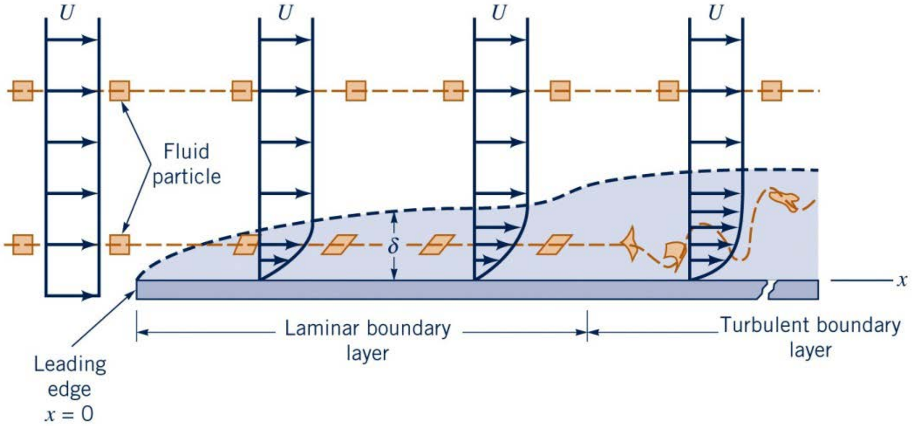
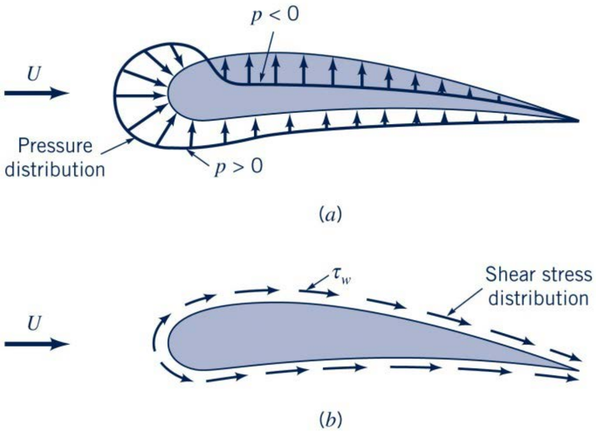
Chapter 7

Specialized Equations in Fluid Dynamics





Chapter 7 Specialized Equations in Fluid Dynamics

Contents

Objectives

- Discuss special cases of flow motion
- Derive equations for creeping motion
- Derive equation for 2-D boundary layers and integral equation
- Study flow resistance and drag force

7.1 Flow Classifications

7.1.1 Various Flows

(1) Laminar flow vs. Turbulent flow

- Laminar flow ~ water moves in parallel streamline (laminas); viscous shear predominates; low Re (Re < 2100)
- Turbulent flow ~ water moves in random, heterogeneous fashion; inertia force predominates; high Re (Re > 4000)

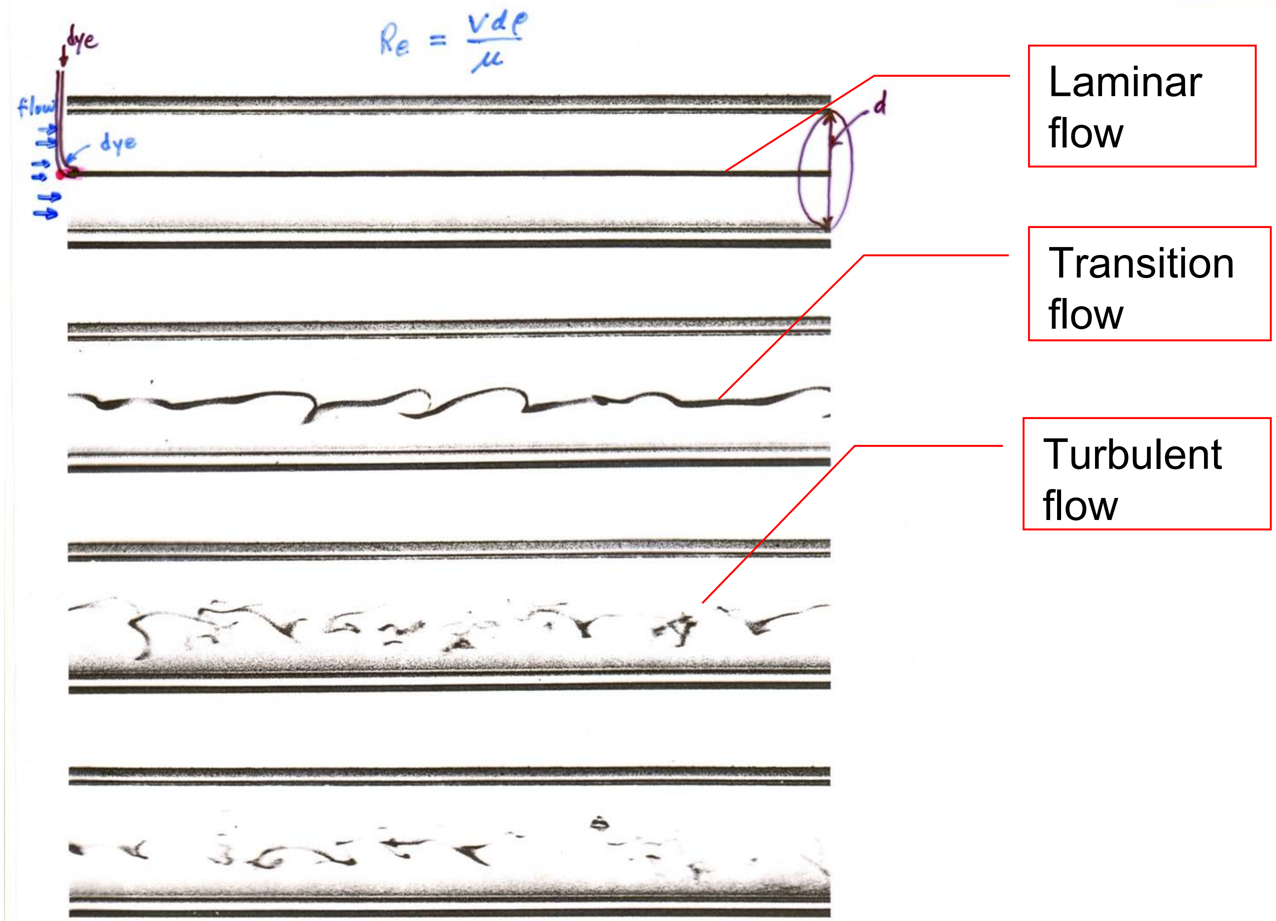
$$Reynolds\ number = \frac{inertia\ force}{viscous\ force} = \frac{Ma}{\tau A} = \frac{\rho l^3 (\frac{v^2}{l})}{\mu \frac{dv}{dy} l^2} = \frac{\rho v^2 l^2}{\mu v l} = \frac{\rho v l}{\mu} = \frac{vl}{\nu}$$

