Estimation of Maximum Response of MDF system Based on Response Spectrum

Linear Elastic Dynamic Equilibrium of multi–degree of freedom (MDOF) systems subject to earthquake motion

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = -\mathbf{M}\mathbf{r}\ddot{u}_g(t)$$

$\mathbf{r}$ = influence vector showing the direction of loading

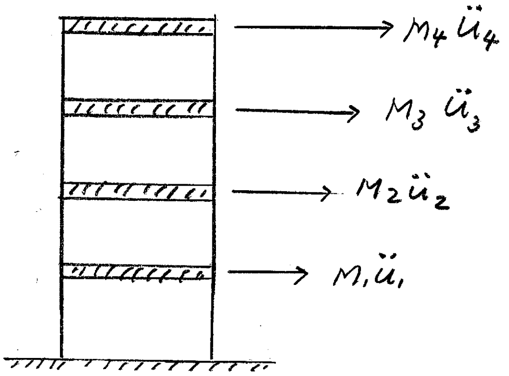
To seek the solution of nonhomogeneous equations,

First find the form of solutions for homogeneous eqs.

Solution by modal superposition analysis

Assuming undamped free vibration,

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = 0$$



Decomposition of dynamic modes

$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{u} = \mathbf{0}$ ⇨ Eigenvalue Problem

$$\left|\mathbf{K} - \omega^2\mathbf{M}\right| = \mathbf{0}$$

ω_i = ith Eigenvalue (ith angular frequency)

ϕ_i = ith Eigenvector (ith mode shape)

$$\mathbf{u} = \phi_1 Y_1 + \phi_2 Y_2 + \phi_3 Y_3 + ---- = \sum_i \phi_i Y_i = \mathbf{\Phi}\mathbf{Y}$$

Multiply the above set of equation by $\mathbf{\Phi}^{\mathrm{T}}$ to obtain an uncoupled set of equations.

orthogonality of eigenvectors: $\phi_i^{\mathrm{T}}\mathbf{M}\phi_j = \phi_i^{\mathrm{T}}\mathbf{K}\phi_j = 0 \quad (i \neq j)$

also assuming $\phi_i^{\mathrm{T}}\mathbf{C}\phi_j = 0 \quad (i \neq j)$

For ith mode,

$$\underbrace{\phi_i^{\mathrm{T}}\mathbf{M}\phi_i}_{M_i^*}\ddot{\mathbf{Y}}_i + \underbrace{\phi_i^{\mathrm{T}}\mathbf{C}\phi_i}_{C_i^*}\dot{\mathbf{Y}}_i + \underbrace{\phi_i^{\mathrm{T}}\mathbf{K}\phi_i}_{K_i^*}\mathbf{Y}_i = -\phi_i^{\mathrm{T}}\mathbf{M}\mathbf{r}\ddot{u}_g(t)$$

