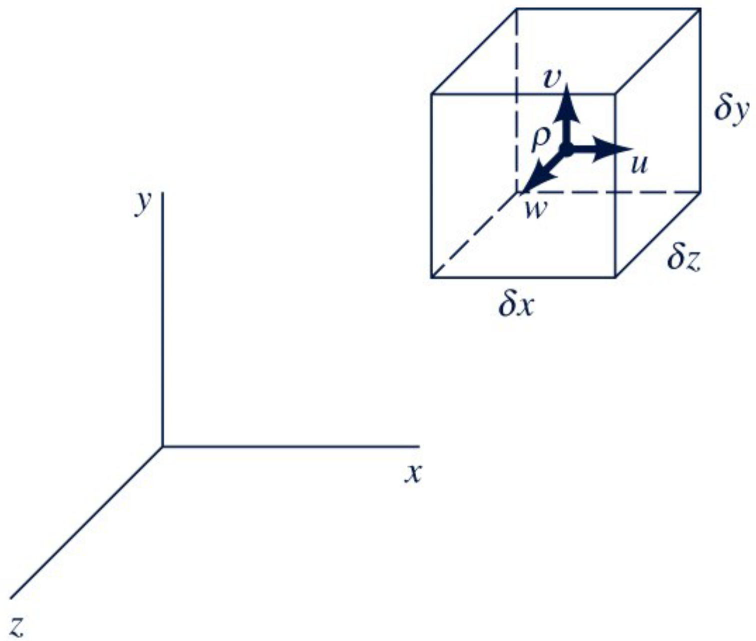
Chapter 6 Equations of Continuity and Motion

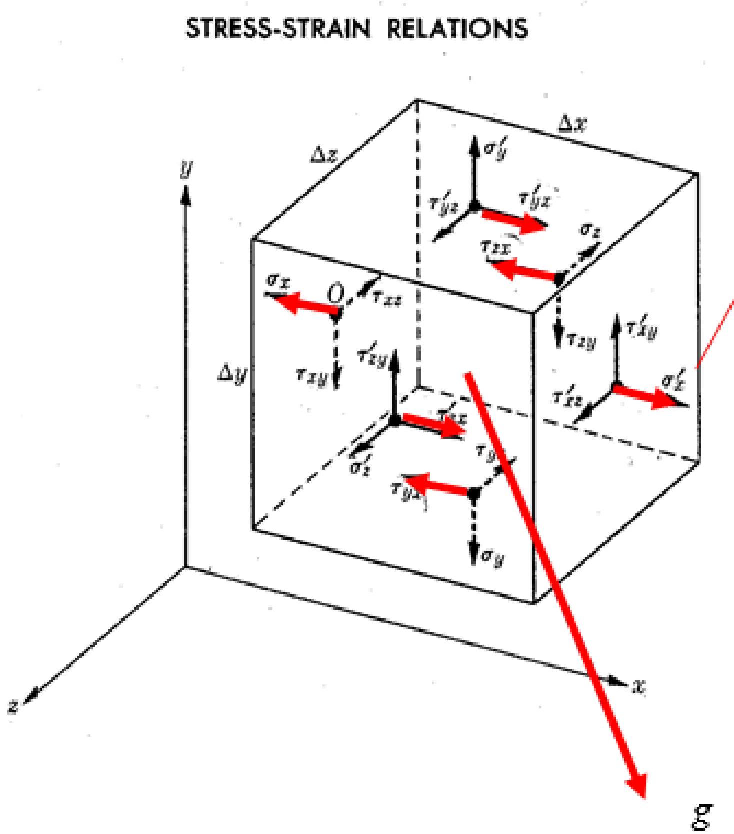
Session 6-2 Equation of motion



Chapter 6 Equations of Continuity and Motion

Contents

6.4 Equations of Motion



$$\sigma_x + \frac{\partial \sigma_x}{\partial x}\Delta x$$

6.4 Equations of Motion

- Apply Newton's 2nd law of motion

$$\sum \vec{F} = m\vec{a} \qquad \text{(A)}$$

$$\Delta F_x = \Delta m a_x$$

- External forces = surface force + body force
  - Surface force:

~ normal force + tangential force

  - Body forces:

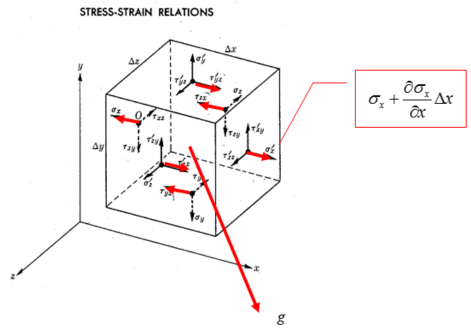
~ due to gravitational or electromagnetic fields, <u>no contact</u>

~ act at the centroid of the element → centroidal force

6.4 Equations of Motion

Consider only gravitational force

$$\vec{g} = \vec{i} g_x + \vec{j} g_y + \vec{k} g_z$$



LHS of (A):

$$\Delta F_x = \left(\rho \Delta x \Delta y \Delta z\right) g_x$$ body force (B)

$$-\sigma_x \Delta y \Delta z + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x}\Delta x\right)\Delta y \Delta z$$ normal force

$$-\tau_{yx} \Delta x \Delta z + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\Delta y\right)\Delta x \Delta z$$ tangential force

$$-\tau_{zx} \Delta x \Delta y + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\Delta z\right)\Delta x \Delta y$$ tangential force