Neither laminar nor turbulent motion would occur in the absence of viscosity.

(2) Creeping motion vs. Boundary layer flow

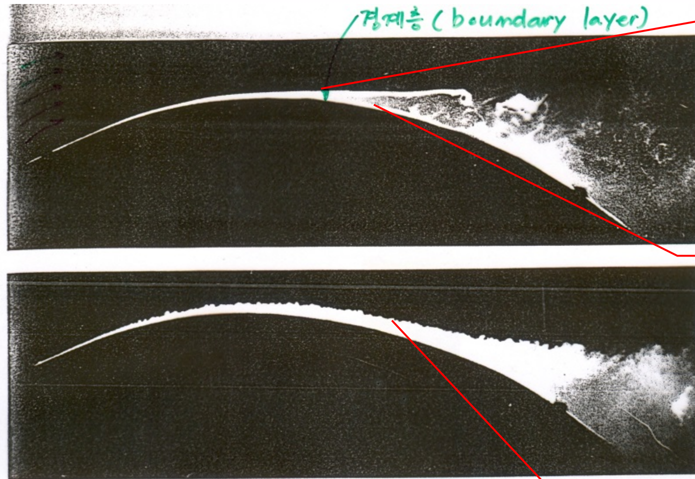
- Creeping motion – high viscosity → low Re
- Boundary layer flow– low viscosity → high Re

7.1 Flow Classifications

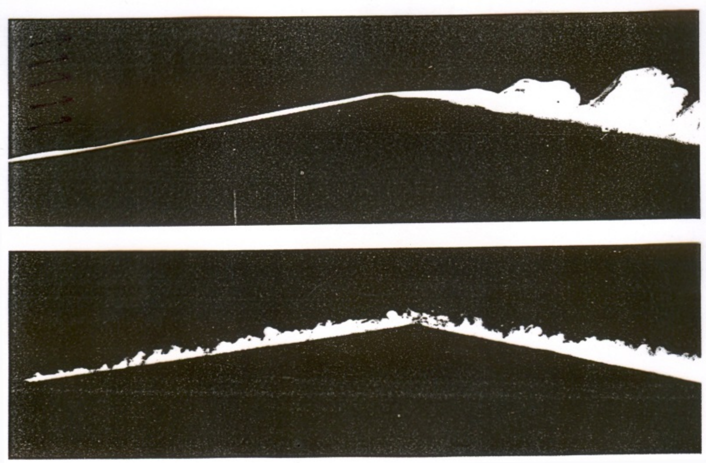


103. Repetition of Reynolds' dye experiment. Osborne Reynolds' celebrated 1883 investigation of stability of flow in a tube was documented by sketches rather than photography. However the original apparatus has survived at the University of Manchester. Using it a century later, N. H. Johannesen and C. Lowe have taken this sequence of photographs. In laminar flow a filament of colored water introduced at a bell-shaped entry extends undisturbed the whole length of the glass tube. Transition is seen in the second of the photographs as the speed is increased; and the last two photographs show fully turbulent flow. Modern traffic in the streets of Manchester made the critical Reynolds number lower than the value 13,000 found by Reynolds.

7.1 Flow Classifications



156. **Comparison of laminar and turbulent boundary layers.** The laminar boundary layer in the upper photograph separates from the crest of a convex surface (cf. figure 38), whereas the turbulent layer in the second photograph remains attached; similar behavior is shown below for a sharp corner. (Cf. figures 55-58 for a sphere.) Titanium tetrachloride is painted on the forepart of the model in a wind tunnel. *Head 1982*

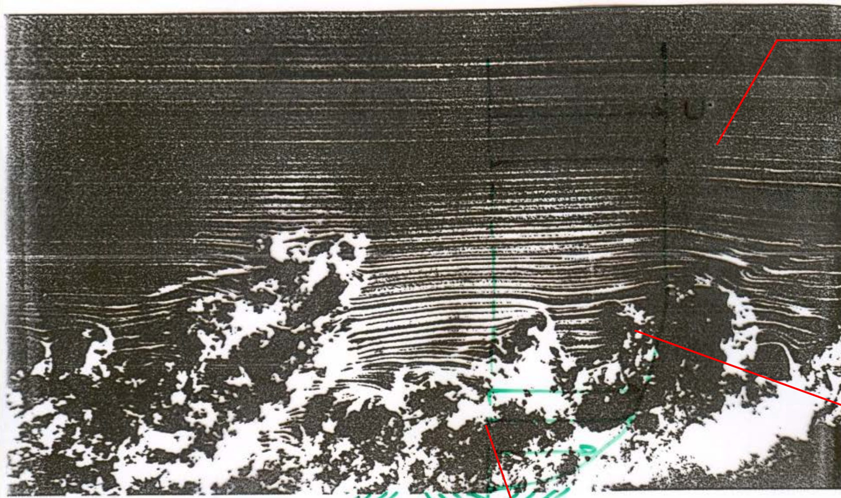


Laminar boundary layer flow

Separation occurs at the crest.