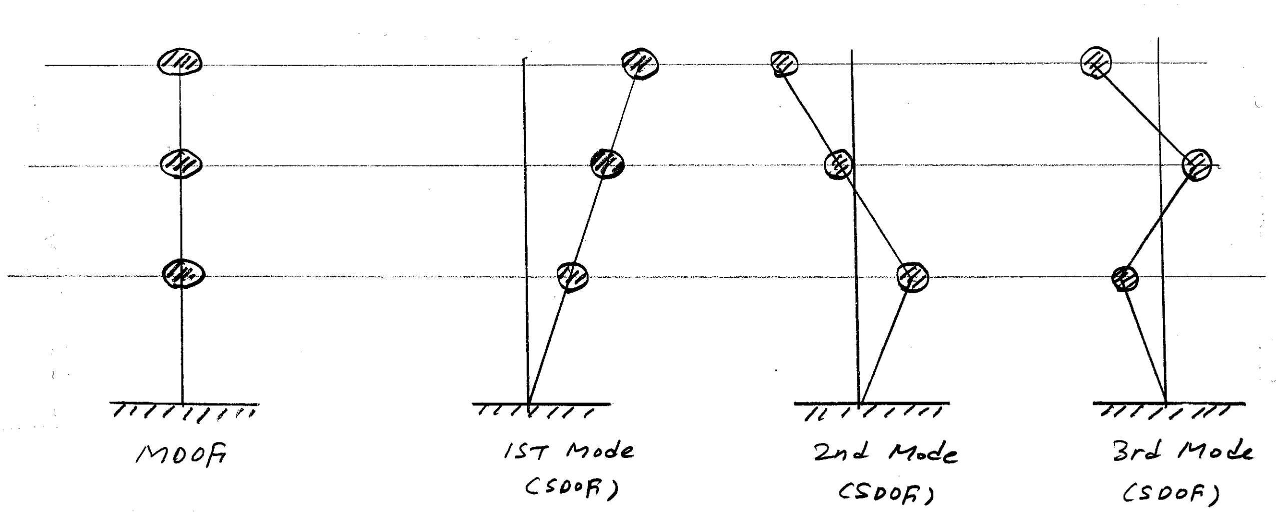
M_i^* = Generalized Mass

C_i^* = Generalized Damping

K_i^* = Generalized Stiffness

Thus, the i th mode dynamic equation is the form of a SDOF system.

By using the modal analysis, the nth MDOF system is decomposed into n number of SDOF systems



$$M_i^*\ddot{\mathbf{Y}}_i + 2\xi_i\omega_i M_i^*\dot{\mathbf{Y}}_i + \omega_i^2 M_i^*\mathbf{Y}_i = -L_i\ddot{u}_g(t)$$

$$\xi_i = c^* / 2\omega_i M_i^* \qquad L_i = -\underset{\sim}{\phi}_i^{\mathrm{T}}\underset{\sim}{\mathbf{M}}\underset{\sim}{\mathbf{r}}$$

or $\ddot{Y}_i + 2\xi_i\omega_i\dot{Y}_i + \omega_i^2 Y_i = -\frac{L_i}{M_i^*}\ddot{u}_g(t)$

Only difference from SDOF equation is the factor of $\dfrac{\underset{\sim}{\phi}_i^{\mathrm{T}}\underset{\sim}{\mathbf{M}}\underset{\sim}{\mathbf{r}}}{\underset{\sim}{\phi}_i^{\mathrm{T}}\underset{\sim}{\mathbf{M}}\underset{\sim}{\phi}_i} = \dfrac{L_i}{M_i^*}$ (= modal participation factor).

Thus, the response of each mode is affected by the modal participation factor as well as the ground acceleration.

$$Y_i \propto \frac{L_i}{M_i^*}\ddot{u}_g(t)$$

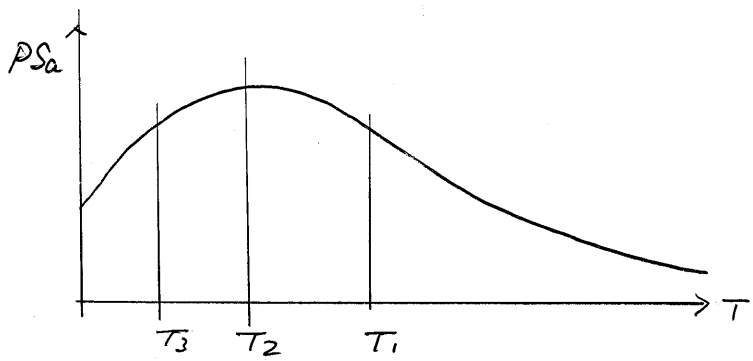
Individual modal responses are added to obtain the final response (modal superposition)

$$\underset{\sim}{\mathbf{u}} = \sum_i \underset{\sim}{\phi}_i Y_i = \underset{\sim}{\mathbf{\Phi}}\underset{\sim}{\mathbf{Y}}$$

Use of Response Spectrum

As we know, to estimate the maximum response of SDOF system, response spectrum can be conveniently used, without performing time history analysis.

