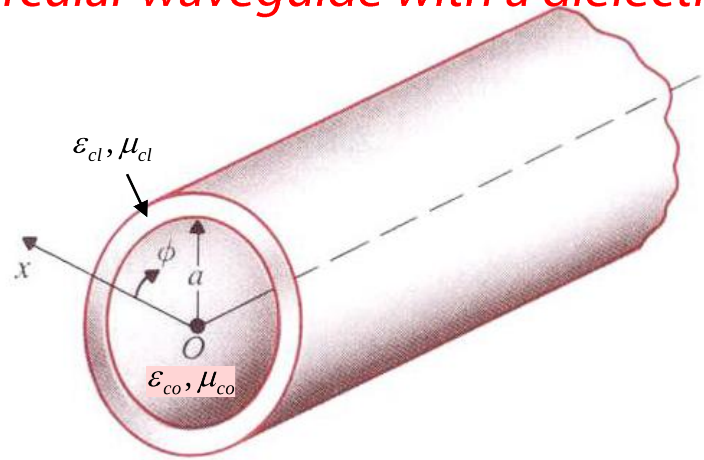
cf.

$$H_n^{(1)}(hr) = J_n(hr) + jN_n(hr)$$

$$H_n^{(2)}(hr) = J_n(hr) - jN_n(hr)$$

Circular Dielectric Waveguides: Optical Fibers

A circular waveguide with a dielectric cladding



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

$$\mu_{co}\varepsilon_{co} > \mu_{cl}\varepsilon_{cl}$$

Let: $\gamma = j\beta$

$$k^2 = \omega^2\mu\varepsilon = \omega^2\begin{bmatrix}\mu_{co}\varepsilon_{co}, & r \le a \\ \mu_{cl}\varepsilon_{cl}, & r > a\end{bmatrix}$$

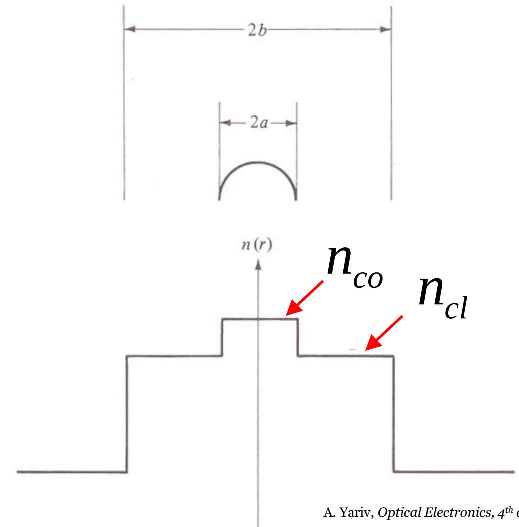
$$\rightarrow h^2 = k_{co}^2 - \beta^2$$

$$\rightarrow q^2 = \beta^2 - k_{cl}^2$$

Let:

$$E_{co,z}(r,\phi,z) = E^0_{co,z}(r,\phi)e^{-\gamma z}$$
$$H_{co,z}(r,\phi,z) = H^0_{co,z}(r,\phi)e^{-\gamma z}$$
$\rightarrow$ Core

$$E_{cl,z}(r,\phi,z) = E^0_{cl,z}(r,\phi)e^{-\gamma z}$$
$$H_{cl,z}(r,\phi,z) = H^0_{cl,z}(r,\phi)e^{-\gamma z}$$
$\rightarrow$ Cladding



A. Yariv, *Optical Electronics, 4th ed. Saunders, 1991.*

Recall:

$$R(r) = \begin{cases} CJ_n(hr) + DN_n(hr) & for \quad h^2 > 0 \\ CI_n(qr) + DK_n(qr) & for \quad h^2 < 0 \end{cases}$$

$$\rightarrow E^0_{co,z}, H^0_{co,z} \propto J_n(hr)\cos n\phi$$
$$\rightarrow E^0_{cl,z}, H^0_{cl,z} \propto K_n(qr)\cos n\phi$$