Turbulent boundary layer flow: no separation

7.1 Flow Classifications



157. **Side view of a turbulent boundary layer.** Here a turbulent boundary layer develops naturally on a flat plate 3.3 m long suspended in a wind tunnel. Streaklines from a smoke wire near the sharp leading edge are illuminated by a vertical slice of light. The Reynolds number is 3500 based on the momentum thickness. The intermittent nature of the outer part of the layer is evident. *Photograph by Thomas Corke, Y. Guezennec, and Hassan Nagib.*

158. **Turbulent boundary layer on a wall.** A fog of tiny oil droplets is introduced into the laminar boundary layer on the test-section floor of a wind tunnel, and the layer then tripped to become turbulent. A vertical sheet of light shows the flow pattern 5.8 m downstream, where the Reynolds number based on momentum thickness is about 4000. *Falco 1977*

External flow: potential flow

Intermittent nature

Boundary layer flow: rotational flow

7.1 Flow Classifications

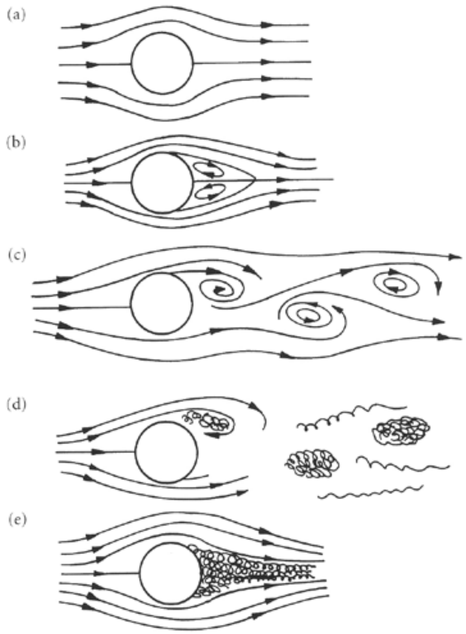


Figure 1.4 Flow behind a cylinder: (a) $Re < 1$; (b) $5 < Re < 40$; (c) $100 < Re < 200$; (d) $Re \sim 10^4$; and (e) $Re \sim 10^6$.

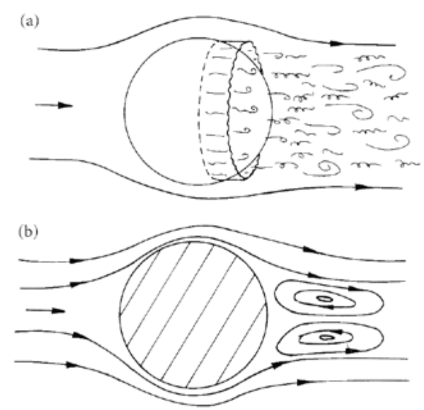


Figure 1.13 Schematic representation of flow over a sphere at $Re = 2 \times 10^4$: (a) snapshot of the flow as illustrated by dye injected into the boundary layer; (b) time-averaged flow pattern as seen in a time-lapse photograph. See also Plate 4 for the actual flow at $Re = 2 \times 10^4$ and 2×10^5.

→ Seo, I. W., and Song, C. G., "Numerical Simulation of Laminar Flow past a Circular Cylinder with Slip Conditions," *International Journal for Numerical Methods in Fluids*, Vol. 68, No. 12, 2012. 4, pp. 1538-1560.

→ 34th IAHR World Congress, Brisbane, Australia, Jun. 26 - Jul. 1 2011

7.1 Flow Classifications

7.1.2 Creeping motion

Creeping motion:

~<u>extreme of laminar motion</u> - viscosity is very high, and <u>velocity is very small</u>.

→ <u>Inertia force can be neglected</u> (*Re* → 0).

→ Convective acceleration and unsteadiness may be neglected.

For incompressible fluid,

Continuity Eq.: $\nabla \cdot \vec{q} = 0$

Navier-Stokes Eq.: $$\cancel{\rho \frac{\partial \vec{q}}{\partial t}} + \cancel{\rho (\vec{q} \cdot \nabla) \vec{q}} = \rho \vec{g} - \nabla p + \mu \nabla^2 \vec{q} + \cancel{\frac{\mu}{3} \nabla (\nabla \cdot \vec{q})}$$

7.1 Flow Classifications

[Ex] - fall of light-weight objects through a mass of molasses

→ Stoke's motion $Re < 1$

- filtration of a liquid through a densely packed bed of fine solid particles (porous media → Darcy's law)

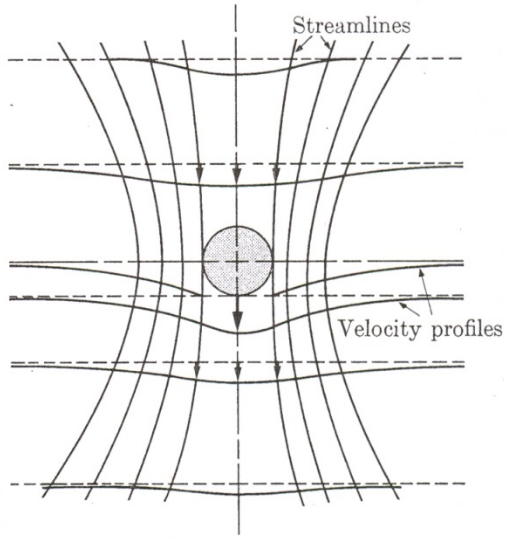


FIG. 8–3. Deformation flow around a falling sphere. (Streamlines and velocity profiles are shown for observer at rest.)