Thus, to estimate the maximum response of each mode (SDOF system), a response spectrum which was prepared beforehand can be used conveniently.



$$Y_{i,\max} = \frac{L_i}{M_i^*} S_d$$

$$= \frac{L_i}{M_i^* \omega_i^2} pS_a$$

S_d = spectral displacement = pS_a / ω_i^2

pS_a = pseudo spectral acceleration (which is estimated from the response spectrum)

difference from the response of real SDOF system is the factor of $\frac{L_i}{M_i^*}$

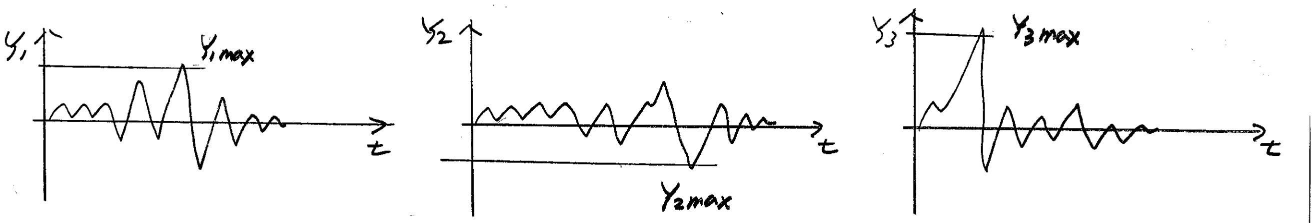
Calculation of maximum response of the MDOF system

In principle, in the modal superposition method, individual modal responses are added to obtain the final response (modal superposition)

$$\underset{\sim}{\mathbf{u}} = \sum_i \underset{\sim}{\phi_i} Y_i = \underset{\sim}{\mathbf{\Phi}} \underset{\sim}{\mathbf{Y}}$$

However, S_d and pS_a estimated from the response spectrum are the maximum response of each mode, and should not occur at the same time.

Thus, to estimate a reasonable maximum response with the spectral values, we need to use an average value of the maxima of the modes.



SRSS method

$\mathbf{u}_{i,\max} = \phi_i Y_{i,\max}$ (ith mode)

The modal maxima are combined using the square root of the sum of the squares

$$\mathbf{u}_{\max} = \sqrt{\sum_i {u_{i\max}}^2}$$

$$\mathbf{u}_{\max} < \sum u_{i\max}$$

Or, more modern techniques such as CQC (Complete Quadratic Combination) can be used.

Other response quantities calculated similarly :

But be careful that $\mathbf{u}_{\max}$ cannot be directly used to estimate the other response quantities because $\mathbf{u}_{\max}$ estimated from SRSS is not accurate maximum displacement.

Thus, the response of each mode should be calculated from the maximum displacement of each mode, and then the response of each member should be estimated by combining the contribution of the modes using SRSS.

$\mathbf{u}_{i,\max} = \phi_i Y_{i,\max}$ (MDOF response in ith mode)

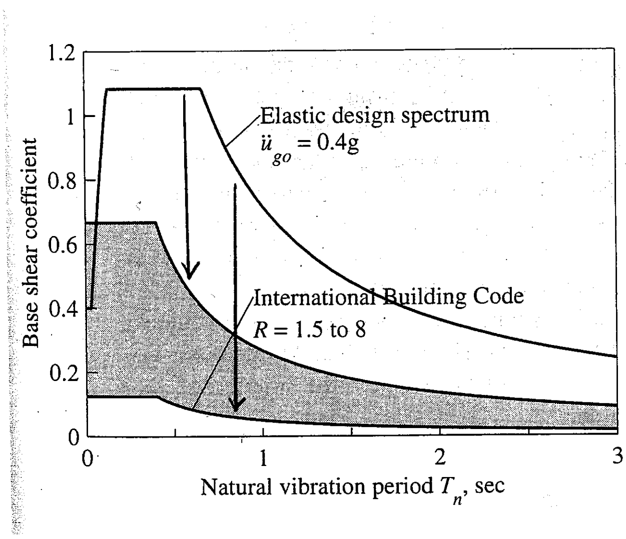
Member actions in ith mode are then computed

$S^j_{i,\max} = K^j u^j_{i,\max}$ (Member end actions for jth member due to ith mode)