Interrelationship among the E and H Fields

Transverse field components described by E_z^0 & H_z^0

From: $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \rightarrow \frac{\partial E_z^0}{\partial y} + \gamma E_y^0 = -j\omega\mu H_x^0$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \rightarrow -\gamma E_x^0 - \frac{\partial E_z^0}{\partial x} = -j\omega\mu H_y^0$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \rightarrow \frac{\partial E_y^0}{\partial x} - \frac{\partial E_x^0}{\partial y} = -j\omega\mu H_z^0$$

From: $\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E}$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \rightarrow \frac{\partial H_z^0}{\partial y} + \gamma H_y^0 = j\omega\varepsilon E_x^0$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \rightarrow -\gamma H_x^0 - \frac{\partial H_z^0}{\partial x} = j\omega\varepsilon E_y^0$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \rightarrow \frac{\partial H_y^0}{\partial x} - \frac{\partial H_x^0}{\partial y} = j\omega\varepsilon E_z^0$$

$$H_x^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial H_z^0}{\partial x} - j\omega\varepsilon \frac{\partial E_z^0}{\partial y}\right)$$

$$H_y^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial H_z^0}{\partial y} + j\omega\varepsilon \frac{\partial E_z^0}{\partial x}\right)$$

$$E_x^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial E_z^0}{\partial x} + j\omega\mu \frac{\partial H_z^0}{\partial y}\right)$$

$$E_y^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial E_z^0}{\partial y} - j\omega\mu \frac{\partial H_z^0}{\partial x}\right)$$

where $h^2 = \gamma^2 + k^2$

TM Modes

Unless n = 0, the B.C. cannot be satisfied only with E_z!

Recall:

$$Z_{TM} = \gamma / j\omega\varepsilon$$

$$\mathbf{E}_T^0 = -\frac{\gamma}{h^2}\nabla_T E_z^0 \rightarrow \nabla_T E_z^0 = \left(\mathbf{a}_r \frac{\partial}{\partial r} + \mathbf{a}_\phi \frac{\partial}{r\partial\phi}\right) E_z^0 \rightarrow \mathbf{H} = \frac{1}{Z_{TM}} \mathbf{a}_z \times \mathbf{E}$$

Core:	$E^0_{co,r} = -\frac{j\beta}{h^2} h A_n J'_n(hr)\cos n\phi$	$H^0_{co,r} = \frac{\omega\varepsilon_{co}}{\beta}(-1)E^0_{co,\phi} = -\frac{j\omega\varepsilon_{co} n}{h^2 r} A_n J_n(hr)\sin n\phi$
	$E^0_{co,\phi} = -\frac{j\beta}{h^2}\frac{(-n)}{r} A_n J_n(hr)\sin n\phi$	$H^0_{co,\phi} = \frac{\omega\varepsilon_{co}}{\beta} E^0_{co,r} = -\frac{j\omega\varepsilon_{co}}{h} A_n J'_n(hr)\cos n\phi$
	$E^0_{co,z} = A_n J_n(hr)\cos n\phi$	$H^0_{co,z} = 0$
Cladding:	$E^0_{cl,r} = \frac{j\beta}{q^2} q C_n K'_n(qr)\cos n\phi$	$H^0_{cl,r} = \frac{\omega\varepsilon_{cl}}{\beta}(-1)E^0_{cl,\phi} = \frac{j\omega\varepsilon_{cl} n}{q^2 r} C_n K_n(qr)\sin n\phi$
	$E^0_{cl,\phi} = \frac{j\beta}{q^2}\frac{(-n)}{r} C_n K_n(qr)\sin n\phi$	$H^0_{cl,\phi} = \frac{\omega\varepsilon_{cl}}{\beta} E^0_{cl,r} = \frac{j\omega\varepsilon_{cl}}{q} C_n K'_n(qr)\cos n\phi$
	$E^0_{cl,z} = C_n K_n(qr)\cos n\phi$	$H^0_{cl,z} = 0$
B.C.:	$E^0_{co,z}(a) = E^0_{cl,z}(a)$	$H^0_{co,z}(a) = H^0_{cl,z}(a)$
	$E^0_{co,\phi}(a) = E^0_{cl,\phi}(a)$	$H^0_{co,\phi}(a) = H^0_{cl,\phi}(a)$

TE Modes

Unless n = 0, the B.C. cannot be satisfied only with H_z!

