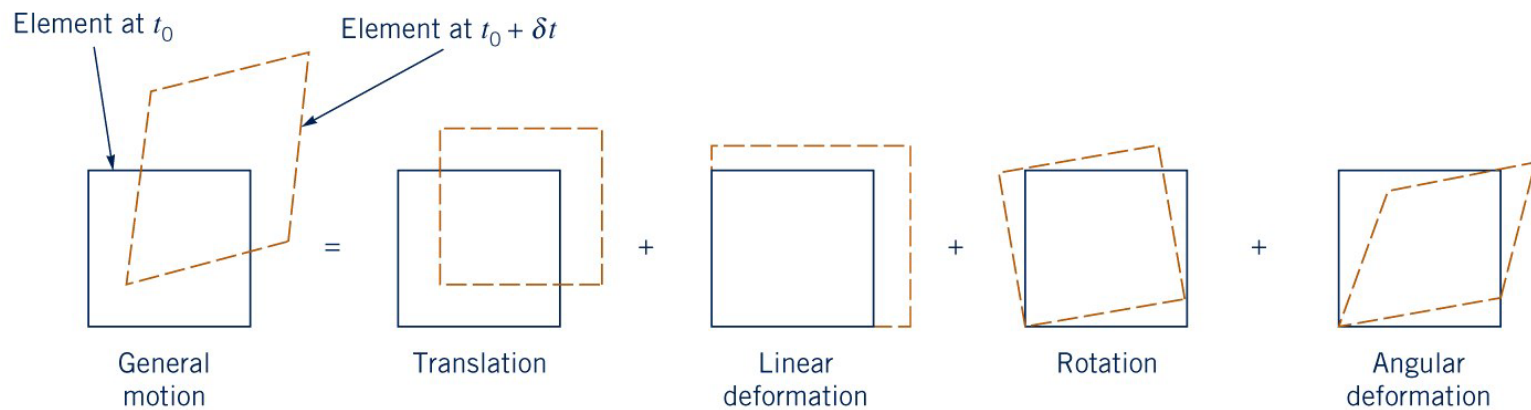


Chapter 5

Stress–Strain Relation



Chapter 5 Stress–Strain Relation

Contents

5.1 General Stress–Strain System

5.2 Relations Between Stress and Strain for Elastic Solids

5.3 Relations Between Stress and Rate of Strain for Newtonian Fluids

Objectives

- Understand tensor systems of stress and strain
- Study difference between displacement and deformation
- Study solid mechanics to deduce stress-rate of strain relations for fluid

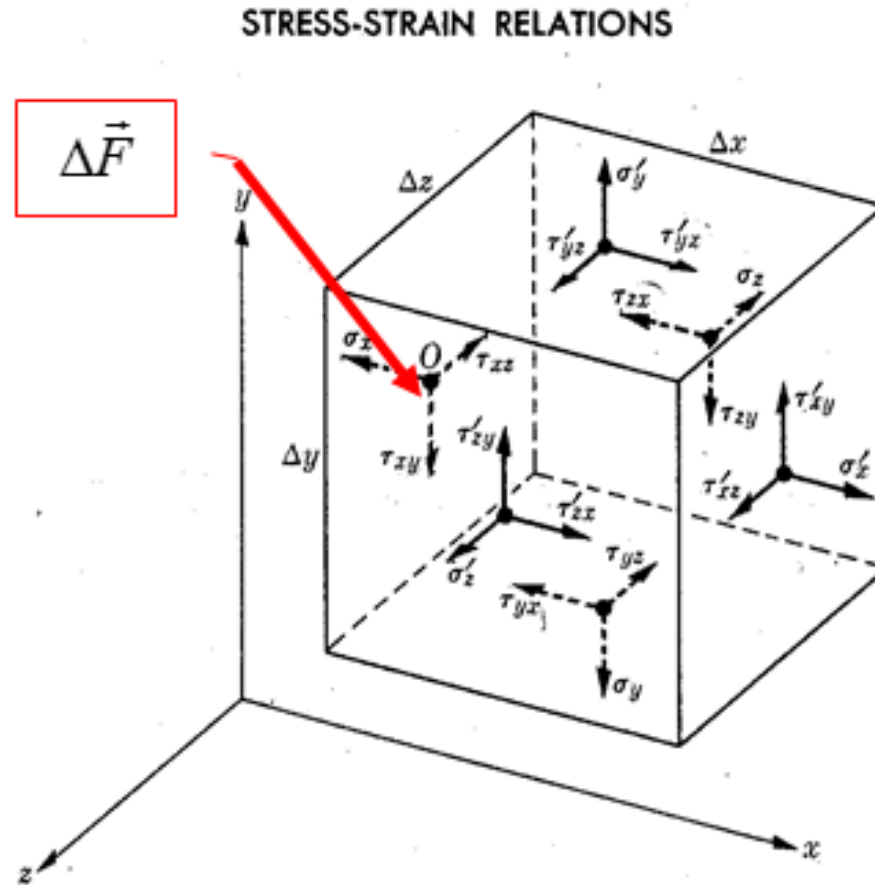
5.1 General Stress–Strain System

Parallelepiped, cube \rightarrow infinitesimal C.V.

5.1.1 Surface Stress

Surface stresses: $\left\{ \begin{array}{l} \text{normal stress} - \sigma_x \\ \text{shear stress} - \tau_{xy}, \tau_{xz} \end{array} \right.$

5.1 General Stress–Strain System



5.1 General Stress–Strain System

$$\sigma_{xx} = \sigma_x = \lim_{\Delta A_x \rightarrow 0} \frac{\Delta F_x}{\Delta A_x}$$

$$(\Delta A_x = \Delta y \Delta z)$$

$$\tau_{yx} = \lim_{\Delta A_y \rightarrow 0} \frac{\Delta F_x}{\Delta A_y}$$

$$(\Delta A_y = \Delta x \Delta z)$$

$$\tau_{zx} = \lim_{\Delta A_z \rightarrow 0} \frac{\Delta F_x}{\Delta A_z}$$

$$(\Delta A_z = \Delta x \Delta y)$$

$$\tau_{xy} = \lim_{\Delta A_x \rightarrow 0} \frac{\Delta F_y}{\Delta A_x}$$

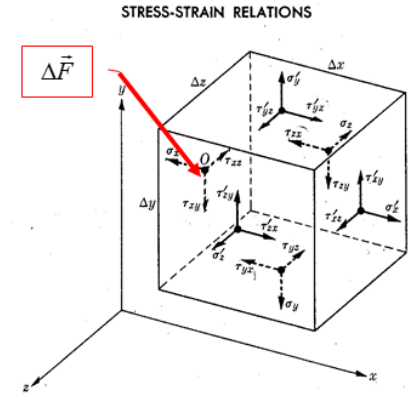
$$\sigma_{yy} = \sigma_y = \lim_{\Delta A_y \rightarrow 0} \frac{\Delta F_y}{\Delta A_y}$$

$$\tau_{zy} = \lim_{\Delta A_z \rightarrow 0} \frac{\Delta F_y}{\Delta A_z}$$

$$\tau_{xz} = \lim_{\Delta A_x \rightarrow 0} \frac{\Delta F_z}{\Delta A_x}$$

$$\tau_{yz} = \lim_{\Delta A_y \rightarrow 0} \frac{\Delta F_z}{\Delta A_y}$$

$$\sigma_{zz} = \sigma_z = \lim_{\Delta A_z \rightarrow 0} \frac{\Delta F_z}{\Delta A_z}$$



where $\Delta F_x, \Delta F_y, \Delta F_z =$ component of force vector $\Delta \vec{F}$