Modal member actions are combined by SRSS or CQC

$$S^j_{i,\max} = \sqrt{\sum_i {S^j_{i,\max}}^2}$$

Design Forces vs. Linear Elastic Response Spectra



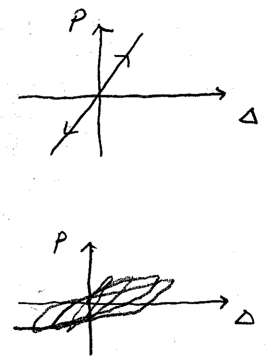
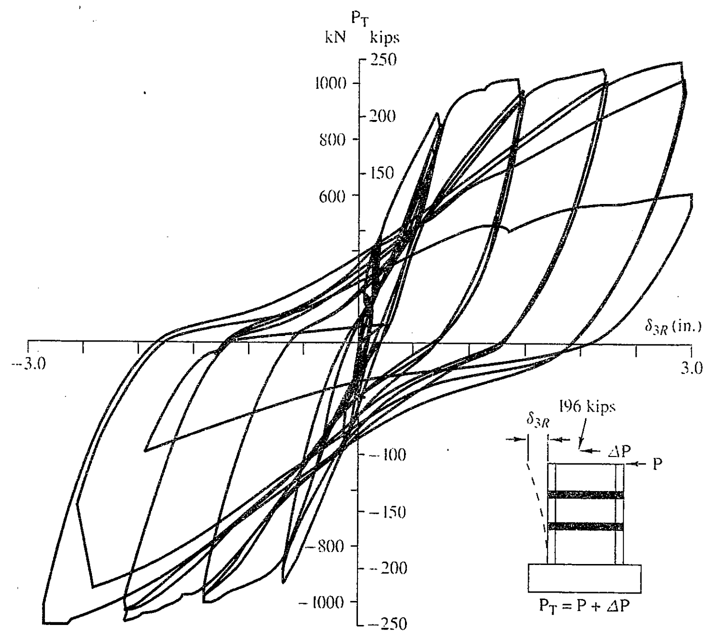


Figure 7.1 Comparison of base shear coefficients from elastic design spectrum and *International Building Code*.

The design spectral accelerations of current design codes (KBC, IBC) are less than those of the linear elastic design response spectrum by a factor of 5 or 6.

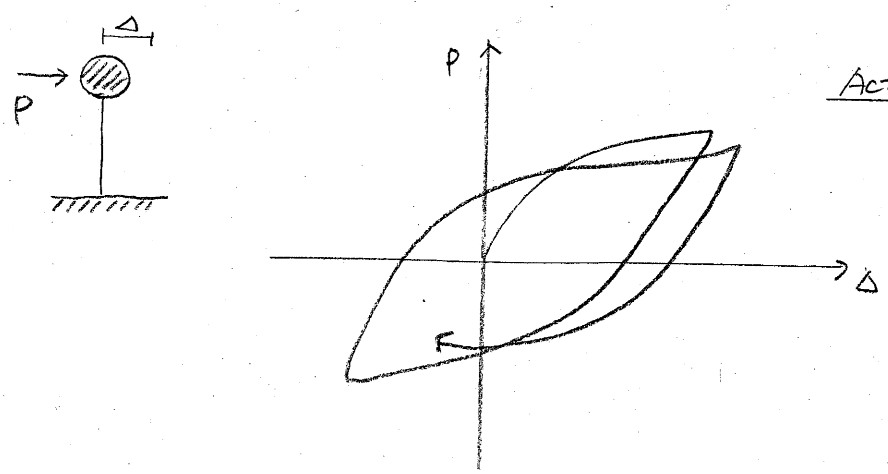
Note that the implied IBC Design Spectrum is less than the linear elastic design Response Spectrum for El Centro EQ by a factor of 5 or 6



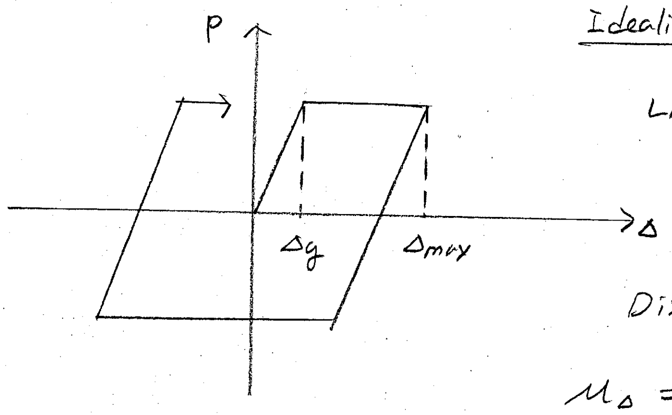
Effects of Inelastic Behavior on EQ Response

We have seen that the elastic response of a typical SDOF (single degree of freedom) system involves forces many times greater than those required for design by typical codes (such as IBC and KBC) , even when safety factors are included.