$$Z_{TE} = j\omega\mu/\gamma$$

Recall:

$$\mathbf{H}_T^0 = -\frac{\gamma}{h^2}\nabla_T H_z^0 \rightarrow \nabla_T H_z^0 = \left(\mathbf{a}_r\frac{\partial}{\partial r} + \mathbf{a}_\phi\frac{\partial}{r\partial\phi}\right)H_z^0 \rightarrow \mathbf{E} = -Z_{TE}\mathbf{a}_z\times\mathbf{H}$$

Core:	$H_{co,r}^0 = -\frac{j\beta}{h^2}hB_nJ_n'(hr)\cos n\phi$	$E_{co,r}^0 = -\frac{\omega\mu_{co}}{\beta}(-1)H_{co,\phi}^0 = \frac{j\omega\mu_{co}n}{h^2r}B_nJ_n(hr)\sin n\phi$
	$H_{co,\phi}^0 = -\frac{j\beta}{h^2}\frac{(-n)}{r}B_nJ_n(hr)\sin n\phi$	$E_{co,\phi}^0 = -\frac{\omega\mu_{co}}{\beta}H_{co,r}^0 = \frac{j\omega\mu_{co}}{h}B_nJ_n'(hr)\cos n\phi$
	$H_{co,z}^0 = B_nJ_n(hr)\cos n\phi$	$E_{co,z}^0 = 0$
Cladding:	$H_{cl,r}^0 = \frac{j\beta}{q^2}qD_nK_n'(qr)\cos n\phi$	$E_{cl,r}^0 = -\frac{\omega\mu_{cl}}{\beta}(-1)H_{cl,\phi}^0 = -\frac{j\omega\mu_{cl}n}{q^2r}D_nK_n(qr)\sin n\phi$
	$H_{cl,\phi}^0 = \frac{j\beta}{q^2}\frac{(-n)}{r}D_nK_n(qr)\sin n\phi$	$E_{cl,\phi}^0 = -\frac{\omega\mu_{cl}}{\beta}H_{cl,r}^0 = -\frac{j\omega\mu_{cl}}{q}D_nK_n'(qr)\cos n\phi$
	$H_{cl,z}^0 = D_nK_n(qr)\cos n\phi$	$E_{cl,z}^0 = 0$
B.C.:	$H_{co,z}^0(a) = H_{cl,z}^0(a)$ $H_{co,\phi}^0(a) = H_{cl,\phi}^0(a)$	$E_{co,z}^0(a) = E_{cl,z}^0(a)$ $E_{co,\phi}^0(a) = E_{cl,\phi}^0(a)$

Hybrid Modes (1)

Core:

TM	TE	TM	TE
$E^0_{co,r} = -\frac{j\beta}{h^2} hA_n J_n'(hr)\cos n\phi$	$E^0_{co,r} = \frac{j\omega\mu_{co} n}{h^2 r} B_n J_n(hr)\sin(n\phi+\phi_0)$	$H^0_{co,r} = -\frac{j\omega\varepsilon_{co} n}{h^2 r} A_n J_n(hr)\sin n\phi$	$H^0_{co,r} = -\frac{j\beta}{h^2} hB_n J_n'(hr)\cos(n\phi+\phi_0)$
$E^0_{co,\phi} = -\frac{j\beta}{h^2}\frac{(-n)}{r} A_n J_n(hr)\sin n\phi$	$E^0_{co,\phi} = \frac{j\omega\mu_{co}}{h} B_n J_n'(hr)\cos(n\phi+\phi_0)$	$H^0_{co,\phi} = -\frac{j\omega\varepsilon_{co}}{h} A_n J_n'(hr)\cos n\phi$	$H^0_{co,\phi} = -\frac{j\beta}{h^2}\frac{(-n)}{r} B_n J_n(hr)\sin(n\phi+\phi_0)$
$E^0_{co,z} = A_n J_n(hr)\cos n\phi$	$E^0_{co,z} = 0$	$H^0_{co,z} = 0$	$H^0_{co,z} = B_n J_n(hr)\cos(n\phi+\phi_0)$

$$\rightarrow E^0_{co,r} = -\frac{j\beta}{h^2}[hA_n J_n'(hr)\cos n\phi - \frac{\omega\mu_{co} n}{\beta r} B_n J_n(hr)\sin(n\phi+\phi_0)]$$