7.1 Flow Classifications

7.1.3 The boundary layer concept

For continuum fluid, there is no slip at the rigid boundary. [Cf] partial slip

→ Fluid velocity relative to the boundary is zero.

→ Velocity gradient $\left(\frac{du}{dy}\right)$ and shear stress have maximum values at the boundary

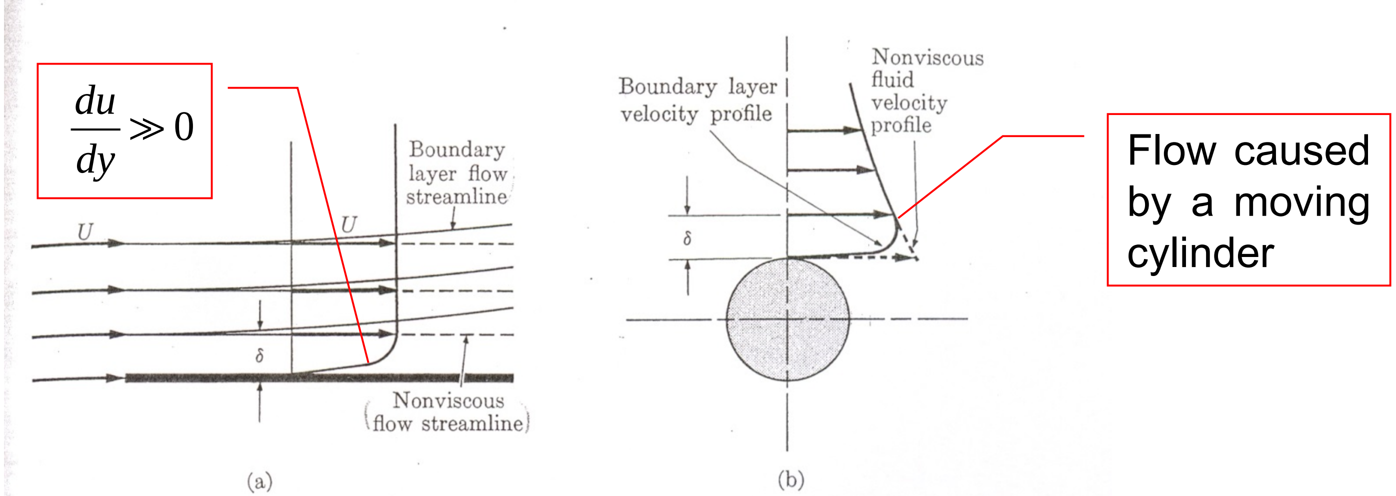


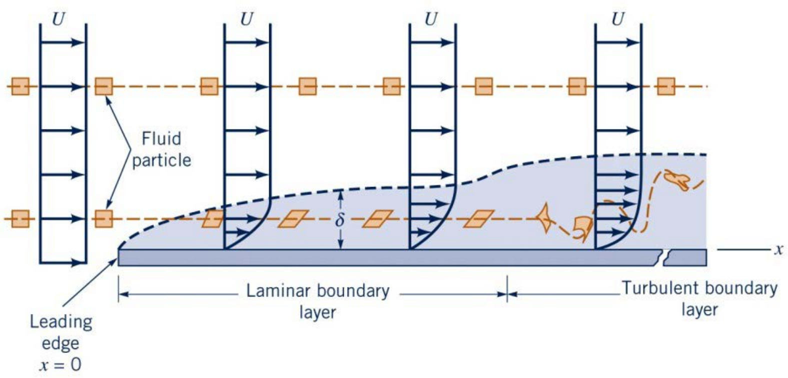
FIG. 8–4. Boundary layer versus slip flow: (a) flat plate; (b) cylinder.

7.1 Flow Classifications

For very low viscosity and high acceleration of the fluid motion

→ Significant viscous shear occurs only within a relatively thin layer next to the boundary.

→ boundary layer flow (Prandtl, 1904)



• Boundary layer flow:

~ inside the boundary layer,

viscous effects override inertia effects.

• Outer flow:

~ outside the layer, the flow will suffer only a minor influence of the viscous forces.

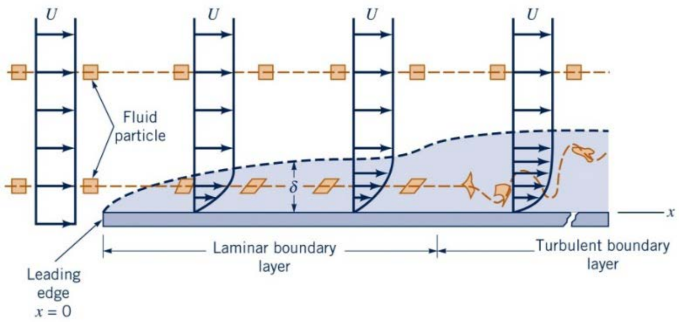
7.1 Flow Classifications

~ Flow will be determined primarily by the relation among inertia, pressure gradient, and body forces.

→ potential flow (irrotational flow)

1) Flow past a thin plate and flow past a circular cylinder

→ Due to flow retardation within boundary-layer thickness δ, displacement of streamlines is necessary to satisfy continuity.