5.1 General Stress–Strain System

- subscripts

σ_x : subscript indicates the direction of stress

τ_{xy} : 1st - direction of the normal to the face on which τ acts

2nd - direction in which τ acts

- general stress system: stress tensor

~ 9 scalar components

$$\begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

5.1 General Stress–Strain System

[Re] Tensor

~ an ordered array of entities which is invariant under coordinate transformation; includes scalars & vectors

~ 3^n

0th order – 1 component, scalar (mass, length, pressure, energy)

1st order – 3 components, vector (velocity, force, acceleration)

2nd order – 9 components (stress, rate of strain, turbulent diffusion)

At three other surfaces,

$$\sigma_x' = \sigma_x + \frac{\partial \sigma_x}{\partial x} \Delta x$$

5.1 General Stress–Strain System

$$\sigma_y' = \sigma_y + \frac{\partial \sigma_y}{\partial y} \Delta y$$

$$\sigma_z' = \sigma_z + \frac{\partial \sigma_z}{\partial z} \Delta z$$

$$\tau_{xy}' = \tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} \Delta x$$

$$\tau_{yx}' = \tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \Delta y$$

$$\tau_{zx}' = \tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \Delta z$$

(5.1)

5.1 General Stress–Strain System

◇ Shear stress is symmetric.

→ Shear stress pairs with subscripts differing in order are equal.

$$\rightarrow \tau_{xy} = \tau_{yx}$$

[Proof]

In static equilibrium, sum of all moments and sum of all forces equal zero for the element.

First, apply Newton's 2nd law

$$\sum F = m \frac{du}{dt}$$

5.1 General Stress–Strain System

Then, consider torque (angular momentum), T

$$\sum T = \frac{d}{dt}(rmv) = \frac{d}{dt}(r^2m\omega) = \frac{d}{dt}(I\omega) = I \frac{d\omega}{dt}$$

where $I =$ moment of inertia $= r^2m$

$r =$ radius of gyration

$\frac{d\omega}{dt} =$ angular acceleration

Thus,

$$\sum T = mr^2 \frac{d\omega}{dt} \tag{A}$$

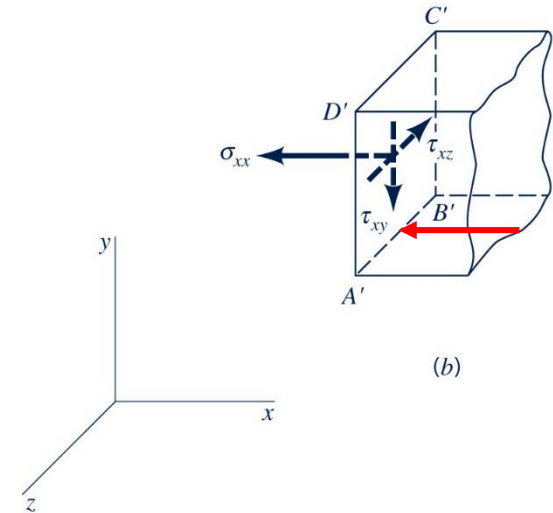
5.1 General Stress–Strain System

Now, take a moment about a centroid axis in the z-direction

$$LHS = \sum T = (\Delta y \Delta z \tau_{xy}) \frac{\Delta x}{2} - (\tau_{yx} \Delta x \Delta z) \frac{\Delta y}{2} = \frac{\Delta x \Delta y \Delta z}{2} (\tau_{xy} - \tau_{yx})$$

$$RHS = \rho dvol r^2 \frac{d\omega}{dt} = \Delta x \Delta y \Delta z \rho r^2 \frac{d\omega}{dt}$$

$$\therefore (\tau_{xy} - \tau_{yx}) \Delta x \Delta y \Delta z = 2 \Delta x \Delta y \Delta z \rho r^2 \frac{d\omega}{dt}$$



5.1 General Stress–Strain System

After canceling terms, this gives

$$\tau_{xy} - \tau_{yx} = 2\rho r^2 \frac{d\omega}{dt}$$

$$\lim_{\Delta x, \Delta y, \Delta z \rightarrow 0} r^2 \rightarrow 0$$

$$\tau_{xy} - \tau_{yx} = 0$$

$$\therefore \tau_{xy} = \tau_{yx}$$

5.1 General Stress–Strain System

[Homework Assignment-Special work]

Due: 1 week from today

1. Fabricate your own “Stress Cube” using paper box.

5.1 General Stress–Strain System

5.1.2 Strain components

Motion and deformation of fluid element

Motion { translation (displacement)
rotation

Deformation { linear deformation
angular deformation

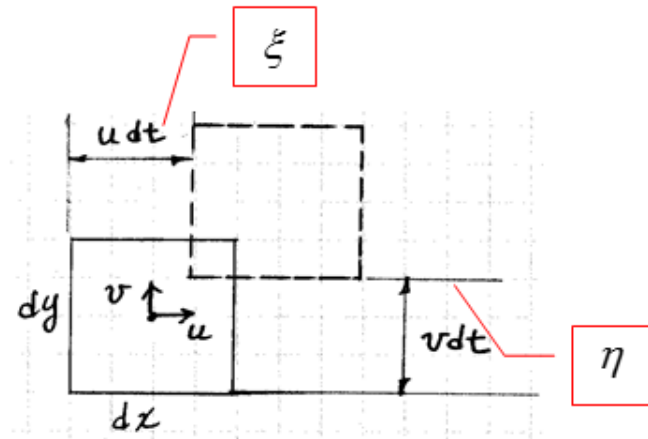
5.1 General Stress–Strain System

(1) Motion

1) Translation: ξ, η

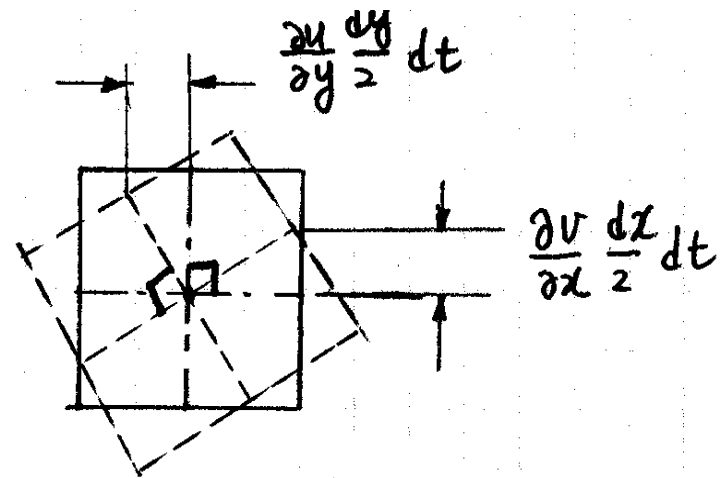
$$\xi = u dt, \quad u = \frac{d\xi}{dt}$$