$$\rightarrow E^0_{co,\phi} = -\frac{j\beta}{h^2}[\frac{(-n)}{r} A_n J_n(hr)\sin n\phi - \frac{\omega\mu_{co}}{\beta} B_n hJ_n'(hr)\cos(n\phi+\phi_0)]$$

$$\rightarrow E^0_{co,z} = A_n J_n(hr)\cos n\phi$$

$$\rightarrow H^0_{co,r} = -\frac{j\beta}{h^2}[\frac{\omega\varepsilon_{co} n}{\beta r} A_n J_n(hr)\sin n\phi + hB_n J_n'(hr)\cos(n\phi+\phi_0)]$$

$$\rightarrow H^0_{co,\phi} = -\frac{j\beta}{h^2}[\frac{\omega\varepsilon_{co}}{\beta} hA_n J_n'(hr)\cos n\phi - \frac{n}{r} B_n J_n(hr)\sin(n\phi+\phi_0)]$$

$$\rightarrow H^0_{co,z} = B_n J_n(hr)\cos(n\phi+\phi_0)$$

Cladding:

TM	TE	TM	TE
$E^0_{cl,r} = \frac{j\beta}{q^2} qC_n K_n'(qr)\cos n\phi$	$E^0_{cl,r} = -\frac{j\omega\mu_{cl} n}{q^2 r} D_n K_n(qr)\sin(n\phi+\phi_0)$	$H^0_{cl,r} = \frac{j\omega\varepsilon_{cl} n}{q^2 r} C_n K_n(qr)\sin n\phi$	$H^0_{cl,r} = \frac{j\beta}{q^2} qD_n K_n'(qr)\cos(n\phi+\phi_0)$
$E^0_{cl,\phi} = \frac{j\beta}{q^2}\frac{(-n)}{r} C_n K_n(qr)\sin n\phi$	$E^0_{cl,\phi} = -\frac{j\omega\mu_{cl}}{q} D_n K_n'(qr)\cos(n\phi+\phi_0)$	$H^0_{cl,\phi} = \frac{j\omega\varepsilon_{cl}}{q} C_n K_n'(qr)\cos n\phi$	$H^0_{cl,\phi} = \frac{j\beta}{q^2}\frac{(-n)}{r} D_n K_n(qr)\sin(n\phi+\phi_0)$
$E^0_{cl,z} = C_n K_n(qr)\cos n\phi$	$E^0_{cl,z} = 0$	$H^0_{cl,z} = 0$	$H^0_{cl,z} = D_n K_n(qr)\cos(n\phi+\phi_0)$

$$\rightarrow E^0_{cl,r} = \frac{j\beta}{q^2}[qC_n K_n'(qr)\cos n\phi - \frac{\omega\mu_{cl} n}{\beta r} D_n K_n(qr)\sin(n\phi+\phi_0)]$$

$$\rightarrow E^0_{cl,\phi} = \frac{j\beta}{q^2}[\frac{(-n)}{r} C_n K_n(qr)\sin n\phi - \frac{\omega\mu_{cl}}{\beta} qD_n K_n'(qr)\cos(n\phi+\phi_0)]$$

$$\rightarrow E^0_{cl,z} = C_n K_n(qr)\cos n\phi$$

$$\rightarrow H^0_{cl,r} = \frac{j\beta}{q^2}[\frac{\omega\varepsilon_{cl} n}{\beta r} C_n K_n(qr)\sin n\phi + qD_n K_n'(qr)\cos(n\phi+\phi_0)]$$

$$\rightarrow H^0_{cl,\phi} = \frac{j\beta}{q^2}[\frac{\omega\varepsilon_{cl}}{\beta} qC_n K_n'(qr)\cos n\phi + \frac{(-n)}{r} D_n K_n(qr)\sin(n\phi+\phi_0)]$$

$$\rightarrow H^0_{cl,z} = D_n K_n(qr)\cos(n\phi+\phi_0)$$

B.C.:

$$E^0_{co,z}(a) = E^0_{cl,z}(a)$$
$$E^0_{co,\phi}(a) = E^0_{cl,\phi}(a)$$

$$H^0_{co,z}(a) = H^0_{cl,z}(a)$$
$$H^0_{co,\phi}(a) = H^0_{cl,\phi}(a)$$

Hybrid Modes (2)