7.1 Flow Classifications

2) Boundary layers in pipes

- Laminar flow between parallel walls → Poiseuille flow (parabolic profile)

- Turbulent flow in pipes → logarithmic profile

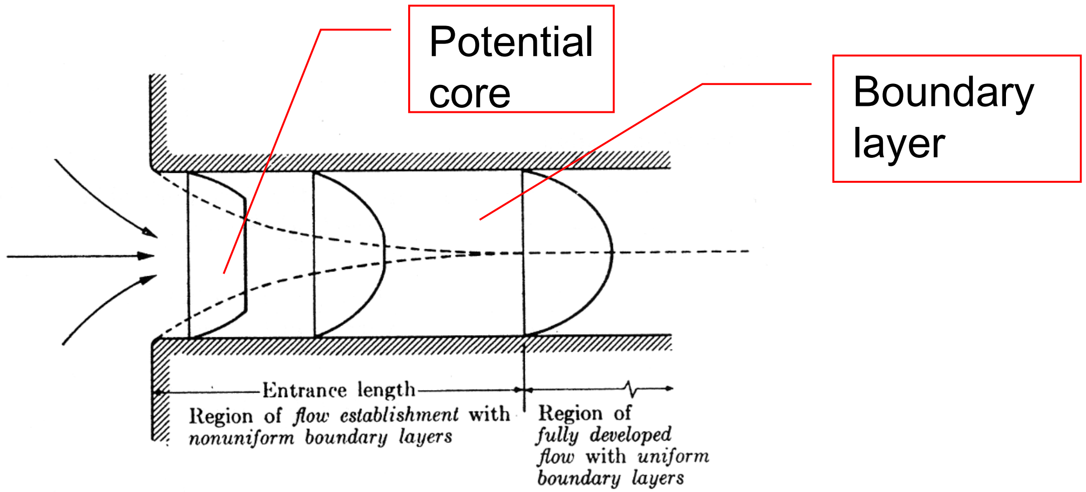


FIG. 8–5. Boundary layers in ducts.

[Re]

Creeping flow: <u>very viscous fluids</u> → only laminar flow

Boundary-layer flow: <u>slightly viscous fluids</u> → both laminar and turbulent flows

7.2 Equations for Creeping Motion and 2-D Boundary Layers

7.2.1 Creeping motion

→ **Study Ch.** 9 (D&H)

Assumptions:

– incompressible fluid

– very slow motion → inertia terms can be neglected.

$$\underbrace{\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\omega\frac{\partial u}{\partial z}}_{aeceleration}=-\underbrace{g\frac{\partial h}{\partial x}}_{body\ force}-\underbrace{\frac{1}{\rho}\frac{\partial p}{\partial x}}_{normal\ force}+\underbrace{\frac{\mu}{\rho}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right]}_{shear\ force}$$

$$= inertia\ effect \rightarrow 0$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

x-Eq.

$$\frac{\partial(p+\gamma h)}{\partial x}=\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

y-Eq.

$$\frac{\partial(p+\gamma h)}{\partial y}=\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \qquad (7.1)$$

z-Eq.

$$+\ \frac{\partial(p+\gamma h)}{\partial z}=\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

$$\boxed{\nabla(p+\gamma h)=\mu\nabla^2\vec{q}} \qquad (7.1a)$$

→ pressure change = combination of viscous effects and gravity

7.2 Equations for Creeping Motion and 2-D Boundary Layers

1) For incompressible fluids in an enclosed system (fluid within fixed boundaries)

$$p = p_d + p_s$$

where P_d= pressure responding to the dynamic effects by acceleration

$$p_s = const. - \gamma h$$ (hydrostatic relation)

where const. depends only on the datum selected.

$$\therefore \quad p = p_d + const - \gamma h$$

Eq. (7.1a) becomes

$$\nabla(p_d + const - \gamma h + \gamma h) = \mu \nabla^2 \vec{q}$$

$$\boxed{\nabla p_d = \mu \nabla^2 \vec{q}} \qquad (7.2)$$

→ Equation of motion for creeping flow

7.2 Equations for Creeping Motion and 2-D Boundary Layers

2) Continuity eq. for constant density

$$\nabla \cdot \vec{q} = 0 \qquad \text{(A)}$$

Solve (7.2) and (A) together with BC's

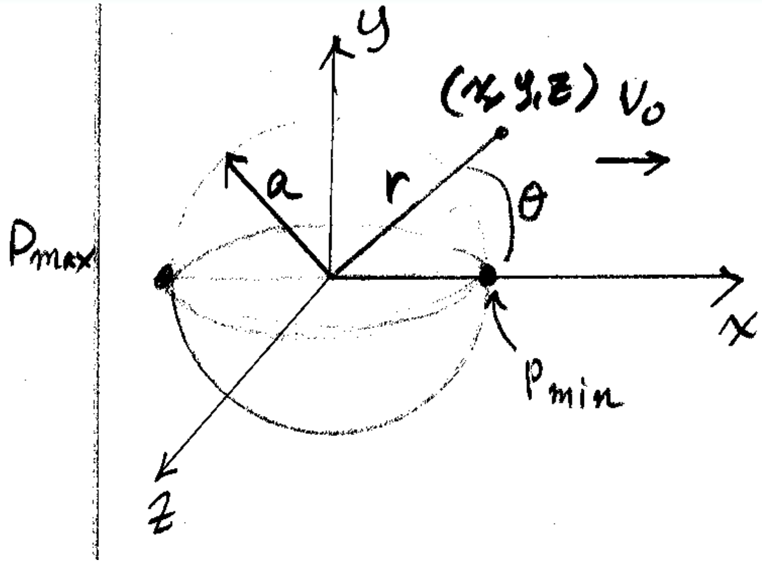
$$\begin{bmatrix} Unknowns = u, v, w, p \\ Eqs. = 3 + 1 \end{bmatrix}$$