6.4 Equations of Motion

Divide (B) by volume of element

$$\frac{\Delta F_x}{\Delta x \Delta y \Delta z} = \rho g_x + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \qquad \text{(C)}$$

RHS of (A):

$$\frac{\Delta m a_x}{\Delta x \Delta y \Delta z} = \rho a_x \qquad \text{(D)}$$

6.4 Equations of Motion

Combine (C) and (D)

$$\boxed{\begin{aligned} \rho g_x + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} &= \rho a_x \\ \rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} &= \rho a_y \\ \rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} &= \rho a_z \end{aligned}} \qquad (6.21)$$

6.4 Equations of Motion

6.4.1 Navier-Stokes equations

– Eq (6.21) ~ general equation of motion

– For Newtonian fluids (with single viscosity coeff.), use stress-strain relation given in (5.29) and (5.30)

→ Navier-Stokes equations

Eq. (5.29):

$$\sigma_x = \underbrace{-p}_{\text{pressure}} + \underbrace{2\mu\frac{\partial u}{\partial x} - \left(\frac{2}{3}\right)\mu(\nabla\cdot\vec{q})}_{\text{normal stress due to fluid deformation and viscosity}}$$

6.4 Equations of Motion

$$\sigma_y = -p + 2\mu \frac{\partial v}{\partial y} - \left(\frac{2}{3}\right)\mu\left(\nabla \cdot \vec{q}\right) \quad (6.22)$$

$$\sigma_z = -p + 2\mu \frac{\partial w}{\partial z} - \left(\frac{2}{3}\right)\mu\left(\nabla \cdot \vec{q}\right) \quad (6.23)$$

Eq. (5.30):

$$\tau_{yx} = \tau_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

6.4 Equations of Motion

Substitute Eqs. (5.29) & (5.30) into (6.21)

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu(\nabla \cdot \vec{q})\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] = \rho a_x$$

Assume constant viscosity (neglect effect of pressure and temperature on viscosity variation)

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\frac{\partial}{\partial x}\left[2\frac{\partial u}{\partial x} - \frac{2}{3}(\nabla \cdot \vec{q})\right] + \mu\frac{\partial}{\partial y}\left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \mu\frac{\partial}{\partial z}\left[\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] = \rho a_x$$

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}}$$

6.4 Equations of Motion

Expand and simplify

normal stress

shear stress

$$L.H.S = \rho g_x - \frac{\partial p}{\partial x} + 2\mu\frac{\partial^2 u}{\partial x^2} - \frac{2}{3}\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z}\right) + \mu\left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z}\right)$$

$$= \rho g_x - \frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + \frac{1}{3}\mu\underline{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z}\right)}$$

$$\boxed{\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)}$$

$$= \rho g_x - \frac{\partial p}{\partial x} + \boxed{\mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + \frac{1}{3}\mu\frac{\partial}{\partial x}(\nabla \cdot \vec{q})}$$

normal stress + shear stress

6.4 Equations of Motion

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + \frac{1}{3}\mu\frac{\partial}{\partial x}\left(\nabla\cdot\vec{q}\right) = \rho a_x$$

$$\rho g_y - \frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] + \frac{1}{3}\mu\frac{\partial}{\partial y}\left(\nabla\cdot\vec{q}\right) = \rho a_y$$

$$\rho g_z - \frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right] + \frac{1}{3}\mu\frac{\partial}{\partial z}\left(\nabla\cdot\vec{q}\right) = \rho a_z \qquad (6.24)$$

→ Navier-Stokes equation for compressible fluids with constant viscosity

6.4 Equations of Motion

◆ Vector form

$$\rho\vec{g} - \nabla p + \mu\nabla^2\vec{q} + \frac{\mu}{3}\nabla\left(\nabla\cdot\vec{q}\right) = \rho\frac{\partial\vec{q}}{\partial t} + \rho\left(\vec{q}\cdot\nabla\right)\vec{q}$$

where $\vec{a} = \frac{d\vec{q}}{dt} = \frac{\partial\vec{q}}{\partial t} + \left(\vec{q}\cdot\nabla\right)\vec{q}$ --- Eq. (2.5)

1) For inviscid (ideal) fluid flow, $(\mu = 0) \rightarrow$ viscous forces are neglected.

$$\rho\vec{g} - \nabla p = \rho\frac{\partial\vec{q}}{\partial t} + \rho\left(\vec{q}\cdot\nabla\right)\vec{q}$$

→ Euler equations for ideal fluid

6.4 Equations of Motion

2) For incompressible fluids, $\nabla \cdot \vec{q} = 0$ (Continuity Eq.)

$$\boxed{\rho \vec{g} - \nabla p + \mu \nabla^2 \vec{q} = \rho \frac{\partial \vec{q}}{\partial t} + \rho \left(\vec{q} \cdot \nabla \right) \vec{q}} \qquad (6.25)$$

Define acceleration due to gravity as

$$\left. \begin{aligned} g_x &= -g \frac{\partial h}{\partial x} \\ g_y &= -g \frac{\partial h}{\partial y} \\ g_z &= -g \frac{\partial h}{\partial z} \end{aligned} \right\} \quad \vec{g} = -g \nabla h$$

6.4 Equations of Motion

where h = vertical direction measured positive upward

For Cartesian axes oriented so that h and z coincide

$$g_x = g_y = 0 \quad , \quad \frac{\partial h}{\partial z} = 1 \qquad (6.26)$$

$$g_z = -g \qquad (6.27)$$

→ minus sign indicates that acceleration due to gravity is in the negative h direction