The resolution of this apparent discrepancy lies in the effect of inelastic response.



Actual P-Δ Relationship



Idealized P-Δ Relationship

Linear Elastic - Pefectly plastic System

Displacement Ductility

$$\mu_\Delta = \frac{|\Delta_{max}|}{|\Delta_y|}$$

Two reasons for

1) EQ is not force but displacement.

2) Heavy damage is permitted because of the low possibility of EQ occurrence.

Calculation of Inelastic Response

$$M\ddot{u} + C\dot{u} + F = -M\ddot{u}_g(t)$$

Use any step-by-step method, for example, central difference method.

$$u_{i+1} = u_i + \Delta t\dot{u}_i + \frac{\Delta t^2}{2}\ddot{u}_i + \cdots$$
$$u_{i-1} = u_i - \Delta t\dot{u}_i + \frac{\Delta t^2}{2}\ddot{u}_i - \cdots$$

Therefore,

$$\dot{u}_i = \frac{u_{i+1} - u_{i-1}}{2\Delta t}$$

$$\ddot{u}_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta t^2}$$

Solution in linear elastic range

Consider a SDOF system, in the linear range, $F_{elastic} = Ku$

From Equations

$$M\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta t^2}\right) + C\left(\frac{u_{i+1} - u_{i-1}}{2\Delta t}\right) + Ku_i = -M\ddot{u}_{g,i}$$

$$u_{i+1}\left(\frac{M}{\Delta t^2} + \frac{C}{2\Delta t}\right) + u_i\left(K - \frac{2M}{2\Delta t^2}\right) + u_{i-1}\left(\frac{M}{\Delta t^2} - \frac{C}{2\Delta t}\right) = -M\ddot{u}_{g,i}$$

$$u_{i+1} = \frac{\left[-M\ddot{u}_{g,i} - u_i\left(K - \frac{2M}{2\Delta t^2}\right) - u_{i-1}\left(\frac{M}{\Delta t^2} - \frac{C}{2\Delta t}\right)\right]}{\left(\frac{M}{\Delta t^2} + \frac{C}{2\Delta t}\right)}$$

Solution in plastic (post-yield) range

Assuming an elastic perfectly plastic behavior,

If the system has yielded, $F_{inelastic} = Ku_y = \pm F_y$

$$M\ddot{u} + C\dot{u} \pm F_y = -M\ddot{u}_g(t)$$

$$\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta t^2}\right) + C\left(\frac{u_{i+1} - u_{i-1}}{2\Delta t}\right) \pm F_y = -M\ddot{u}_{g,i}$$

$$u_{i+1}\left(\frac{M}{\Delta t^2} + \frac{C}{2\Delta t}\right) - u_i\left(\frac{2M}{2\Delta t^2}\right) + u_{i-1}\left(\frac{M}{\Delta t^2} - \frac{C}{2\Delta t}\right) \pm F_y = -M\ddot{u}_{g,i}$$

$$u_{i+1} = \frac{\left[\mp F_y - M\ddot{u}_{g,i} - u_i\left(-\frac{2M}{2\Delta t^2}\right) - u_{i-1}\left(\frac{M}{\Delta t^2} - \frac{C}{2\Delta t}\right)\right]}{\left(\frac{M}{\Delta t^2} + \frac{C}{2\Delta t}\right)}$$

Typical elasto-plastic response results are shown on the next page in the form of response spectra