[Ex] **Stoke's motion**: *Re* < 1

~ very slow flow past a fixed sphere → <u>refer Figs. 9.1-9.3 (D&H)</u>

~ solid sphere falling through a very viscous infinite fluid

7.2 Equations for Creeping Motion and 2-D Boundary Layers



- Solution:

$$u = V_0\left[\frac{3}{4}\frac{ax^2}{r^3}\left(\frac{a^2}{r^2}-1\right)-\frac{1}{4}\frac{a}{r}\left(3+\frac{a^2}{r^2}\right)+1\right]$$

$$v = V_0\frac{3}{4}\frac{axy}{r^3}\left(\frac{a^2}{r^2}-1\right)$$

$$w = V_0\frac{3}{4}\frac{axz}{r^3}\left(\frac{a^2}{r^2}-1\right) \quad (9.4)$$

$$p_d = -\frac{3}{2}\mu\frac{ax}{r^3}V_0$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

- Pressure distribution: Eq. (9.4) → Fig 9.4

$$p\Big|_{r=a} = -\frac{3}{2}\mu\frac{x}{a^2}V_0 = -\frac{3}{2}\frac{\mu V_0}{a}\cos\theta \;\; (\because x = a\cos\theta)$$

$$\therefore p_{\max}\Big|_{x=-a} = \frac{3}{2}\frac{\mu V_0}{a} \qquad \sim upstream\;\; stagnation\;\; point$$

$$p_{\min}\Big|_{x=a} = -\frac{3}{2}\frac{\mu V_0}{a} \qquad \sim downstream\;\; stagnation\;\; point$$

- Shear stress:

$$\tau_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta} + \frac{\partial v_\theta}{\partial r}\right) \qquad (9.12)$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

where $v_r = V_0 \cos\theta \left(1 - \frac{3}{2}\frac{a}{r} + \frac{1}{2}\frac{a^3}{r^3}\right)$

$$v_\theta = -V_0 \sin\theta \left(1 - \frac{3}{4}\frac{a}{r} - \frac{1}{4}\frac{a^3}{r^3}\right)$$

$$\therefore \tau_{r\theta}\big|_{r=a} = \frac{3}{2}\frac{\mu V_0}{a}\sin\theta \qquad (9.13)$$

- Drag on the sphere

Eq. (8.22):

$$D = \underbrace{+\int_0^\pi \tau_{r\theta}\sin\theta ds}_{D_f \,=\, frictional\ drag} - \underbrace{\int_0^\pi p\cos\theta ds}_{pressure\ drag \,=\, D_p}$$

where $ds = 2\pi a^2 \sin\theta d\theta$

$$\therefore D = \underbrace{4\pi a\mu V_0}_{frictional\ drag} + \underbrace{2\pi a\mu V_0}_{pressure\ drag} = 6\pi a\mu V_0$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

Eq. (8.27):

$$D = C_D \rho \frac{V_0^2}{2} A = C_D \rho \frac{V_0^2}{2} \pi a^2$$

$$\therefore 6\pi a \mu V_0 = C_D \rho \frac{V_0^2}{2} \pi a^2$$

$$\therefore C_D = \frac{12\mu}{\rho V_0 a} = \frac{24}{\rho V_0 D / \mu} = \frac{24}{\text{Re}} \qquad (9.17)$$

→ Fig. 9.5: valid if Re < 1;

for Re > 1 we cannot neglect inertia effect.

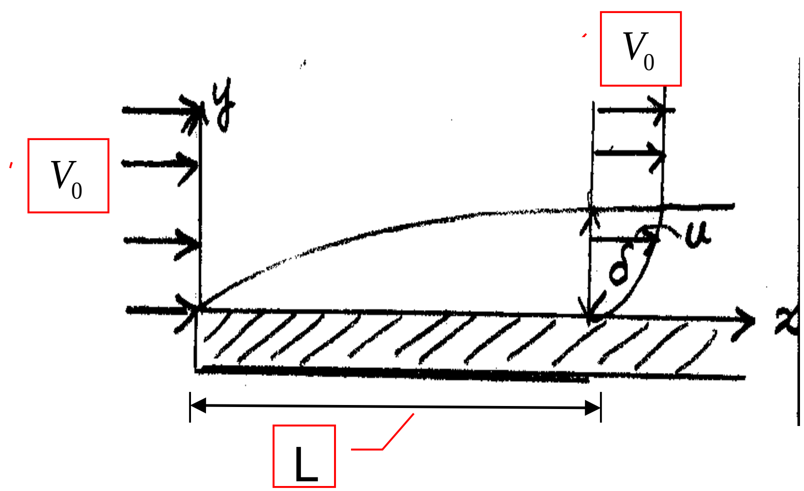
7.2 Equations for Creeping Motion and 2-D Boundary Layers

7.2.2 Equations for 2-D boundary layers

(1) Two-dimensional boundary layer equations: Prandtl

→ simplification of the N-S Eq. using order-of-magnitude arguments

→ 2D dimensionless N-S eq. for incompressible fluid (omit gravity)



L, V_0 - constant reference values

7.2 Equations for Creeping Motion and 2-D Boundary Layers

Within thin and small curvature boundary layer

$$u \gg v, \qquad x \gg y$$

$$\frac{\partial u}{\partial y} \gg \frac{\partial u}{\partial x}$$

$$\frac{\partial p}{\partial y} \textit{ is small} \sim \textit{may be neglected}$$

dimensionless boundary-layer thickness δ°

$$\delta^\circ = \frac{\delta}{L} \rightarrow \boxed{\delta^\circ \ll 1}$$

$\therefore$ scale for decreasing order

$$\frac{1}{\delta^{\circ 2}} > \frac{1}{\delta^\circ} > 1 > \delta^\circ > \delta^{\circ 2}$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