Then, N-S equation for incompressible fluids and isothermal flows are

6.4 Equations of Motion

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-g\frac{\partial h}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right]$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-g\frac{\partial h}{\partial y}-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right]$$

$$\underbrace{\frac{\partial w}{\partial t}}_{\text{Local acceleration}}+\underbrace{u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}}_{\text{Convective acceleration}}=\underbrace{-g\frac{\partial h}{\partial z}}_{\text{Body force per mass}}\underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial z}}_{\text{Pressure force per mass}}+\underbrace{\frac{\mu}{\rho}\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right]}_{\text{Viscosity force per mass}} \qquad (6.28)$$

6.4 Equations of Motion

Eq. (6.28): unknowns – *u, v, w, p*

→ We need one more equation to obtain a solution when the boundary conditions are specified.

→ Eq. of continuity for incompressible fluid

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

♦ Boundary conditions

1) kinematic BC: velocity normal to any rigid boundary (wall) equal the boundary velocity (velocity = 0 for stationary boundary)

2) physical BC: no slip condition (continuum stick to a rigid boundary)

→ tangential velocity relative to the wall vanish at the wall surface

6.4 Equations of Motion

♦ General solutions for Navier-Stocks equations are not available because of the nonlinear, 2nd-order nature of the partial differential equations.

→ Only particular solutions may be obtained by simplifications.

→ Numerical solutions are usually sought.

♦ Navier-Stocks equations in cylindrical coordinates for constant density and viscosity

r - component:

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{{v_\theta}^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)$$

$$=\rho g_r-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right\}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right]$$

6.4 Equations of Motion

θ - component:

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}-\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)$$

$$=\rho g_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r v_\theta\right)\right\}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right]$$

z - component:

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)$$

$$=\rho g_z-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right] \qquad (6.29)$$

6.4 Equations of Motion

Continuity eq. for incompressible fluid

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta)+\frac{\partial}{\partial z}(v_z)=0 \qquad (6.30)$$

Normal & shear stresses for constant density and viscosity

$$\sigma_r=-p+2\mu\frac{\partial v_r}{\partial r}$$

$$\sigma_\theta=-p+2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}\right)$$

$$\sigma_z=-p+2\mu\frac{\partial v_z}{\partial z}$$

6.4 Equations of Motion

$$\tau_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau_{\theta z} = \mu\left[\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right]$$

$$\tau_{zr} = \mu\left[\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right]$$

6.5 Examples of Laminar Motion

- N-S equations are important in viscous flow problems.

♦ Laminar motion

~ orderly state of flow in which macroscopic fluid particles move in layers

~ viscosity effect is dominant

♦ Laminar flow through a tube (pipe) of constant diameter

~ instantaneous velocity at any point is always unidirectional (along the axis of the tube)

~ no-slip condition @ boundary wall

~ apply concept of the Newtonian viscosity

~ velocity gradient gives rise to viscous force within the fluid

~ low Re

6.5 Examples of Laminar Motion

[Re] Reynolds number = inertial force / viscous force = destabilizing force / stabilizing force

♦ Viscous force

~ dissipative

~ have a stabilizing or damping effect on the motion

~ use Reynolds number

[Cf] Turbulent flow

~ unstable flow

~ instantaneous velocity is no longer unidirectional

~ destabilizing force > stabilizing force

~ high Re

6.5 Examples of Laminar Motion

6.5.1 Laminar flow between parallel plates

Consider the two-dimensional, steady, laminar flow between parallel plates in which either of two surfaces is moving at constant velocity and there is also an external pressure gradient.

♦ Assumptions:

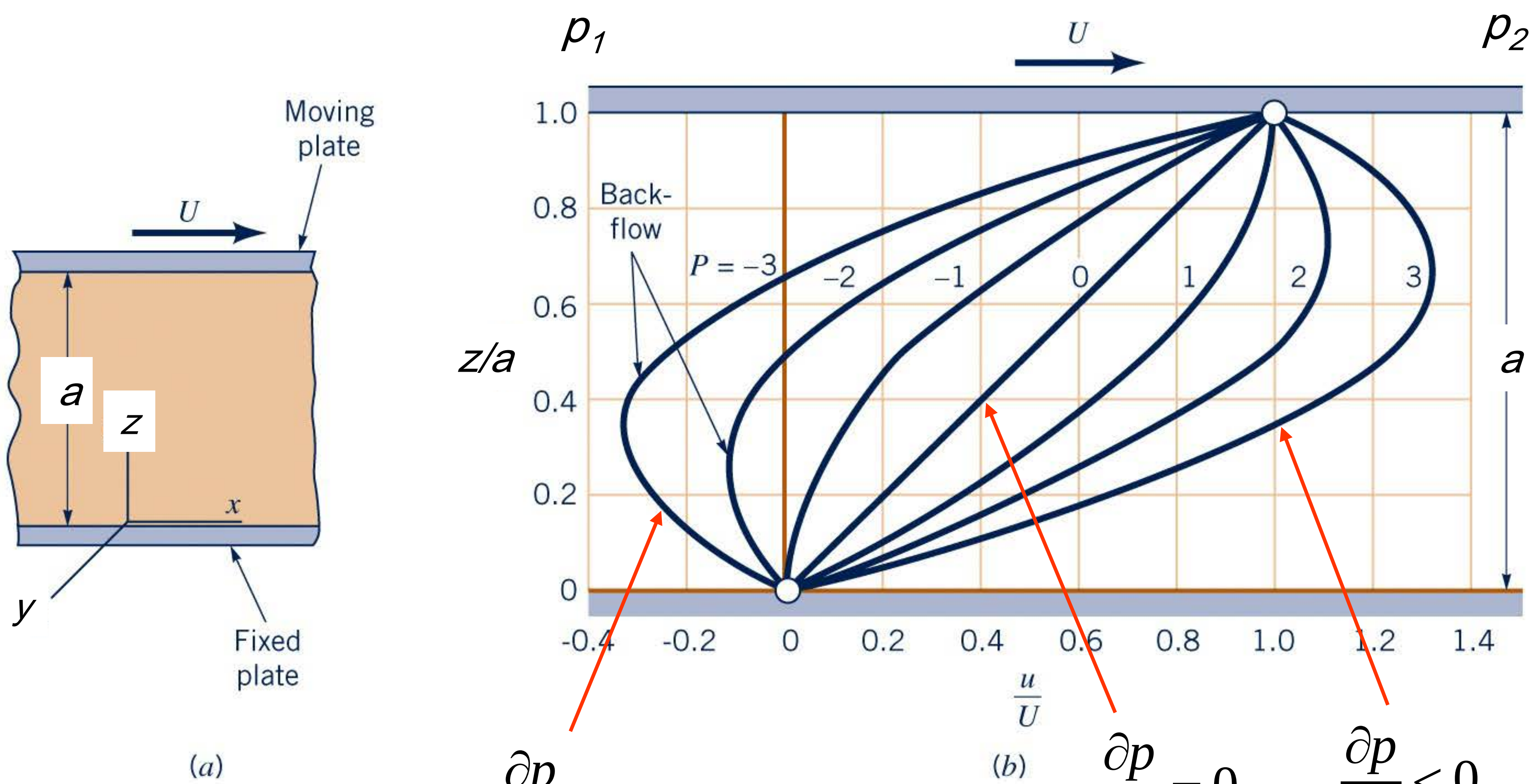
2-D flow (x, z) $\rightarrow$ $v = 0\ ;\ \dfrac{\partial(\ )}{\partial y} = 0$

steady flow $\rightarrow$ $\dfrac{\partial(\ )}{\partial t} = 0$

parallel flow $\rightarrow$ $w = 0\ ;\ \dfrac{\partial w}{\partial(\ )} = 0$

z-axis coincides with *h* $\rightarrow$ $\dfrac{\partial h}{\partial x} = \dfrac{\partial h}{\partial y} = 0\ ;\ \dfrac{\partial h}{\partial z} = 1$

6.5 Examples of Laminar Motion



$$\frac{\partial p}{\partial x} < 0$$

$$\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial p}{\partial x} < 0$$

6.5 Examples of Laminar Motion

♦ External pressure gradient

$$p_1 > p_2$$

i) $\dfrac{\partial p}{\partial x} < 0$ $\rightarrow$ pressure gradient assists the viscously induced motion to overcome the shear force at the lower surface

ii) $\dfrac{\partial p}{\partial x} > 0$ $\rightarrow$ pressure gradient resists the motion which is induced by the motion of the upper surface

$$p_1 < p_2$$

6.5 Examples of Laminar Motion

Continuity eq. for two-dimensional, parallel flow:

$$\frac{\partial u}{\partial x} + \cancel{\frac{\partial w}{\partial z}} = 0$$

$$\rightarrow \begin{cases} \dfrac{\partial^2 u}{\partial x^2} = 0 \\ u = f\left(z\right) \text{ only} \end{cases}$$

6.5 Examples of Laminar Motion

N-S Eq.:

Steady flow

Continuity eq. for incompressible fluid

2D flow

parallel flow

$$x-dir.: \cancel{\frac{\partial u}{\partial t}} + u\cancel{\frac{\partial u}{\partial x}} + v\cancel{\frac{\partial u}{\partial y}} + w\cancel{\frac{\partial u}{\partial z}}$$

$$= -g\cancel{\frac{\partial h}{\partial x}} - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left[\cancel{\frac{\partial^2 u}{\partial x^2}} + \cancel{\frac{\partial^2 u}{\partial y^2}} + \frac{\partial^2 u}{\partial z^2}\right]$$

$$\therefore \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial z^2}\right) \qquad (6.31a)$$

6.5 Examples of Laminar Motion

$$z-dir.:\ \cancel{\frac{\partial w}{\partial t}}+u\cancel{\frac{\partial w}{\partial x}}+v\cancel{\frac{\partial w}{\partial y}}+w\cancel{\frac{\partial w}{\partial z}}$$

$$=-g\frac{\partial h}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\mu}{\rho}\left[\cancel{\frac{\partial^2 w}{\partial x^2}}+\cancel{\frac{\partial^2 w}{\partial y^2}}+\cancel{\frac{\partial^2 w}{\partial z^2}}\right]$$

$$\therefore\quad 0=-g-\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad (6.31b)$$

(6.31b): $\dfrac{\partial p}{\partial z}=-\rho g=-\gamma$

$$\therefore\quad p=-\gamma z+f(x) \qquad (6.32)$$

6.5 Examples of Laminar Motion

→ hydrostatic pressure distribution normal to flow

→ For any orientation of z-axis. in case of a parallel flow, pressure is distributed hydrostatically in a direction normal to the flow.