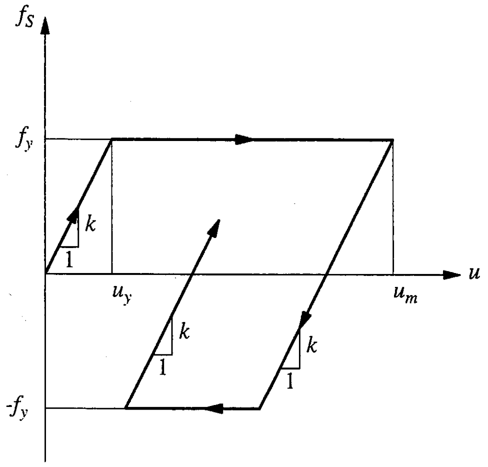
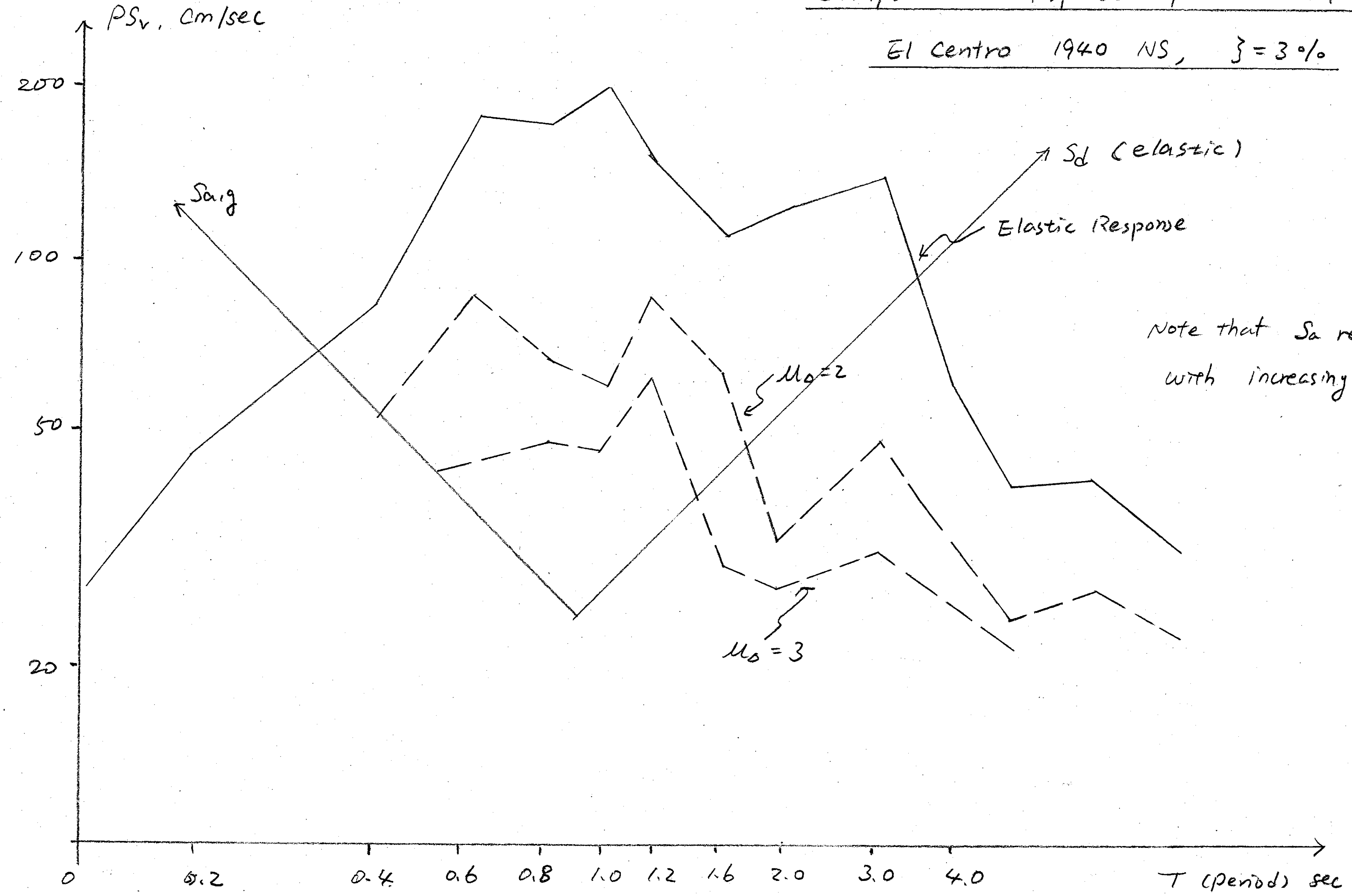


Figure 7.1.3 Elastoplastic force–deformation relation.