Order of magnitude

$$x^\circ \sim O\,(1) \qquad x^\circ = \frac{x}{L}$$

$$y^\circ \sim O\,(\delta^\circ) \qquad y^\circ = \frac{y}{L}$$

$$u^\circ \sim O\,(1) \qquad u^\circ = \frac{u}{V_0}$$

$$v^\circ \sim O\,(\delta^\circ) \qquad v^\circ = \frac{v}{V_0}$$

$$\frac{\partial u^\circ}{\partial x^\circ} \sim O\,(1) \qquad p^\circ = \frac{p}{\rho V_0^2}$$

$$\frac{\partial v^\circ}{\partial y^\circ} \sim O\,(1) \leftarrow continuity \left(\frac{\partial v^\circ}{\partial y^\circ} = -\frac{\partial u^\circ}{\partial x^\circ}\right)$$

$$\frac{\partial u^\circ}{\partial y^\circ} \sim O\,(\frac{1}{\delta^\circ})$$

$$\frac{\partial v^\circ}{\partial x^\circ} \sim O\,(\delta^\circ)$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

$$\frac{\partial^2 u^\circ}{\partial (x^\circ)^2} = \frac{\partial}{\partial x^\circ}\left(\frac{\partial u^\circ}{\partial x^\circ}\right) \sim O(1)$$

$$\frac{\partial^2 v^\circ}{\partial (y^\circ)^2} = \frac{\partial}{\partial y^\circ}\left(\frac{\partial v^\circ}{\partial y^\circ}\right) \sim O(\frac{1}{\delta^\circ})$$

$$\frac{\partial u^\circ}{\partial t^\circ} = \frac{\partial u^\circ}{\partial x^\circ}\frac{\partial x^\circ}{\partial t^\circ} = u^\circ \frac{\partial u^\circ}{\partial x^\circ} \sim O(1)$$

$$\frac{\partial v^\circ}{\partial t^\circ} = \frac{\partial v^\circ}{\partial x^\circ}\frac{\partial x^\circ}{\partial t^\circ} = u^\circ \frac{\partial v^\circ}{\partial x^\circ} \sim O(\delta^\circ)$$

$$\mathrm{Re} = \frac{\rho v y}{\mu} \sim O(\delta^{\circ 2})$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

$$x: \frac{\partial u^\circ}{\partial t^\circ} + u^\circ \frac{\partial u^\circ}{\partial x^\circ} + v^\circ \frac{\partial u^\circ}{\partial y^\circ} = -\frac{\partial p^\circ}{\partial x^\circ} + \frac{1}{\mathrm{Re}}\left(\cancel{\frac{\partial^2 u^\circ}{\partial x^{\circ 2}}} + \frac{\partial^2 u^\circ}{\partial y^{\circ 2}} \right) \tag{7.3}$$

1   1x1 $\delta^\circ \times 1/\delta^\circ$ $\delta^{\circ 2}(1+1/\delta^{\circ 2}) \rightarrow 1$

$$y: \cancel{\frac{\partial v^\circ}{\partial t^\circ}} + u^\circ \cancel{\frac{\partial v^\circ}{\partial x^\circ}} + v^\circ \cancel{\frac{\partial v^\circ}{\partial y^\circ}} = -\cancel{\frac{\partial p^\circ}{\partial y^\circ}} + \frac{1}{\mathrm{Re}}\left(\cancel{\frac{\partial^2 v^\circ}{\partial x^{\circ 2}}} + \cancel{\frac{\partial^2 v^\circ}{\partial y^{\circ 2}}} \right)$$

δ° $1 \times \delta^\circ$ $\delta^\circ \times 1$ $\delta^{\circ 2}(\delta^\circ + 1/\delta^\circ) \rightarrow \delta^\circ$

$$Continuity: \frac{\partial u^\circ}{\partial x^\circ} + \frac{\partial v^\circ}{\partial y^\circ} = 0$$

1   1

7.2 Equations for Creeping Motion and 2-D Boundary Layers

Therefore, eliminate all terms of order less than unity in Eq. (7.3) and revert to dimensional terms

$$\boxed{\begin{gathered}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}\\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\end{gathered}} \qquad (7.7)$$

→ Prandtl's 2-D boundary-layer equation

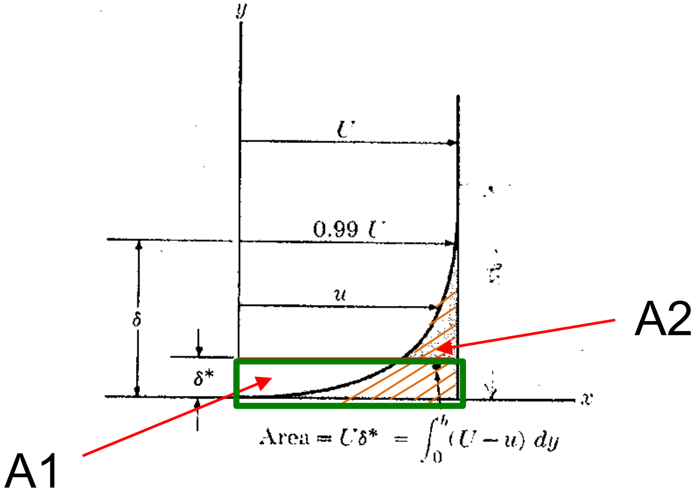
$$BC: 1)\quad y=0\,;\, u=0, v=0$$

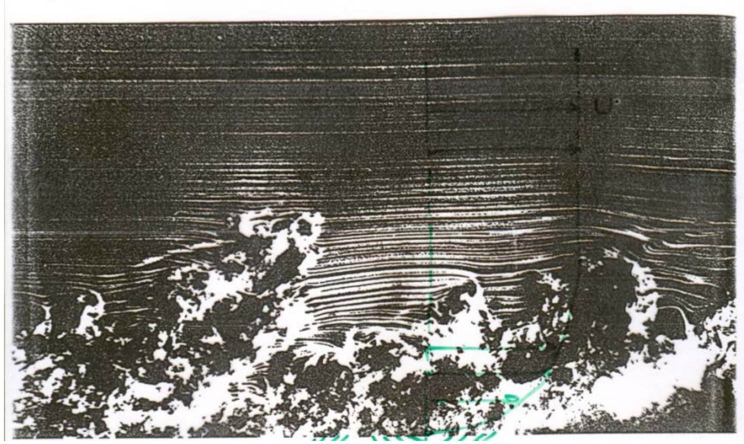
$$2)\quad y=\infty\,;\, u=U(x) \qquad (7.8)$$