(6.31a): $\dfrac{\partial p}{\partial x} \rightarrow \dfrac{dp}{dx}$ ~ independent of z

$$\therefore \quad \underbrace{\frac{dp}{dx}}_{\text{Pressure drop}} = \underbrace{\mu \frac{\partial^2 u}{\partial z^2}}_{\text{Energy loss due to viscosity}} \qquad \text{(A)}$$

6.5 Examples of Laminar Motion

Integrate (A) twice w.r.t. *z*

$$\iint \frac{dp}{dx} dzdz = \iint \mu \frac{\partial^2 u}{\partial z^2} dzdz$$

$$\int \frac{dp}{dx} z\, dz = \int \mu \frac{\partial u}{\partial z} dz + \int C_1\, dz$$

$$\frac{dp}{dx}\frac{z^2}{2} = \mu u + C_1 z + C_2 \qquad (6.33)$$

Use the boundary conditions,

i) $z = 0, \quad u = 0 \rightarrow \dfrac{dp}{dx} \times 0 = \mu(0) + C_2 \qquad \therefore\ C_2 = 0$

6.5 Examples of Laminar Motion

ii) $z = a, \quad u = U \rightarrow \frac{dp}{dx}\frac{a^2}{2} = \mu U + C_1 a$

$$\therefore \quad C_1 = \frac{1}{a}\left(\frac{dp}{dx}\frac{a^2}{2} - \mu U\right)$$

$\therefore$ (6.33) becomes

$$\frac{dp}{dx}\frac{z^2}{2} = \mu u + \frac{1}{a}\left(\frac{dp}{dx}\frac{a^2}{2} - \mu U\right)z$$

$$\therefore \quad \mu u = \frac{z}{a}\mu U - \frac{dp}{dx}\left(\frac{az}{2} - \frac{z^2}{2}\right)$$

6.5 Examples of Laminar Motion

$$\boxed{u(z) = u = \frac{U}{a}z - \frac{a}{2\mu}\frac{dp}{dx}\left(1 - \frac{z}{a}\right)z} \tag{6.34}$$

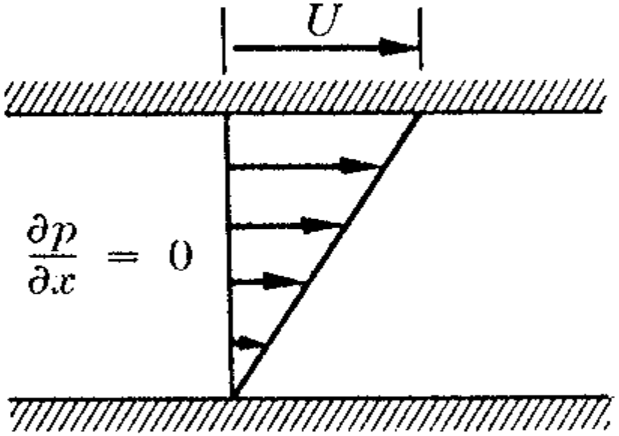
Velocity driven

Pressure driven

i) If $\dfrac{dp}{dx} = 0$ $\rightarrow$ Couette flow (plane Couette flow)

$$u = \frac{U}{a}z \tag{6.35}$$

$\rightarrow$ driving mechanism = U (velocity)



6.5 Examples of Laminar Motion

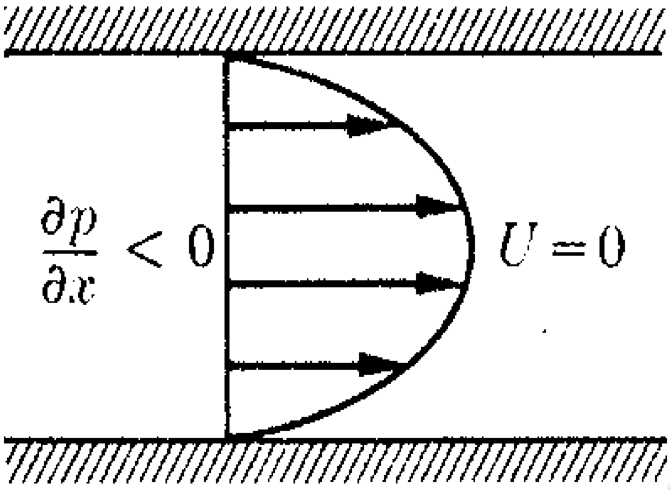
ii) If $U = 0 \rightarrow$ 2-D Poiseuille flow (plane Poiseuille flow)

$$u = \frac{1}{2\mu}\frac{dp}{dx}(z-a)z \quad \sim \text{parabolic} \tag{6.36}$$

→ driving mechanism = external pressure gradient, $\dfrac{dp}{dx}$

$u_{\max}$ @ $z = \dfrac{a}{2}$

$$u_{\max} = -\frac{a^2}{8\mu}\frac{dp}{dx} \tag{6.37}$$



6.5 Examples of Laminar Motion

V = average velocity

$$= \frac{Q}{A} = \frac{2}{3} u_{\max} = -\frac{a^2}{12\mu} \frac{dp}{dx} \tag{6.38}$$

[Re] detail

$$Q = \int_0^a u\, dz = \int_0^a \frac{1}{2\mu} \frac{dp}{dx} \left(z^2 - az \right) dz = -\frac{1}{12\mu} \frac{dp}{dx} a^3$$

$$A = a \times 1 \qquad \therefore\ V = \frac{Q}{A} = -\frac{a^2}{12\mu} \frac{dp}{dx} = \frac{2}{3} u_{\max}$$