Comparative Response Spectra for Forces

El Centro 1940 NS, $\zeta = 3\%$



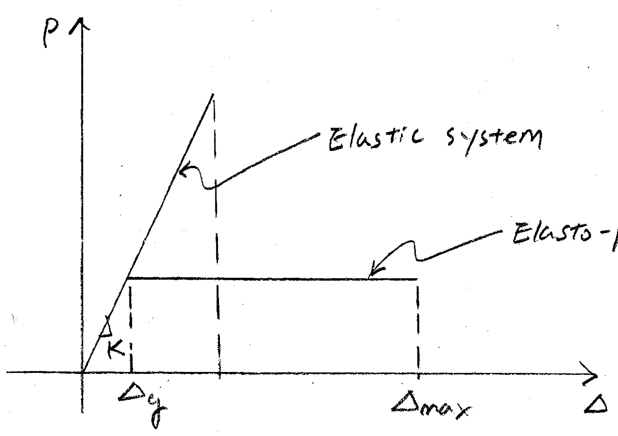
Note that Sa redues with increasing μ_o

Comments

1) Response spectrum for forces is reduced in longer period ranges as a function of ductility
2) Response spectrum for displacements is increased in short-period ranges as a function of ductility

Elasto-plastic system

$$\Delta_m = \Delta_y \mu = (P_y / K)\mu$$



3) Response of inelastic system can be calculated using an elastic system, subjected to a reduced response spectrum (for forces), or an increased spectrum (for displacements).

On the basis of the observations, inelastic force and displacement spectrum of structures can be estimated by using the linear elastic response spectrum and the ductility of the structure.

Equal Energy Principle & Equal Displacement Principle

Equal energy principle is valid for long-period structures
Equal displacement principle is valid for short-period structures.

μ : ductility capacity of a structure

R : strength modification factor = F_E / F_{IE}

In the equal displacement principle,

For long period structures,

$$R = \mu \qquad F_{IE} = F_E / R = F_E / \mu$$

$$\Delta_{IE} = \Delta_E$$

In the equal energy principle,

For short period structures,

$$A_2 = (F_E / R)(\Delta_{IE} - \Delta_y R) \qquad A_1 = (F_E - F_E / R)(\Delta_y R - \Delta_y)/2$$

$$A_1 = A_2 \qquad \frac{\Delta_{IE}}{\Delta_y} = \frac{(R^2+1)}{2} = \mu$$

$$R = \sqrt{2\mu - 1} \qquad F_{IE} = F_E / R = F_E / \sqrt{2\mu - 1}$$

$$\Delta_{IE} = \mu \Delta_y = \mu \Delta_E / \sqrt{2\mu - 1}$$

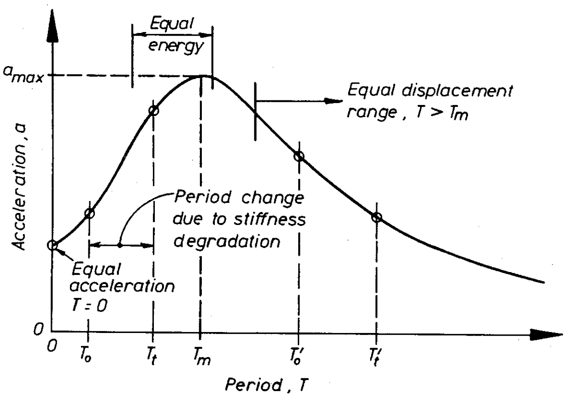


Fig. 2.21 Influence of period on ductile force reduction.