$$\eta = v dt, \quad v = \frac{d\eta}{dt}$$



2) Rotation ← Shear flow

$$\omega_x = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$



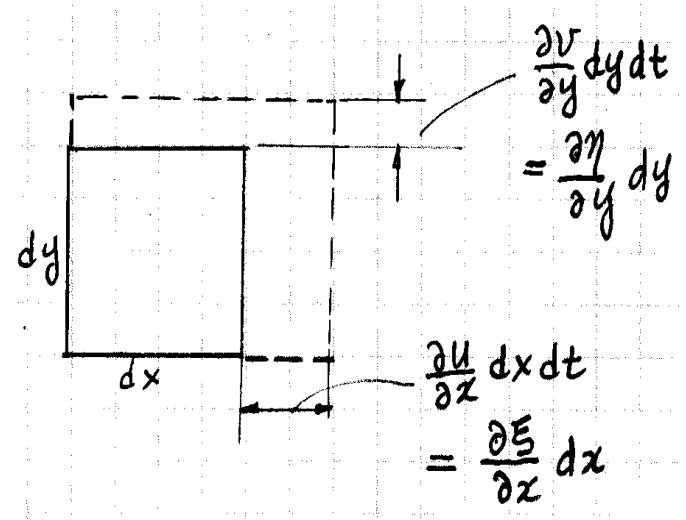
5.1 General Stress–Strain System

(2) Deformation

1) Linear deformation – normal strain

$$\varepsilon_x = \frac{\partial \xi}{\partial x}$$

$$\varepsilon_y = \frac{\partial \eta}{\partial y}$$



i) For compressible fluid, changes in temperature or pressure cause change in volume.

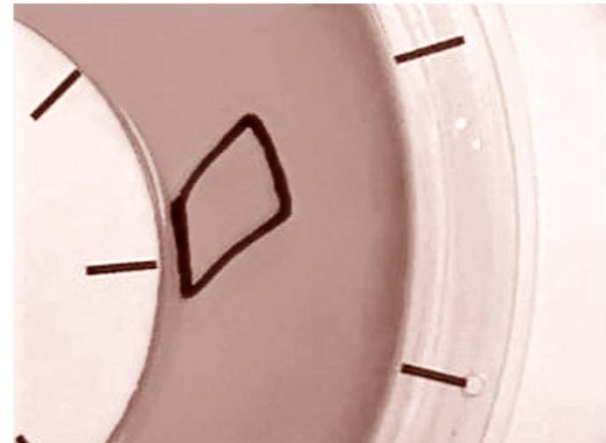
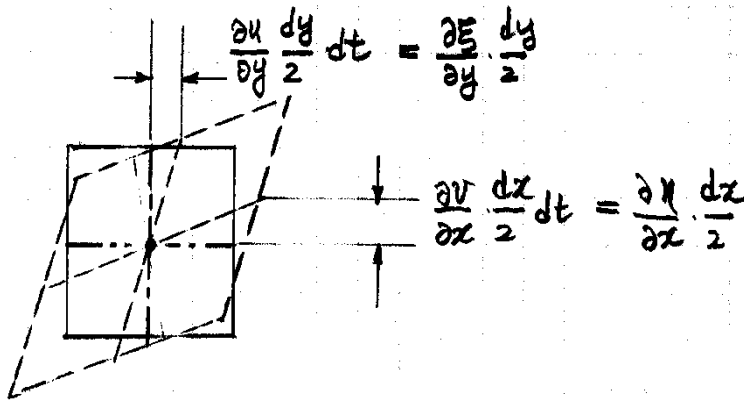
ii) For incompressible fluid, if length in 2-D increases, then length in another 1-D decreases in order to make total volume unchanged.

5.1 General Stress–Strain System

2) Angular deformation– shear strain

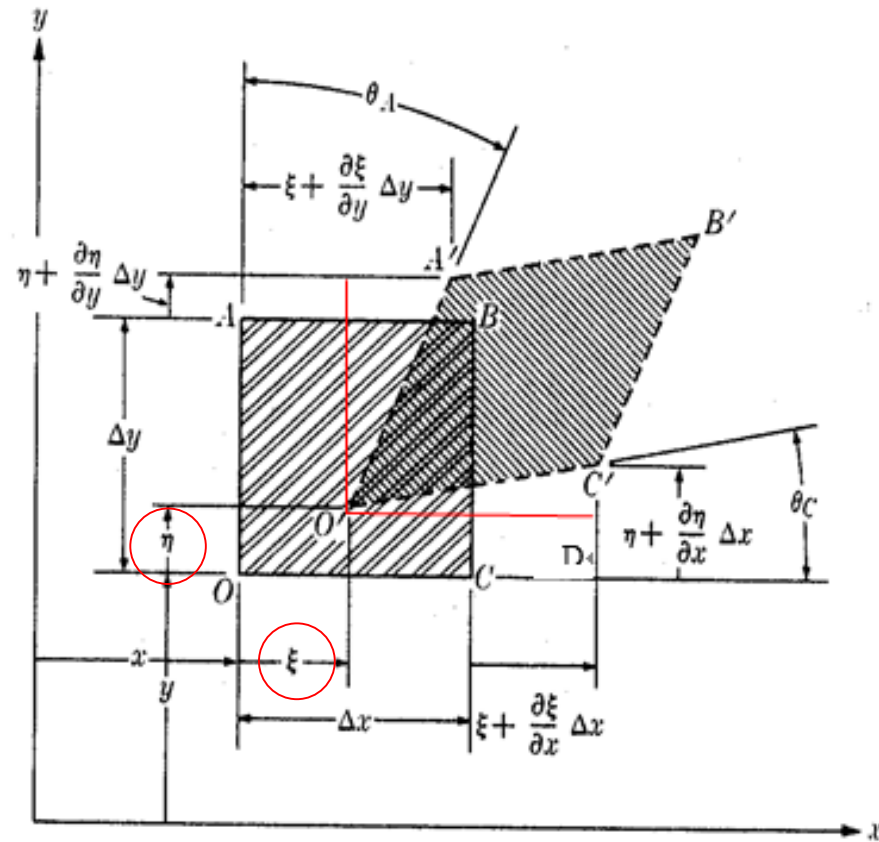
$$\gamma_{xy} = \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}$$

- Strain
 - normal strain: ε ← linear deformation
 - shear strain: γ ← angular deformation



5.1 General Stress–Strain System

Consider a small element $OABC$



5.1 General Stress–Strain System

i) Displacement (translation): ξ, η, ζ

$$O(x, y, z) \rightarrow O'(x + \xi, y + \eta, z + \zeta)$$

$$C(x + \Delta x, y + \Delta y, z + \Delta z) \rightarrow$$

$$C' \left(x + \Delta x + \xi + \frac{\partial \xi}{\partial x} \Delta x, y + \eta + \frac{\partial \eta}{\partial x} \Delta x, z + \zeta + \frac{\partial \zeta}{\partial x} \Delta x \right)$$