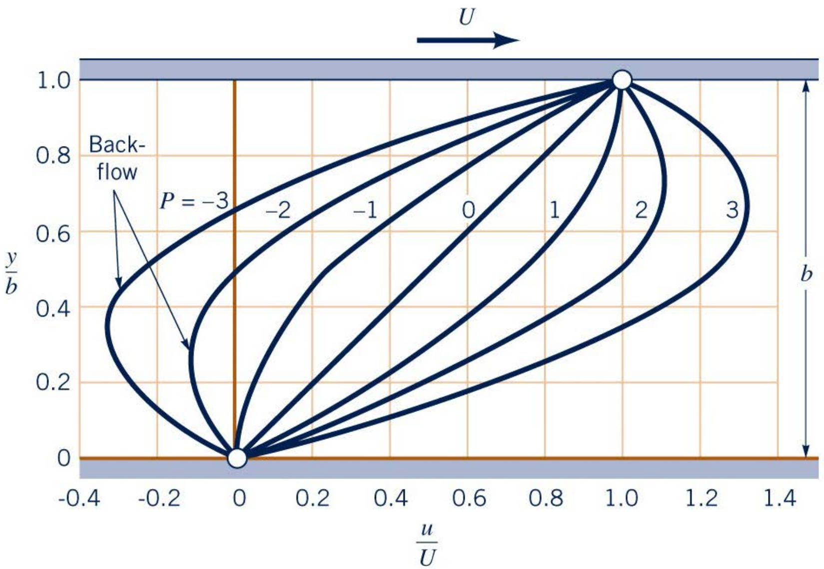
6.5 Examples of Laminar Motion

[Re] Dimensionless form

$$\frac{u}{U}=\frac{z}{a}-\frac{a^2}{2\mu U}\frac{dp}{dx}\frac{z}{a}\left(1-\frac{z}{a}\right)$$

$$P=-\frac{a^2}{2\mu U}\frac{dp}{dx}$$

$$\frac{u}{U}=\frac{z}{a}+P\frac{z}{a}\left(1-\frac{z}{a}\right)$$



(b)

6.5 Examples of Laminar Motion

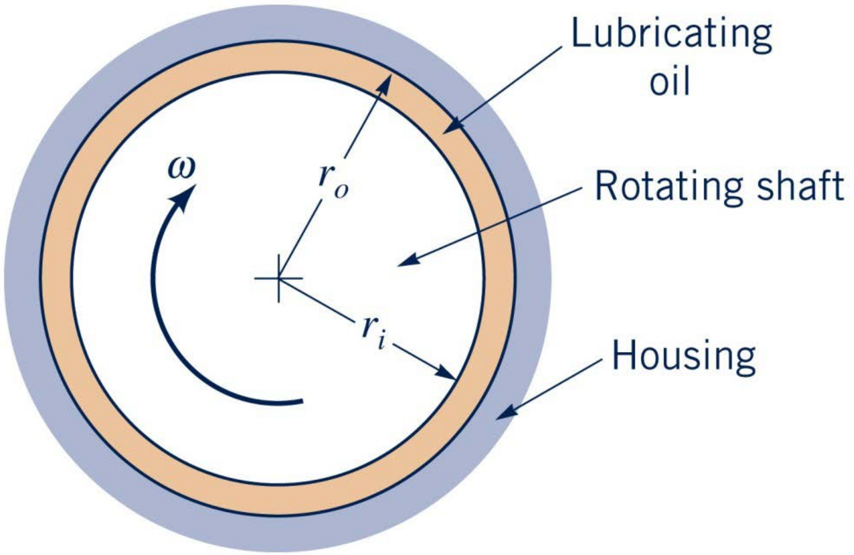
[Cf] Couette flow in the narrow gap of a journal bearing

Flow between closely spaced concentric cylinders in which one cylinder is fixed and the other cylinder rotates with a constant angular velocity, ω

$$U = r_i \omega$$

$$a = r_o - r_i$$

$$\tau \approx \mu \frac{U}{a}$$



6.5 Examples of Laminar Motion

6.5.2 Laminar flow in a circular tube of constant diameter

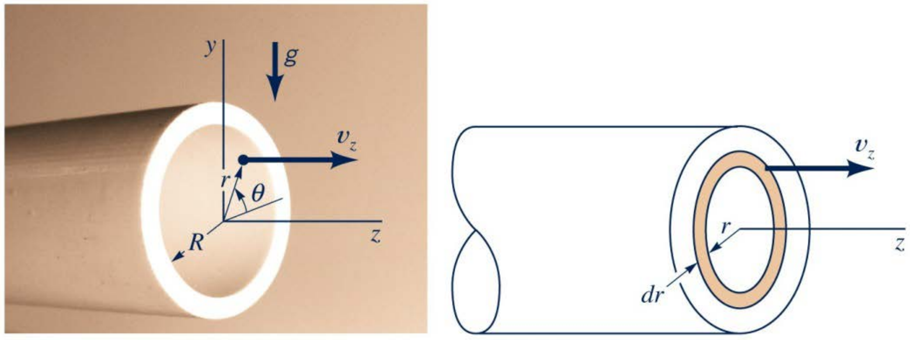
$$\frac{\partial p}{\partial x} < 0$$

→ Hagen-Poiseuille flow

→ Poiseuille flow: steady laminar flow due to pressure drop along a tube

Assumptions:

– use cylindrical coordinates



(a) (b)

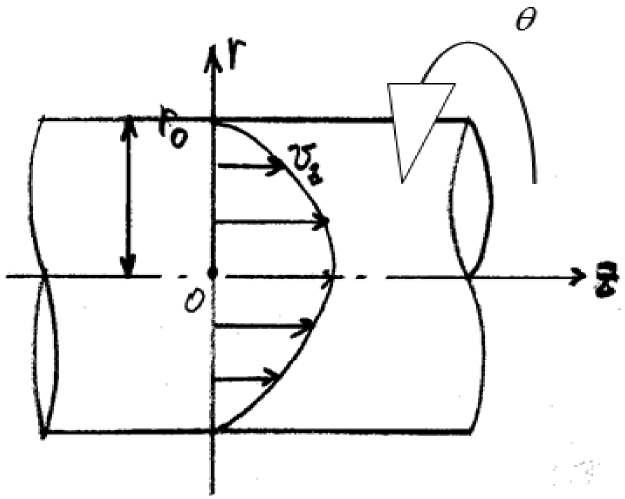
6.5 Examples of Laminar Motion

parallel flow $\rightarrow$ $v_r = 0$, $v_\theta = 0$ $\qquad v_z \neq 0$

Continuity eq. $\rightarrow$ $\dfrac{\partial v_z}{\partial z} = 0 \qquad \dfrac{\partial v_z}{\partial r} \neq 0$

paraboloid $\rightarrow$ $\dfrac{\partial v_z}{\partial \theta} = 0$

steady flow $\rightarrow$ $\dfrac{\partial v_z}{\partial t} = 0$



6.5 Examples of Laminar Motion

Eq. (6.29c) becomes

$$0 = \underline{-\frac{\partial p}{\partial z} + \rho g_z} + \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r} \right) \qquad \text{(A)}$$

By the way,

$$-\frac{\partial p}{\partial z} + \rho g_z = -\frac{\partial}{\partial z}\left(p + \gamma h \right) = -\frac{d}{dz}\left(p + \gamma h \right)$$

$$\left[\rho g_z = -\rho g \frac{\partial h}{\partial z} \right]$$

independent of *r*

$r-$comp. Eq. $\rightarrow$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial r} + g_r$$

$$\rightarrow \frac{\partial}{\partial r}\left(p + \gamma r \right) = 0$$

6.5 Examples of Laminar Motion

Then (A) becomes

$$\frac{d}{dz}(p+\gamma h)=\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)$$

$$\frac{1}{\mu}\frac{d}{dz}(p+\gamma h)r=\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) \qquad \text{(B)}$$

Integrate (B) twice w.r.t. *r*

$$\frac{1}{\mu}\frac{d}{dz}(p+\gamma h)\frac{r^2}{2}=r\frac{\partial v_z}{\partial r}+C_1 \qquad \text{(C)}$$

6.5 Examples of Laminar Motion

$$\frac{1}{2\mu}\frac{d}{dz}\left(p+\gamma h\right)r=\frac{\partial v_z}{\partial r}+\frac{C_1}{r}$$

$$\frac{1}{2\mu}\frac{d}{dz}\left(p+\gamma h\right)\frac{r^2}{2}=v_z+C_1\ln r+C_2 \qquad \text{(D)}$$

Using BCs

$r=0\,,\ v_z=v_{z_{\max}}$ → (C) : $C_1=0$

$r=r_0\,,\ v_z=0$ → (D) : $C_2=\frac{1}{2\mu}\frac{d}{dz}\left(p+\gamma h\right)\frac{{r_0}^2}{2}$ (D1)

6.5 Examples of Laminar Motion

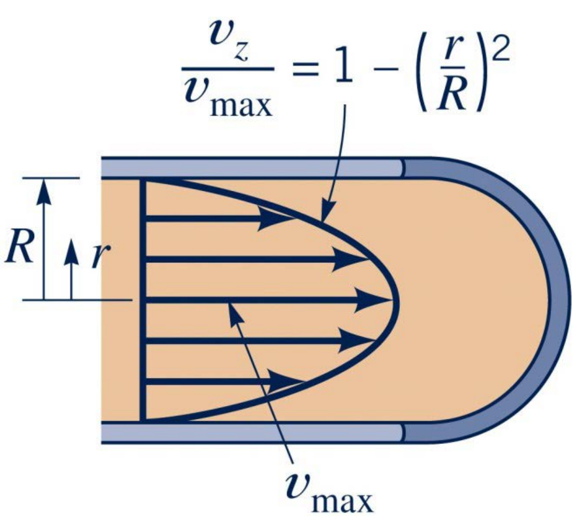
Then, substitute (D1) into (D) to obtain v_z

piezometric pressure

$$\therefore \quad v_z = \frac{1}{4\mu}\left[-\frac{d}{dz}(p+\gamma h)\right]\left(r_0^{\,2} - r^2\right) \qquad (6.39)$$

$$v_z = -\frac{d}{dz}\underline{(p+\gamma h)}\frac{r_0^{\,2}}{4\mu}\left[1-\left(\frac{r}{r_0}\right)^2\right]$$