Boundary condition:

$$E^0_{co,z}(a) = E^0_{cl,z}(a) \quad \rightarrow A_n J_n(ha)\cos n\phi = C_n K_n(qa)\cos n\phi \quad \rightarrow A_n = C_n \frac{K_n(qa)}{J_n(ha)}$$

$$E^0_{co,\phi}(a) = E^0_{cl,\phi}(a) \quad \rightarrow -\frac{j\beta}{h^2}\left[\frac{(-n)}{a} A_n J_n(ha)\sin n\phi - \frac{\omega\mu_{co}}{\beta} B_n h J'_n(ha)\cos(n\phi+\phi_0)\right] = \frac{j\beta}{q^2}\left[\frac{(-n)}{a} C_n K_n(qa)\sin n\phi - \frac{\omega\mu_{cl}}{\beta} q D_n K'_n(qa)\cos(n\phi+\phi_0)\right]$$

$$H^0_{co,z}(a) = H^0_{cl,z}(a) \quad \rightarrow B_n J_n(ha)\cos(n\phi+\phi_0) = D_n K_n(qa)\cos(n\phi+\phi_0) \quad \rightarrow D_n = B_n \frac{J_n(ha)}{K_n(qa)}$$

$$H^0_{co,\phi}(a) = H^0_{cl,\phi}(a) \quad \rightarrow -\frac{j\beta}{h^2}\left[\frac{\omega\varepsilon_{co}}{\beta} h A_n J'_n(ha)\cos n\phi - \frac{n}{a} B_n J_n(ha)\sin(n\phi+\phi_0)\right] = \frac{j\beta}{q^2}\left[\frac{\omega\varepsilon_{cl}}{\beta} q C_n K'_n(qa)\cos n\phi + \frac{(-n)}{a} D_n K_n(qa)\sin(n\phi+\phi_0)\right]$$

$$\rightarrow A_n \frac{n}{h^2 a} J_n(ha)\sin n\phi + B_n \frac{\omega\mu_{co}}{h\beta} J'_n(ha)\cos(n\phi+\phi_0) + \frac{n}{q^2 a} C_n K_n(qa)\sin n\phi + D_n \frac{\omega\mu_{cl}}{q\beta} K'_n(qa)\cos(n\phi+\phi_0) = 0$$

$$\rightarrow A_n \frac{\omega\varepsilon_{co}}{h\beta} J'_n(ha)\cos n\phi - B_n \frac{n}{h^2 a} J_n(ha)\sin(n\phi+\phi_0) + C_n \frac{\omega\varepsilon_{cl}}{q\beta} K'_n(qa)\cos n\phi - D_n \frac{n}{q^2 a} K_n(qa)\sin(n\phi+\phi_0) = 0$$

$$\rightarrow B_n\left[\frac{\omega\mu_{co}}{h\beta} J'_n(ha) + \frac{\omega\mu_{cl}}{q\beta}\frac{K'_n(qa)}{K_n(qa)} J_n(ha)\right]\cos(n\phi+\phi_0) + C_n\left[\frac{n}{q^2 a} + \frac{n}{h^2 a}\right] K_n(qa)\sin n\phi = 0$$

$$\rightarrow -B_n\left[\frac{n}{h^2 a} + \frac{n}{q^2 a}\right] J_n(ha)\sin(n\phi+\phi_0) + C_n\left[\frac{\omega\varepsilon_{co}}{h\beta}\frac{J'_n(ha)}{J_n(ha)} K_n(qa) + \frac{\omega\varepsilon_{cl}}{q\beta} K'_n(qa)\right]\cos n\phi = 0$$