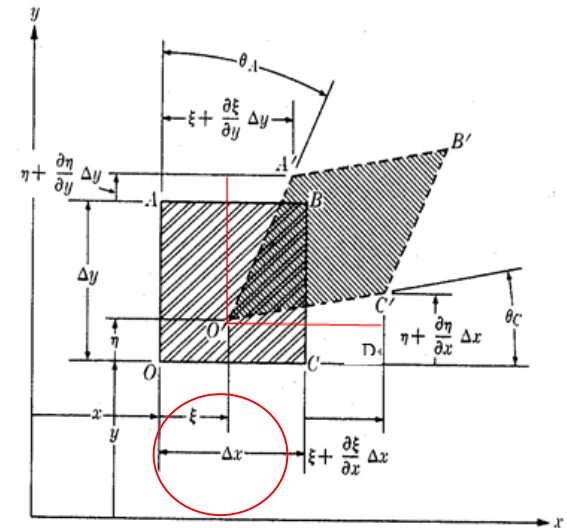
ii) Deformation: due to system of external forces

$$OABC \rightarrow O'A'B'C'$$

(1) Deformation

Normal strain, ε

$$\varepsilon = \frac{\text{change in length}}{\text{original length}}$$



5.1 General Stress–Strain System

$O'C'$

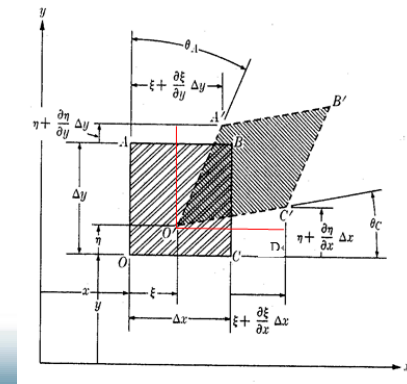
OC

$$\varepsilon_x = \lim_{\Delta x \rightarrow 0} \frac{O'C' - OC}{OC} = \lim_{\Delta x \rightarrow 0} \frac{\left\{ \left(x + \Delta x + \xi + \frac{\partial \xi}{\partial x} \Delta x \right) - (x + \xi) \right\} - \Delta x}{\Delta x} = \frac{\partial \xi}{\partial x}$$

$$\varepsilon_y = \lim_{\Delta y \rightarrow 0} \frac{O'A' - OA}{OA} = \lim_{\Delta y \rightarrow 0} \frac{\left\{ \left(y + \Delta y + \eta + \frac{\partial \eta}{\partial y} \Delta y \right) - (y + \eta) \right\} - \Delta y}{\Delta y} = \frac{\partial \eta}{\partial y}$$

$$\varepsilon_z = \frac{\partial \zeta}{\partial z}$$

~ ε is positive when element elongates
under deformation



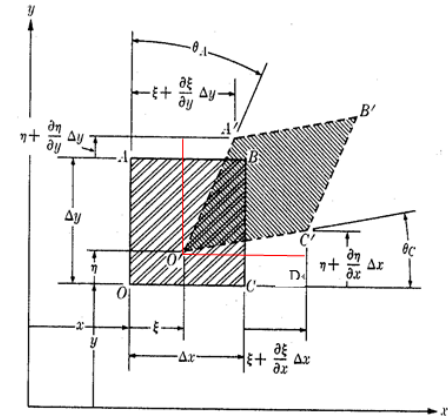
5.1 General Stress–Strain System

2) Shear strain, γ

~ change in angle between two originally perpendicular elements

For xy -plane

$$\gamma_{xy} = \lim_{\Delta x, \Delta y \rightarrow 0} (\theta_c + \theta_A) \cong \lim_{\Delta x, \Delta y \rightarrow 0} (\tan \theta_c + \tan \theta_A)$$



$$= \lim_{\Delta x, \Delta y \rightarrow 0} \left\{ \frac{\frac{\partial \eta}{\partial x} \Delta x}{\Delta x + \frac{\partial \xi}{\partial x} \Delta x} + \frac{\frac{\partial \xi}{\partial y} \Delta y}{\Delta y + \frac{\partial \eta}{\partial y} \Delta y} \right\} = \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}$$

C'D A'E
O'D O'E

$$\left(\because \Delta x \frac{\partial \xi}{\partial x} < \Delta x \right)$$

5.1 General Stress–Strain System

$$\gamma_{xy} = \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y} \quad (5.2)$$

$$\gamma_{yz} = \frac{\partial \zeta}{\partial y} + \frac{\partial \eta}{\partial z} \quad (5.3)$$

$$\gamma_{zx} = \frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial x} \quad (5.4)$$

(2) displacement vector $\vec{\delta}$

$$\vec{\delta} = \xi \vec{i} + \eta \vec{j} + \zeta \vec{k}$$

5.1 General Stress–Strain System

(3) Volume dilation

$$e = \frac{\text{change of volume of deformed element}}{\text{original volume}}$$

$$e = \frac{d(\Delta V)}{\Delta V} = \frac{\left(\Delta x + \frac{\partial \xi}{\partial x} \Delta x\right) \left(\Delta y + \frac{\partial \eta}{\partial y} \Delta y\right) \left(\Delta z + \frac{\partial \zeta}{\partial z} \Delta z\right) - \Delta x \Delta y \Delta z}{\Delta x \Delta y \Delta z}$$

$$\cong \frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = \varepsilon_x + \varepsilon_y + \varepsilon_z \quad (5.5)$$

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z \quad (5.6)$$

$$e = \frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = \nabla \cdot \vec{\delta} \quad \text{--- divergence} \quad (5.7)$$

5.2 Relations between Stress and Strain for Elastic Solids

5.2.1 Normal Stresses

Hooke's law: Stress is linear with strain.

$$\sigma_x \propto \varepsilon_x^\circ$$

$$\sigma_x = E \varepsilon_x^\circ$$

$$\varepsilon_x^\circ = \frac{1}{E} \sigma_x \quad (5.8)$$

in which E = Young's modulus of elasticity

ε_x° = elongation in the x - *dir* due to normal stress, σ_x

$$y - \text{dir.} : \varepsilon_y^\circ = \frac{\sigma_y}{E}$$

$$z - \text{dir.} : \varepsilon_z^\circ = \frac{\sigma_z}{E}$$

5.2 Relations between Stress and Strain for Elastic Solids

Now, we have to consider other elongations because of lateral contraction of matter under tension.

ε_x' = elongation in the x -dir. due to σ_y

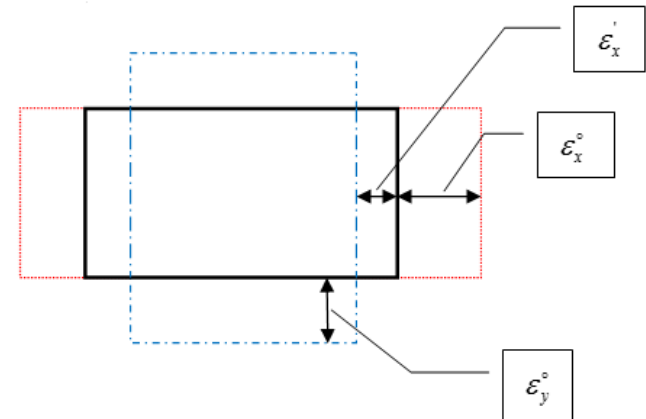
ε_x'' = elongation in the x -dir. due to σ_z