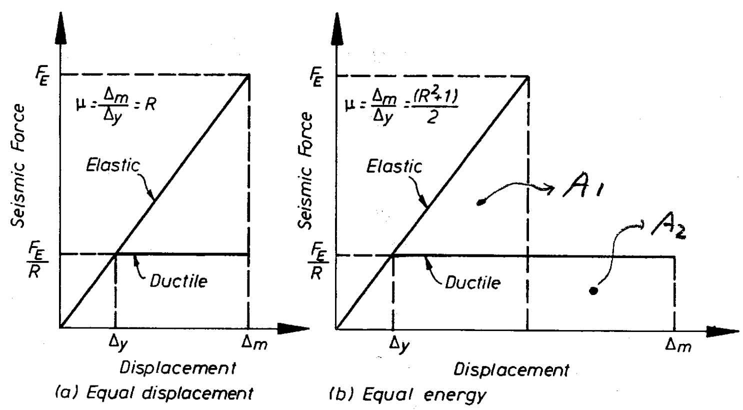
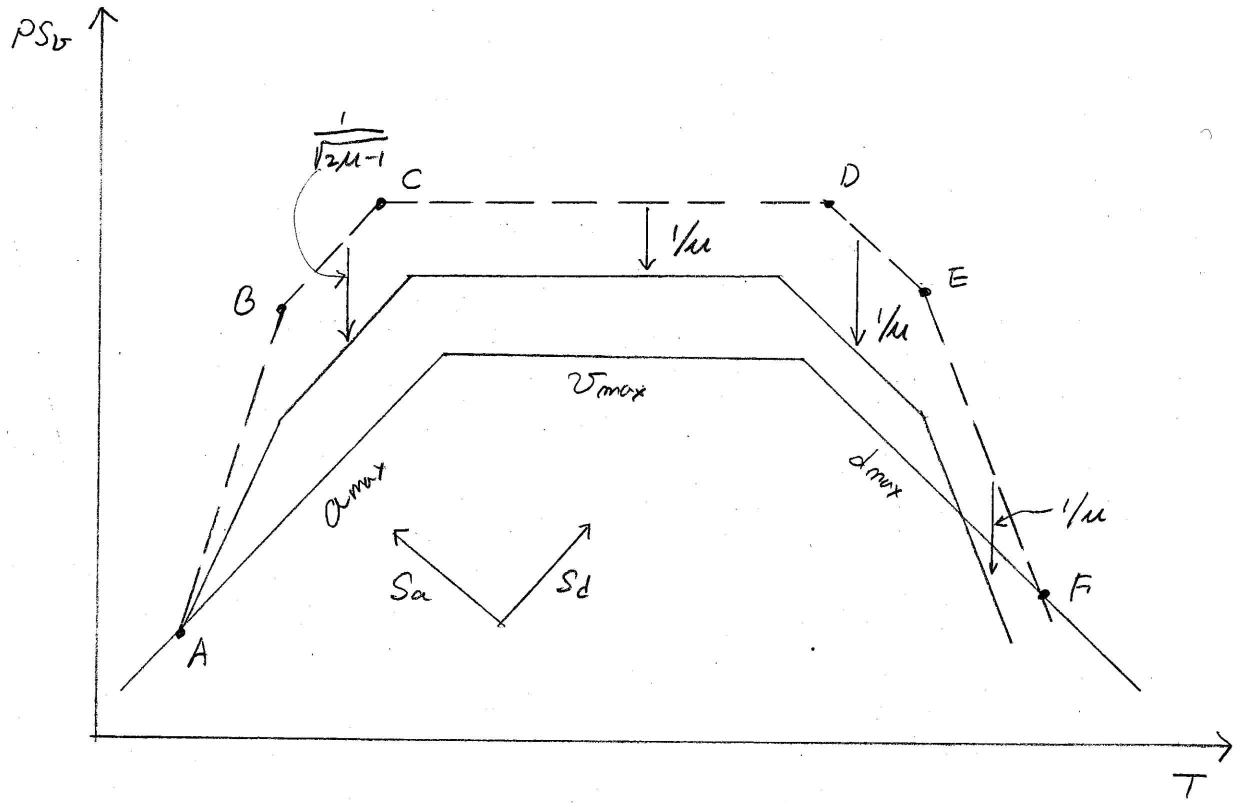


Fig. 2.22 Relationship between ductility and force reduction factor.

Force Response Spectrum

Reduced spectrum for forces

1) Reduce by $1/\mu$ in constant Sd and Sv regions.
2) Reduce by $1/\sqrt{2\mu-1}$ in constant Sa region
3) Linear interpolation between T_A and T_B
4) Reduce by $1/\mu$ between T_E and T_F



Displacement Response Spectrum

Increased spectrum for displacements

1) Same as LEDRS in constant Sd and Sv regions
2) Increase by μ where Sa = PGA.
3) Increase by $\mu/\sqrt{2\mu-1}$ in constant Sa region
4) Linear interpolation between T_A and T_B