→ *For a non-trivial solution to exist:* $\rightarrow \phi_0 = \pm\frac{\pi}{2}$

$$\rightarrow \left[\frac{\omega\mu_{co}}{h\beta} J'_n(ha) + \frac{\omega\mu_{cl}}{q\beta}\frac{K'_n(qa)}{K_n(qa)} J_n(ha)\right]\left[\frac{\omega\varepsilon_{co}}{h\beta}\frac{J'_n(ha)}{J_n(ha)} K_n(qa) + \frac{\omega\varepsilon_{cl}}{q\beta} K'_n(qa)\right]\cos n\phi\cos(n\phi+\phi_0) + \left[\frac{n}{h^2 a} + \frac{n}{q^2 a}\right]^2 J_n(ha) K_n(qa)\sin n\phi\sin(n\phi+\phi_0) = 0$$

→ *Transcendental equation for modes:*

$$\rightarrow \left[\frac{\mu_{co}}{ha}\frac{J'_n(ha)}{J_n(ha)} + \frac{\mu_{cl}}{qa}\frac{K'_n(qa)}{K_n(qa)}\right]\left[\frac{\varepsilon_{co}}{ha}\frac{J'_n(ha)}{J_n(ha)} + \frac{\varepsilon_{cl}}{qa}\frac{K'_n(qa)}{K_n(qa)}\right] = n^2\left[\left(\frac{1}{ha}\right)^2 + \left(\frac{1}{qa}\right)^2\right]^2\left(\frac{\beta}{\omega}\right)^2$$

Characteristics of Hybrid Modes (1)

Transcendental equation for modes:

$$\left[\frac{\mu_{co}}{ha}\frac{J_n'(ha)}{J_n(ha)}+\frac{\mu_{cl}}{qa}\frac{K_n'(qa)}{K_n(qa)}\right]\left[\frac{\varepsilon_{co}}{ha}\frac{J_n'(ha)}{J_n(ha)}+\frac{\varepsilon_{cl}}{qa}\frac{K_n'(qa)}{K_n(qa)}\right]=n^2\left[\left(\frac{1}{ha}\right)^2+\left(\frac{1}{qa}\right)^2\right]^2\left(\frac{\beta}{\omega}\right)^2$$

For non-magnetic media: $\rightarrow \mu_{co}=\mu_{cl}=\mu_0$

$$\rightarrow\left[\frac{J_n'(ha)}{haJ_n(ha)}+\frac{K_n'(qa)}{qaK_n(qa)}\right]\left[\frac{n_{co}^2J_n'(ha)}{haJ_n(ha)}+\frac{n_{cl}^2K_n'(qa)}{qaK_n(qa)}\right]=n^2\left[\left(\frac{1}{ha}\right)^2+\left(\frac{1}{qa}\right)^2\right]^2\left(\frac{\beta}{k_0}\right)^2$$

Solution:

$$\rightarrow\frac{J_n'(ha)}{haJ_n(ha)}=-\frac{n_{co}^2+n_{cl}^2}{2n_{co}^2}\frac{K_n'(qa)}{qaK_n(qa)}\pm\left\{\left(\frac{n_{co}^2-n_{cl}^2}{2n_{co}^2}\right)^2\left(\frac{K_n'(qa)}{qaK_n(qa)}\right)^2+\frac{n^2}{n_{co}^2}\left(\frac{\beta}{k_0}\right)^2\left[\frac{1}{(ha)^2}+\frac{1}{(qa)^2}\right]^2\right\}^{1/2}$$

$$J_n'(x)=-J_{n+1}'(x)+\frac{n}{x}J_n(x)$$

$$J_n'(x)=J_{n-1}'(x)-\frac{n}{x}J_n(x)$$

EH modes: $$\rightarrow\frac{J_{n+1}(ha)}{haJ_n(ha)}=\frac{n_{co}^2+n_{cl}^2}{2n_{co}^2}\frac{K_n'(qa)}{qaK_n(qa)}+\frac{n}{(ha)^2}-\left[\left(\frac{n_{co}^2-n_{cl}^2}{2n_{co}^2}\right)^2\left(\frac{K_n'(qa)}{qaK_n(qa)}\right)^2+\frac{n^2}{n_{co}^2}\left(\frac{\beta}{k_0}\right)^2\left(\frac{1}{h^2a^2}+\frac{1}{q^2a^2}\right)^2\right]^{1/2}$$