Now, define

$$\varepsilon_x' = -n\varepsilon_y^\circ = -n\frac{\sigma_y}{E} \quad (5.9)$$

$$\varepsilon_x'' = -n\varepsilon_z^\circ = -n\frac{\sigma_z}{E} \quad (5.10)$$

where $n = \text{Poisson's ratio}$

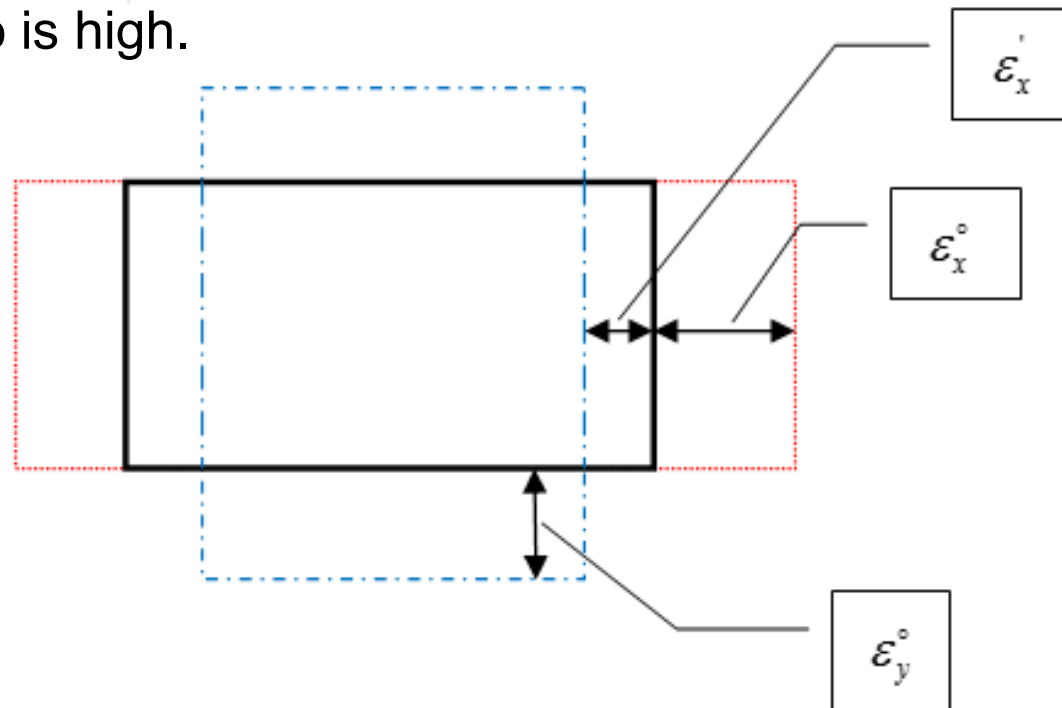


5.2 Relations between Stress and Strain for Elastic Solids

Poisson's ratio: $\nu = 0 \sim 0.5$

water ~ 0.5 ; metal ~ 0.3

cork: Poisson's ratio is high.



5.2 Relations between Stress and Strain for Elastic Solids

Thus, total strain ε_x is

$$\varepsilon_x = \varepsilon_x^{\circ} + \varepsilon_x' + \varepsilon_x'' = \frac{\sigma_x}{E} - \frac{n}{E}(\sigma_y + \sigma_z) = \frac{1}{E}[\sigma_x - n(\sigma_y + \sigma_z)]$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - n(\sigma_z + \sigma_x)] \quad (5.11)$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - n(\sigma_x + \sigma_y)] \quad (5.12)$$

5.2 Relations between Stress and Strain for Elastic Solids

5.2.2 Shear Stress

~ Hooke's law $\tau_{xy} = G \gamma_{xy}$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} = \frac{\partial \zeta}{\partial y} + \frac{\partial \eta}{\partial z}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G} = \frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial x}$$

(5.13)

5.2 Relations between Stress and Strain for Elastic Solids

where $G =$ shear modulus of elasticity

$$G = \frac{E}{2(1+n)} \quad (5.14)$$

■ Volume dialation

$$\begin{aligned} e = \varepsilon_x + \varepsilon_y + \varepsilon_z &= \frac{1}{E} \left[\sigma_x - n(\sigma_y + \sigma_z) \right] \\ &+ \frac{1}{E} \left[\sigma_y - n(\sigma_z + \sigma_x) \right] \\ &+ \frac{1}{E} \left[\sigma_z - n(\sigma_x + \sigma_y) \right] \\ &= \frac{1}{E} \left[(1-2n)(\sigma_x + \sigma_y + \sigma_z) \right] \end{aligned} \quad (5.15)$$

5.2 Relations between Stress and Strain for Elastic Solids

- $\bar{\sigma}$ = arithmetic mean of 3 normal stresses

$$\bar{\sigma} = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) \quad (5.16)$$

Combine Eqs. (5.12), (5.14) and (5.15)

$$\sigma_x = 2G \left[\varepsilon_x + \frac{ne}{1-2n} \right] \quad (5.17)$$

5.2 Relations between Stress and Strain for Elastic Solids

Therefore

$$\begin{aligned}\sigma_x - \bar{\sigma} &= 2G \left(\varepsilon_x - \frac{e}{3} \right) \\ \sigma_y - \bar{\sigma} &= 2G \left(\varepsilon_y - \frac{e}{3} \right) \\ \sigma_z - \bar{\sigma} &= 2G \left(\varepsilon_z - \frac{e}{3} \right)\end{aligned}\tag{5.18}$$

$$\tau_{xy} = \tau_{yx} = G \left(\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y} \right)$$

5.2 Relations between Stress and Strain for Elastic Solids

$$\begin{aligned}\tau_{zy} = \tau_{yz} &= G \left(\frac{\partial \zeta}{\partial y} + \frac{\partial \eta}{\partial z} \right) \\ \tau_{xz} = \tau_{zx} &= G \left(\frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial x} \right)\end{aligned}\quad (5.19)$$

[Proof] Derivation of Eqs. (5.17) & (5.18)

$$(5.15) \rightarrow e = \frac{1}{E}(1-2n)(\sigma_x + \sigma_y + \sigma_z) \quad (A)$$

$$(5.12) \rightarrow \varepsilon_x = \frac{1}{E} \left[\sigma_x - n(\sigma_y + \sigma_z) \right] \quad (B)$$

$$(5.14) \rightarrow G = \frac{E}{2(1+n)} \rightarrow E = 2G(1+n) \quad (C)$$

5.2 Relations between Stress and Strain for Elastic Solids

i) Combine (A) and (B)

$$+ \left\{ \begin{aligned} \frac{n}{(1+2n)} \times e &= \frac{n}{(1-2n)} \frac{(1-2n)}{E} (\sigma_x + \sigma_y + \sigma_z) = \frac{n}{E} (\sigma_x + \sigma_y + \sigma_z) \\ \varepsilon_x &= \frac{1}{E} [\sigma_x - n(\sigma_y + \sigma_z)] \end{aligned} \right.$$

$$\frac{n}{(1-2n)} e + \varepsilon_x = \frac{1+n}{E} \sigma_x$$

$$\therefore \sigma_x = \frac{E}{1+n} \left[\varepsilon_x + \frac{n}{(1-2n)} e \right] \quad (D)$$

5.2 Relations between Stress and Strain for Elastic Solids

Substitute (C) into (D)

$$\therefore \sigma_x = 2G \left[\varepsilon_x + \frac{n}{(1-2n)} e \right] \quad \rightarrow \quad \text{Eq. (5.17)}$$

ii) Subtract (5.16) from (5.17)

$$\sigma_x - \bar{\sigma} = 2G \left[\varepsilon_x + \frac{n}{(1-2n)} e \right] - \frac{1}{3} (\sigma_x + \sigma_y + \sigma_z) \quad (\text{E})$$