→ equation of a <u>paraboloid of revolution</u>



6.5 Examples of Laminar Motion

(1) maximum velocity, $v_{z_{\max}}$

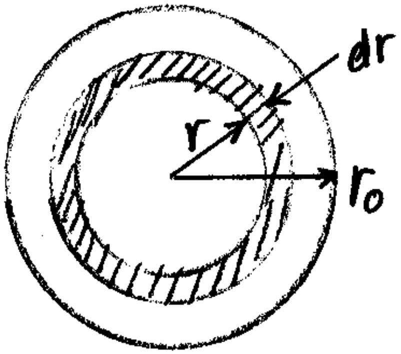
$$v_{z_{\max}} \quad @ \quad r = 0 \qquad (6.40)$$

$$v_{z_{\max}} = -\frac{d}{dz}\left(p + \gamma h\right)\frac{{r_0}^2}{4\mu} \qquad (6.41)$$

(2) mean velocity, V_z

$$dQ = v_z dA$$

$$= \frac{1}{4\mu}\left[-\frac{d}{dz}\left(p + \gamma h\right)\right]\left({r_0}^2 - r^2\right)2\pi r dr$$

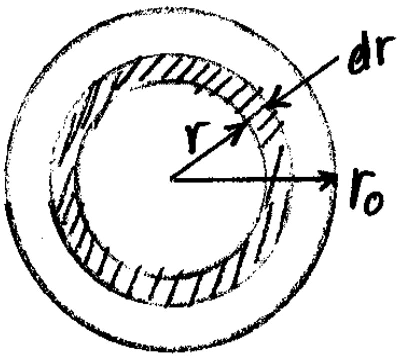


6.5 Examples of Laminar Motion

$$Q=\int_0^{r_0}\frac{1}{4\mu}\left[-\frac{d}{dz}(p+\gamma h)\right]\left(r_0^{\ 2}-r^2\right)2\pi r dr$$

$$=\frac{\pi}{2\mu}\left[-\frac{d}{dz}(p+\gamma h)\right]\left[r_0^{\ 2}\frac{r^2}{2}-\frac{r^4}{4}\right]_r^{r_0}$$

$$=\frac{\pi r_0^{\ 4}}{8\mu}\left[-\frac{d}{dz}(p+\gamma h)\right]$$



$$\therefore \quad V_z=\frac{Q}{A}=\frac{Q}{\pi r_0^{\ 2}}=\frac{r_0^{\ 2}}{8\mu}\left[-\frac{d}{dz}(p+\gamma h)\right]=\frac{v_{z_{\max}}}{2} \qquad \text{(E)}$$

[Cf] For 2 - D Poiseuille flow $V=\frac{2}{3}u_{\max}$

6.5 Examples of Laminar Motion

(3) Head loss per unit length of pipe

Total head = piezometric head + velocity head

Here, velocity head is constant.

Thus, total head loss = piezometric head change

$$\frac{h_f}{L} \equiv \frac{1}{\gamma}\left[-\frac{d}{dz}(p+\gamma h)\right] = \frac{8\mu V_z}{\gamma r_0^{\ 2}} = \frac{32\mu V_z}{\gamma D^2} \qquad (6.42)$$

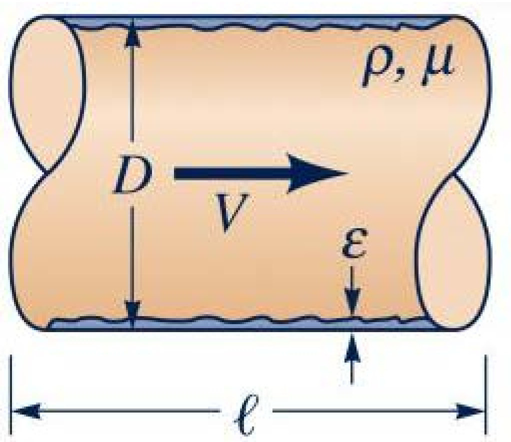
(E)

where $D = 2r_0 = \text{diameter}$

6.5 Examples of Laminar Motion

[Re] Consider Darcy-Weisbach Eq.

$$\frac{h_f}{L} = f\frac{1}{D}\frac{V_z^{\ 2}}{2g} \qquad \text{(F)}$$



h_f = head loss due to friction

f = friction factor

Combine (6.42) and (F)

$$\frac{32\mu V_z}{\gamma D^2} = f\frac{1}{D}\frac{V_z^{\ 2}}{2g} \qquad (6.43)$$

6.5 Examples of Laminar Motion

$$f = \frac{64}{V_z} \frac{\nu}{D} = \frac{64}{V_z D / \nu} = \frac{64}{\mathrm{Re}}$$ → For laminar flow (6.44)

(4) Shear stress

$$\tau_{zr} = \mu \left(\cancel{\frac{\partial v_r}{\partial z}} + \frac{\partial v_z}{\partial r} \right) = \mu \frac{\partial v_z}{\partial r} \qquad \text{(G)}$$

Differentiate (6.39) w.r.t. r

$$\frac{\partial v_z}{\partial r} = \frac{d}{dz}(p + \gamma h) \frac{1}{2\mu} r \qquad \text{(H)}$$

6.5 Examples of Laminar Motion

Combine (G) and (H)

Linear profile

$$\boxed{\tau_{zr} = \frac{1}{2}\frac{d}{dz}(p+\gamma h)r} \qquad (6.45)$$

At center and walls

$$r = 0\,, \quad \tau_{zr} = 0$$

$$r = r_0\,, \quad \tau_{zr} = \frac{1}{2}\frac{d}{dz}(p+\gamma h)r_0 = \tau_{zr_{\max}}$$