HE modes: $$\rightarrow\frac{J_{n-1}(ha)}{haJ_n(ha)}=-\frac{n_{co}^2+n_{cl}^2}{2n_{co}^2}\frac{K_n'(qa)}{qaK_n(qa)}+\frac{n}{(ha)^2}-\left[\left(\frac{n_{co}^2-n_{cl}^2}{2n_{co}^2}\right)^2\left(\frac{K_n'(qa)}{qaK_n(qa)}\right)^2+\frac{n^2}{n_{co}^2}\left(\frac{\beta}{k_0}\right)^2\left(\frac{1}{h^2a^2}+\frac{1}{q^2a^2}\right)^2\right]^{1/2}$$

Characteristics of Hybrid Modes (2)

EH modes: $\rightarrow \dfrac{J_{n+1}(ha)}{haJ_n(ha)} = \dfrac{n_{co}^2+n_{cl}^2}{2n_{co}^2}\dfrac{K_n'(qa)}{qaK_n(qa)} + \dfrac{n}{(ha)^2} - \left[\left(\dfrac{n_{co}^2-n_{cl}^2}{2n_{co}^2}\right)^2\left(\dfrac{K_n'(qa)}{qaK_n(qa)}\right)^2 + \dfrac{n^2}{n_{co}^2}\left(\dfrac{\beta}{k_0}\right)^2\left(\dfrac{1}{h^2a^2}+\dfrac{1}{q^2a^2}\right)^2\right]^{1/2}$

$E_z \uparrow H_z \downarrow$ $\rightarrow$ TM-like

HE modes: $\rightarrow \dfrac{J_{n-1}(ha)}{haJ_n(ha)} = -\dfrac{n_{co}^2+n_{cl}^2}{2n_{co}^2}\dfrac{K_n'(qa)}{qaK_n(qa)} + \dfrac{n}{(ha)^2} - \left[\left(\dfrac{n_{co}^2-n_{cl}^2}{2n_{co}^2}\right)^2\left(\dfrac{K_n'(qa)}{qaK_n(qa)}\right)^2 + \dfrac{n^2}{n_{co}^2}\left(\dfrac{\beta}{k_0}\right)^2\left(\dfrac{1}{h^2a^2}+\dfrac{1}{q^2a^2}\right)^2\right]^{1/2}$

$E_z \downarrow H_z \uparrow$ $\rightarrow$ TE-like

For $n = 0$:

TM modes: $\rightarrow \dfrac{J_1(ha)}{haJ_0(ha)} = \dfrac{n_{cl}^2}{n_{co}^2}\dfrac{K_0'(qa)}{qaK_0(qa)} = -\dfrac{n_{cl}^2}{n_{co}^2}\dfrac{K_1(qa)}{qaK_0(qa)}$

TE modes: $\rightarrow \dfrac{J_1(ha)}{haJ_0(ha)} = \dfrac{K_0'(qa)}{qaK_0(qa)} = -\dfrac{K_1(qa)}{qaK_0(qa)}$

To be solved numerically

Cutoff Conditions

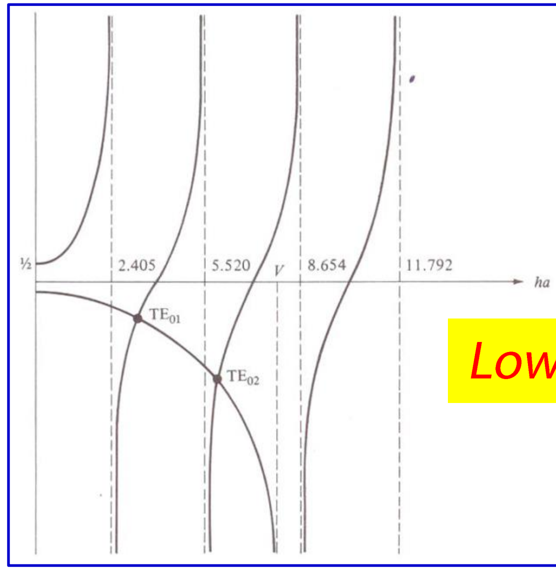
Graphical method

TM_{0m}: $\rightarrow \dfrac{J_1(ha)}{haJ_0(ha)} = -\dfrac{n_{cl}^2}{n_{co}^2}\dfrac{K_1(qa)}{qaK_0(qa)}$

TE_{0m}: $\rightarrow \dfrac{J_1(ha)}{haJ_0(ha)} = -\dfrac{K_1(qa)}{qaK_0(qa)}$

$$If\ q = 0,\ ha = \frac{2\pi a}{\lambda_0}\sqrt{n_{co}^2 - n_{cl}^2} \equiv V$$

V < 2.405 → Single-mode!