Substitute (A) into (E); (A): $\sigma_x + \sigma_y + \sigma_z = \frac{E}{(1-2n)} e$

5.2 Relations between Stress and Strain for Elastic Solids

$$\begin{aligned}
 \therefore \text{RHS of } (E) &= 2G \left[\varepsilon_x + \frac{n}{(1-2n)} e \right] - \frac{1}{3} \frac{E}{(1-2n)} e \\
 &= 2G \varepsilon_x + \left[\frac{2Gn}{(1-2n)} - \frac{1}{3} \frac{2G(1+n)}{(1-2n)} \right] e = 2G \left\{ \varepsilon_x \left[\frac{n}{(1-2n)} - \frac{\frac{1+n}{3}}{(1+2n)} \right] e \right\} \\
 &= 2G \left\{ \varepsilon_x + \frac{-\frac{1}{3}(1-2n)}{(1-2n)} e \right\} = 2G \left(\varepsilon_x - \frac{1}{3} e \right) \quad \rightarrow \text{Eq. (5.18)}
 \end{aligned}$$

5.3 Relations between Stress and Rate of Strain for Newtonian Fluids 36/61

Experimental evidence suggests that, in fluid, stress is linear with time rate of strain.

$$\rightarrow stress \propto \frac{\partial}{\partial t}(strain)$$

→ Newtonian fluid (**Newton's law of viscosity**)

[Cf] For solid,

$$stress \propto strain$$

5.3 Relations between Stress and Rate of Strain for Newtonian Fluids 37/61

5.3.1 Normal stress

For solid, Eq. (5.18) can be used as

$$\text{Hookeian elastic solid: } \sigma_x - \bar{\sigma} = 2 \left(\frac{F}{L^2} \right) \left(\varepsilon_x - \frac{e}{3} \right)$$

G

By analogy,

$$\text{Newtonian fluid: } \sigma_x - \bar{\sigma} = 2 \left(\frac{Ft}{L^2} \right) \frac{\partial}{\partial t} \left(\varepsilon_x - \frac{e}{3} \right) \quad (5.20)$$

Time rate of strain

Now set $\mu \equiv \frac{Ft}{L^2} = \text{dynamic viscosity}$

5.3 Relations between Stress and Rate of Strain for Newtonian Fluids 38/61

Then,

$$\sigma_x - \bar{\sigma} = 2\mu \frac{\partial \varepsilon_x}{\partial t} - \frac{2}{3}\mu \frac{\partial e}{\partial t} \quad (5.21)$$

By the way,

$$\varepsilon_x = \frac{\partial \xi}{\partial x}, \quad e = \nabla \cdot \vec{\delta}$$

$u = \frac{\partial \xi}{\partial t}, \quad v = \frac{\partial \eta}{\partial x}, \quad w = \frac{\partial \zeta}{\partial t} \quad (\xi, \eta, \zeta = \text{displacement})$

Therefore,

$$\frac{\partial \varepsilon_x}{\partial t} = \frac{\partial}{\partial t} \left(\frac{\partial \xi}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial \xi}{\partial t} \right) = \frac{\partial u}{\partial x} \quad (5.22)$$

5.3 Relations between Stress and Rate of Strain for Newtonian Fluids 39/61

$$\frac{\partial e}{\partial t} = \nabla \cdot \frac{\partial \vec{\delta}}{\partial t} = \nabla \cdot \vec{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \quad (5.23)$$

$$\begin{aligned} \vec{\delta} &= \xi \vec{i} + \eta \vec{j} + \zeta \vec{k} \\ \vec{q} &= \frac{\partial \vec{\delta}}{\partial t} = u \vec{i} + v \vec{j} + w \vec{k} \\ \nabla \cdot \vec{q} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \end{aligned}$$

Eq. (5.21) becomes

$$\sigma_x = \bar{\sigma} + 2\mu \frac{\partial u}{\partial x} - \frac{2}{3}\mu (\nabla \cdot \vec{q})$$

5.3 Relations between Stress and Rate of Strain for Newtonian Fluids 40/61

For compressible fluid,

$$\sigma_x = \bar{\sigma} + 2\mu \frac{\partial u}{\partial x} - \frac{2}{3}\mu(\nabla \cdot \vec{q})$$

$$\sigma_y = \bar{\sigma} + 2\mu \frac{\partial v}{\partial y} - \frac{2}{3}\mu(\nabla \cdot \vec{q})$$

$$\sigma_z = \bar{\sigma} + 2\mu \frac{\partial w}{\partial z} - \frac{2}{3}\mu(\nabla \cdot \vec{q})$$

(5.24)

For incompressible fluid,

$$\frac{de}{dt} = \nabla \cdot \vec{q} = 0 \quad \leftarrow \text{time rate of volume expansion}=0$$

5.3 Relations between Stress and Rate of Strain for Newtonian Fluids 41/61

$$\rightarrow \nabla \cdot \vec{q} = 0 \quad \rightarrow \text{Continuity Eq.}$$

Therefore, Eq. (5.24) becomes

$$\sigma_x = \bar{\sigma} + 2\mu \frac{\partial u}{\partial x}$$

$$\sigma_y = \bar{\sigma} + 2\mu \frac{\partial v}{\partial y}$$

$$\sigma_z = \bar{\sigma} + 2\mu \frac{\partial w}{\partial z}$$

5.3 Relations between Stress and Rate of Strain for Newtonian Fluids 42/61

5.3.2. Shear stress

μ

By following the same analogy

$$\tau_{xy} = G \left(\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y} \right) = \left(\frac{Ft}{L^2} \right) \frac{\partial}{\partial t} \left(\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y} \right)$$

$$= \mu \frac{\partial}{\partial x} \left(\frac{\partial \eta}{\partial t} \right) + \frac{\partial}{\partial y} \left(\frac{\partial \xi}{\partial t} \right) = \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)$$

$$\frac{\partial \eta}{\partial t} = v$$

$$\frac{\partial \xi}{\partial t} = u$$

5.3 Relations between Stress and Rate of Strain for Newtonian Fluids 43/61

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)$$

$$\tau_{zy} = \tau_{yz} = \mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)$$