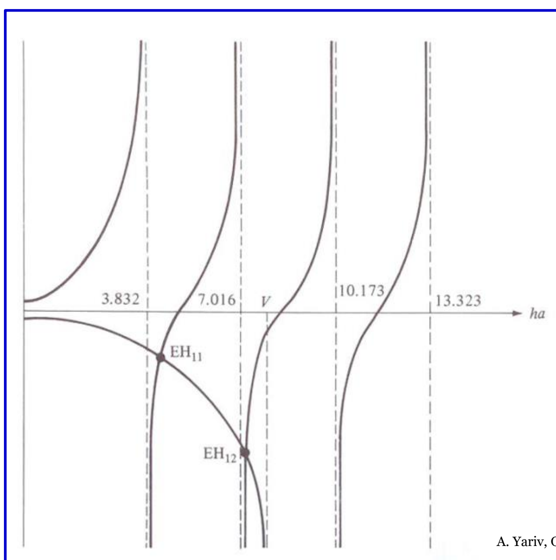
Lowest cutoff!

A. Yariv, Optical Electronics, 4th ed., HBJ, 1991.

EH_{1m}: $\rightarrow \dfrac{J_2(ha)}{haJ_1(ha)} = \dfrac{n_{co}^2 + n_{cl}^2}{2n_{co}^2}\dfrac{K_1'(qa)}{qaK_1(qa)} + \dfrac{1}{(ha)^2} - \left[\left(\dfrac{n_{co}^2 - n_{cl}^2}{2n_{co}^2}\right)^2\left(\dfrac{K_1'(qa)}{qaK_1(qa)}\right)^2 + \dfrac{1}{n_{co}^2}\left(\dfrac{\beta}{k_0}\right)^2\left(\dfrac{1}{h^2a^2} + \dfrac{1}{q^2a^2}\right)^2\right]^{1/2}$

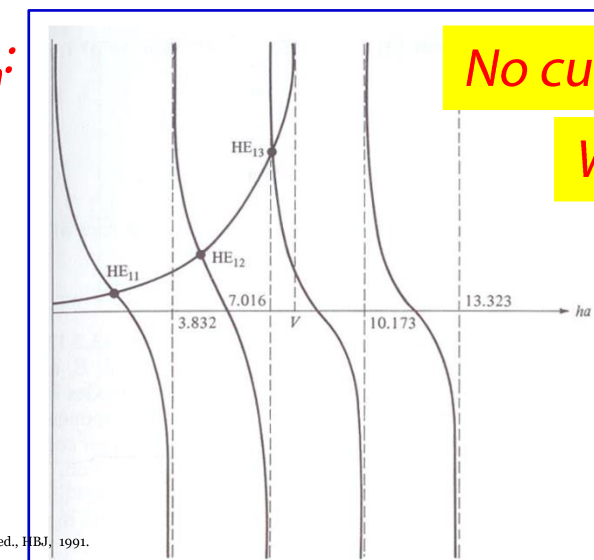
HE_{1m}: $\rightarrow \dfrac{J_0(ha)}{haJ_1(ha)} = -\dfrac{n_{co}^2 + n_{cl}^2}{2n_{co}^2}\dfrac{K_1'(qa)}{qaK_1(qa)} + \dfrac{1}{(ha)^2} - \left[\left(\dfrac{n_{co}^2 - n_{cl}^2}{2n_{co}^2}\right)^2\left(\dfrac{K_1'(qa)}{qaK_1(qa)}\right)^2 + \dfrac{1}{n_{co}^2}\left(\dfrac{\beta}{k_0}\right)^2\left(\dfrac{1}{h^2a^2} + \dfrac{1}{q^2a^2}\right)^2\right]^{1/2}$

EH_{1m}:



A. Yariv, Optical Electronics, 4th ed., HBJ, 1991.

HE_{1m}:



No cutoff!

Why?