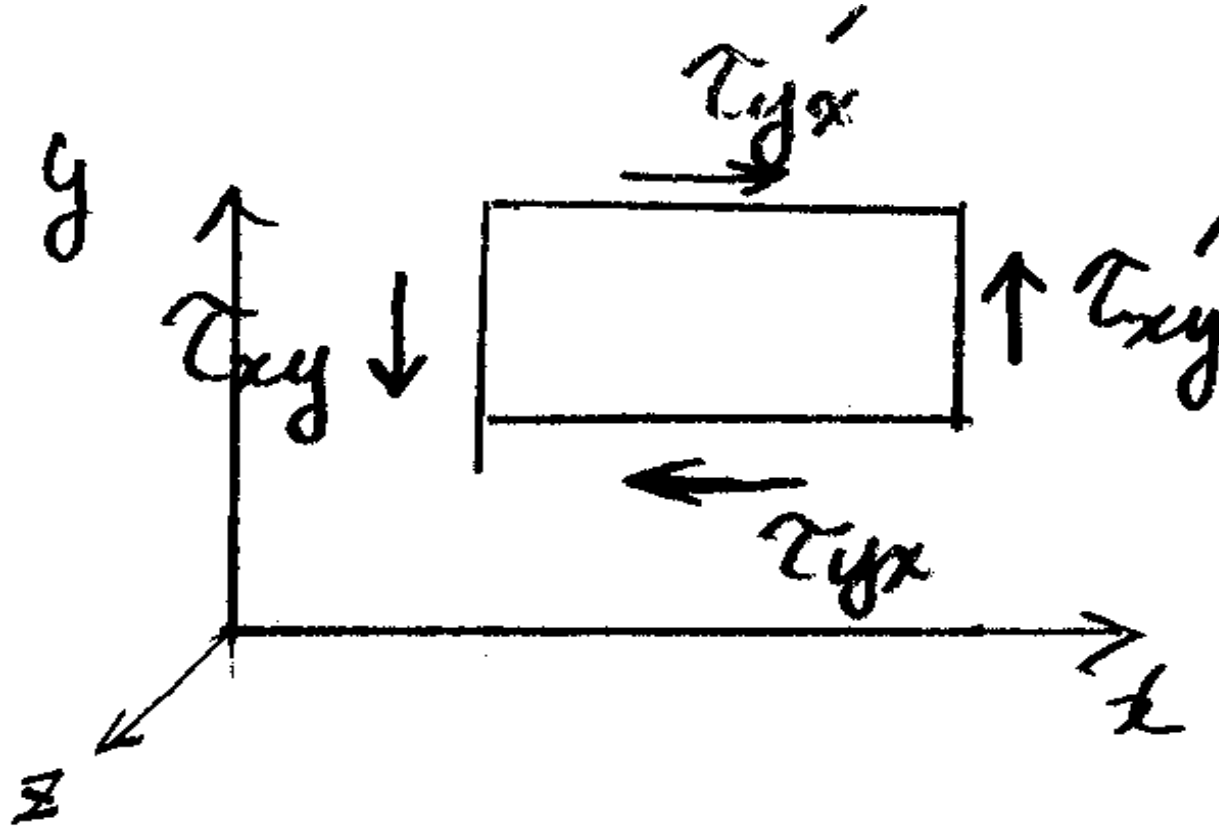
$$\tau_{xz} = \tau_{zx} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)$$

(5.25)

[Appendix 1]

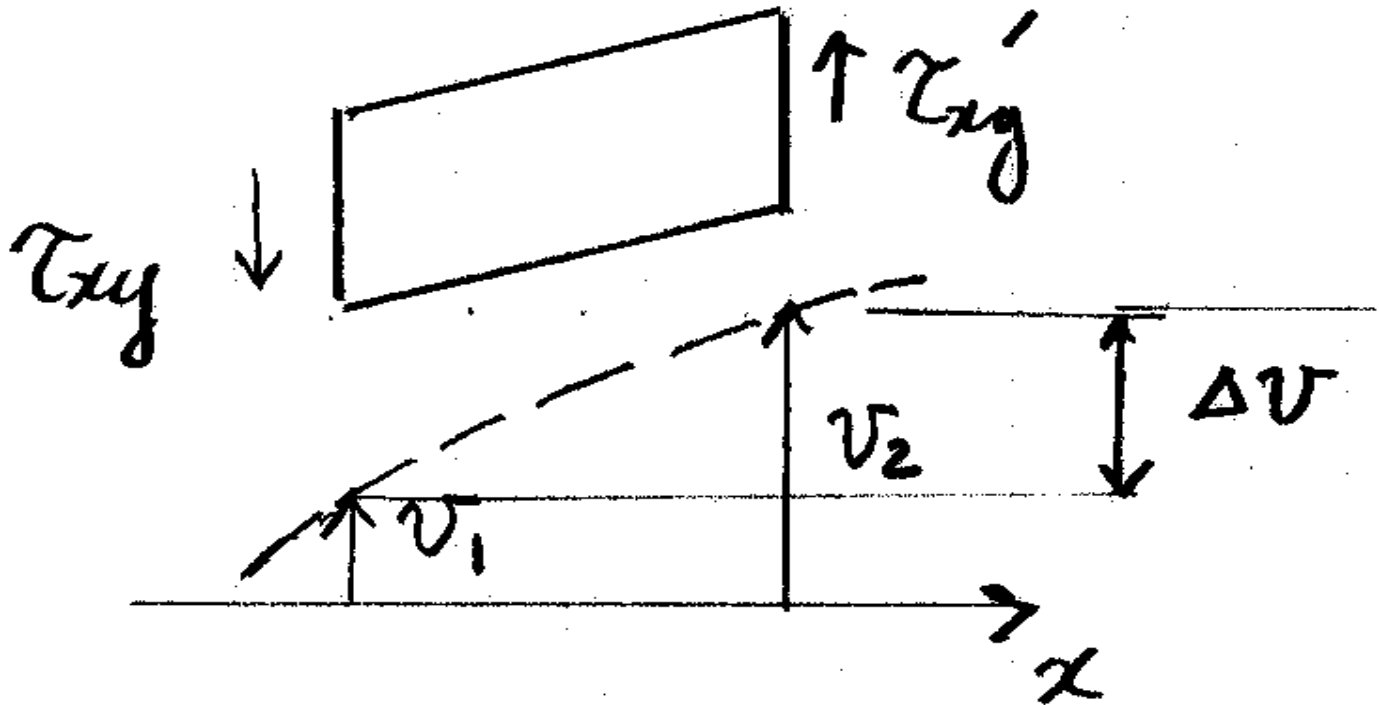
$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)$$

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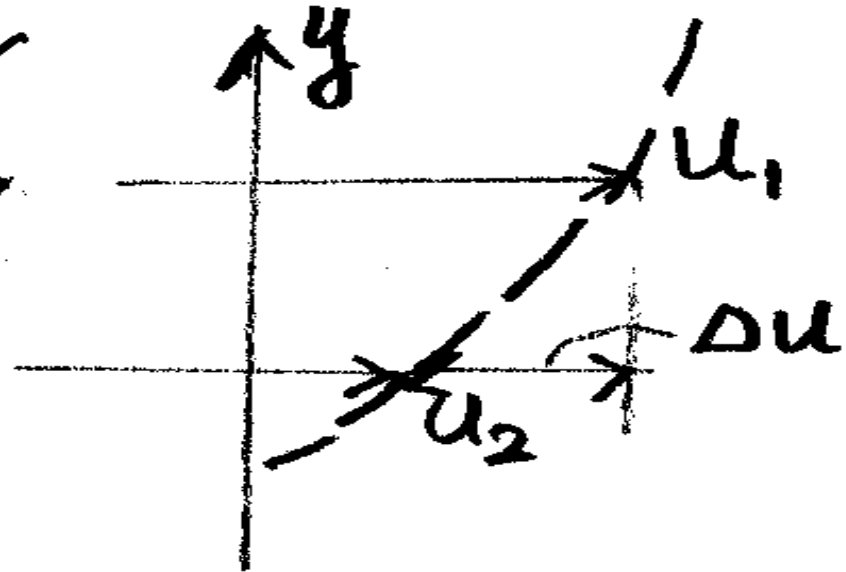
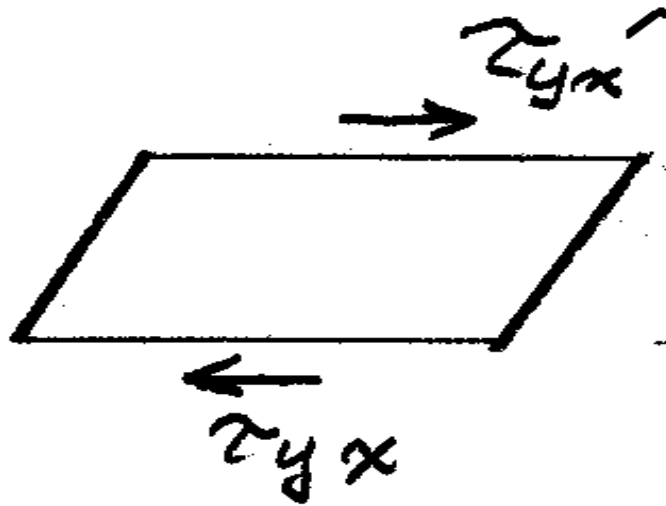
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i) τ_{xy} , τ_{xy}'



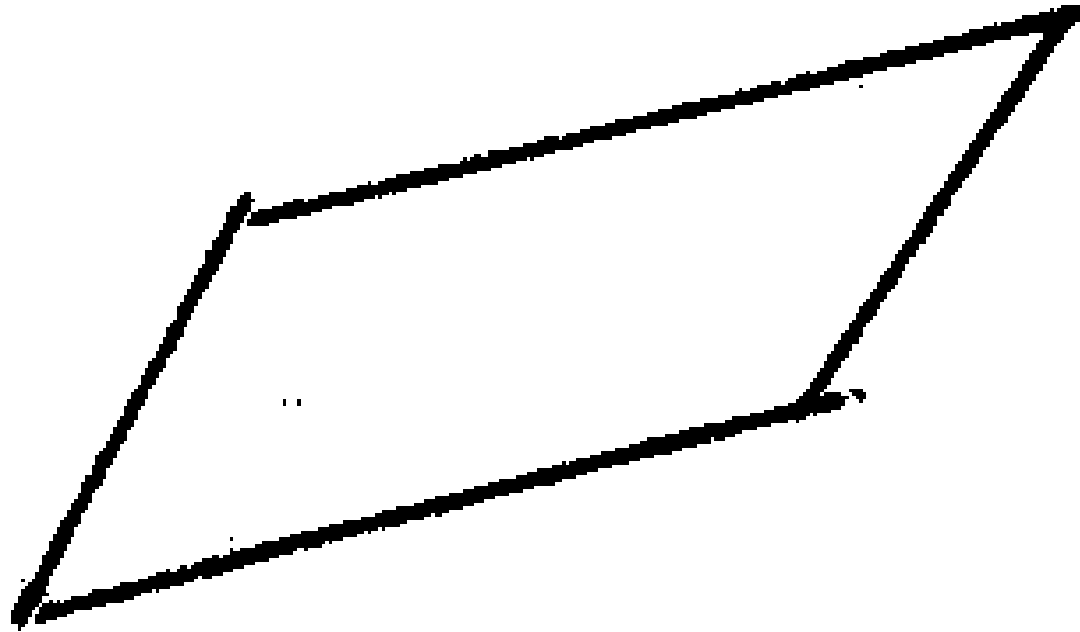
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ii) τ_{yx} , τ'_{yx}



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iii) composition



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- Relation between thermodynamic pressure p and mean normal stress $\bar{\sigma}$

1) Assume viscous effects are completely represented by the viscosity μ for incompressible fluid

$$\bar{\sigma} = -p = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) \quad (5.26)$$

~ minus sign accounts for pressure (compression)

2) For compressible fluid

$$\bar{\sigma} = -p + \mu'(\nabla \cdot \vec{q})$$

in which $\mu' =$ **2nd coefficient of viscosity** associated solely with dilation

= bulk viscosity

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Since, dilation effect is small for most cases

$$\mu'(\nabla \cdot \vec{q}) \rightarrow 0 \quad \therefore \bar{\sigma} = -p$$

For zero-dilation viscosity effects ($\mu' = 0$), (5.24) becomes

$$\sigma_x = -p + 2\mu \frac{\partial u}{\partial x} - \left(\frac{2}{3}\right)\mu(\nabla \cdot \vec{q}) \quad (5.27)$$

$$\sigma_y = -p + 2\mu \frac{\partial v}{\partial y} - \left(\frac{2}{3}\right)\mu(\nabla \cdot \vec{q}) \quad (5.28)$$

$$\sigma_z = -p + 2\mu \frac{\partial w}{\partial z} - \left(\frac{2}{3}\right)\mu(\nabla \cdot \vec{q}) \quad (5.29)$$

Normal stress

pressure

Viscous effects

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- Shear stresses in a real fluid

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)$$

$$\tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)$$

$$\tau_{xz} = \tau_{zx} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)$$

(5.30)

For zero viscous effects ($\mu = 0$) \rightarrow inviscid fluids in motion and for all fluids at rest

$$\sigma_x = \sigma_y = \sigma_z = \bar{\sigma} = -p$$

$$\tau_{xy} = \tau_{yz} = \tau_{zx} = 0$$

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[Appendix 2] Bulk viscosity and thermodynamic pressure

→ Boundary-Layer Theory (Schlichting, 1979) pp. 61-63

$$\bar{\sigma} = -p + \mu'(\nabla \cdot \vec{q})$$

If fluid is compressed, expanded or made to oscillate at a finite rate, work done in a thermodynamically reversible process per unit volume is

$$W = p \nabla \cdot \vec{q} = p \frac{de}{dt} \sim \text{dissipation of energy}$$

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where μ' = bulk viscosity of fluid that represents that property which is responsible for energy dissipation in a fluid of uniform temperature during a change in volume at a finite rate

= second property of a compressible, isotropic, Newtonian fluid

[Cf] μ = shear viscosity = first property

$$\mu' = 0, \quad p = -\bar{\sigma}$$

$$\mu' \neq 0, \quad p \neq -\bar{\sigma}$$

Direct measurement of bulk viscosity is very difficult.

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[Appendix 3] Normal stress

Normal stress = pressure + deviation from it

$$\sigma_x = -p + \sigma_x'$$

$$\sigma_y = -p + \sigma_y'$$

$$\sigma_z = -p + \sigma_z'$$

Thus, stress matrix becomes

$$\begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} + \begin{pmatrix} \sigma_x' & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y' & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z' \end{pmatrix}$$

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Normal stresses are proportional to the volume change (compressibility) and corresponding components of linear deformation, a, b, c .

Thus,

$$\sigma_x = -p + \lambda(a + b + c) + 2\mu a$$

$$\sigma_y = -p + \lambda(a + b + c) + 2\mu b$$

$$\sigma_z = -p + \lambda(a + b + c) + 2\mu c$$

where λ = compressibility coefficient

5.3 Relations between Stress and Rate of Strain for 55/61 Newtonian Fluids

Homework Assignment # 5

Due: 1 week from today

1. (5-3) Consider a fluid element under a general state of stress as illustrated in Fig. 5-1. Given that the element is in a gravity field, show that the equilibrium requirement between surface, body and inertial forces leads to the equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x = \rho a_x$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \rho g_y = \rho a_y$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \rho g_x = \rho a$$