Unknowns: *u, v, p*; Eqs. = 2 → needs assumptions for *p*

7.2 Equations for Creeping Motion and 2-D Boundary Layers

7.2.3 Boundary - layer thickness definitions





Intermittent nature

(1) Boundary-layer thickness,

~ The point separating the boundary layer from the <u>zone of negligible viscous influence</u> is not a sharp one. → very intermittent

δ = distance to the point where the velocity is within 1% of the free-stream velocity, U

$$@\, y = \delta \rightarrow u_{\delta} = 0.99U$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

(2) Mass displacement thickness, $\delta^* \, (\delta_1)$

~ is the thickness of an imaginary layer of fluid of velocity *U*.

~ is the thickness of mass flux rate equal to the amount of defect

$$A_1 = A_2$$

$$\rho U \delta^* = \underbrace{\rho \int_0^h (U - u)\,dy}_{mass\ defect} \qquad h \geq \delta$$

$$\boxed{\therefore \;\; \delta^* = \int_0^h \left(1 - \frac{u}{U}\right) dy} \qquad (7.9)$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

[Re] mass flux = mass/time

$$= \rho Q = \rho U A = \rho U \delta^* \times 1$$

(3) Momentum thickness, $\theta \,(\delta_2)$

→ Velocity retardation within δ causes a reduction in the rate of momentum flux.

→ θ is the thickness of an imaginary layer of fluid of velocity U for which the momentum flux rate equals the reduction caused by the velocity profile.

$$\rho \theta U^2 = \rho \int_0^h (U - u) u dy = \rho \int_0^h (Uu - u^2) dy$$

$$\boxed{\therefore \theta = \int_0^h \frac{u}{U} \left(1 - \frac{u}{U}\right) dy} \qquad (7.10)$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

[Re] momentum in θ = $mass \times velocity = \rho\theta U \times U = \rho\theta U^2$

momentum in shaded area = $\int [\rho(U-u)\times u]dy$

$$\boxed{\delta > \delta^* > \theta}$$

(4) Energy thickness, δ_3

$$\frac{1}{2}\rho U^3\delta_3 = \frac{1}{2}\int_0^h \rho u(U^2-u^2)dy$$

$$\boxed{\therefore \delta_3 = \int_0^h \frac{u}{U}(1-\frac{u^2}{U^2})dy}$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

[Re]

1) Batchelor (1985):

displacement thickness = distance through which streamlines just outside the boundary layer are displaced laterally by the <u>retardation of fluid</u> in the boundary layer.

2) Schlichting (1979):

displacement thickness = distance by which the external streamlines are shifted owing to the formation of the boundary layer.

7.2 Equations for Creeping Motion and 2-D Boundary Layers

7.2.4 Integral momentum equation for 2-D boundary layers

Integrate Prandtl's 2-D boundary-layer equations

$$\cancel{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \cancel{-\frac{1}{\rho}\frac{\partial p}{\partial x}} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Assumptions:

constant density $d\rho = 0$

steady motion $\frac{\partial ()}{\partial t} = 0$

pressure gradient = 0 $\frac{\partial p}{\partial x} = 0$

BC's:

$$@\, y = h\,; \quad \tau = 0, \quad u = U$$

$$@\, y = 0\,; \quad \tau = \tau_0, \quad u = 0$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

Prandtl's 2-D boundary-layer equations become as follows:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} \qquad \text{(A)}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad \text{(B)}$$

Integrate Eq. (A) w.r.t. y

$$\int_{y=0}^{y=h\geq\delta}\left(\underbrace{u\frac{\partial u}{\partial x}}_{①}+\underbrace{v\frac{\partial u}{\partial y}}_{②}\right)dy=\underbrace{\frac{\mu}{\rho}\int_{y=0}^{y=h}\frac{\partial^2 u}{\partial y^2}\,dy}_{③} \qquad \text{(C)}$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

$$③ = \mu \int_0^h \frac{\partial^2 u}{\partial y^2} dy = \int_0^h \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) dy = \int_0^h \frac{\partial \tau}{\partial y} dy = [\tau]_0^h$$

$$= \tau\Big|_{y=h} - \tau\Big|_{y=0} = 0 - \tau_0 = -\tau_0$$

$$② = \int_0^h v \frac{\partial u}{\partial y} dy = \underbrace{\int_0^h \frac{\partial uv}{\partial y} dy}_{④} - \underbrace{\int_0^h u \frac{\partial v}{\partial y} dy}_{⑤} \qquad \text{(D)}$$

[Re] Integration by parts: $\int v u' dy = vu - \int v' u dy$

$$④ = \int_0^h \frac{\partial uv}{\partial y} dy = [uv]_0^h = U v_h - 0 = Uv$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

Continuity Eq.: $\dfrac{\partial v}{\partial y} = -\dfrac{\partial u}{\partial x}$

$$\rightarrow \quad v = -\int_0^h \frac{\partial u}{\partial x} dy \qquad \text{(i)}$$

Substitute (i) into ⑤

$$⑤ = \int_0^h u\left(-\frac{\partial u}{\partial x}\right) dy = -\int_0^h u \frac{\partial u}{\partial x} dy \qquad \text{(ii)}$$

Substitute (i) into ④

$$④ = Uv = -U\int_0^h \frac{\partial u}{\partial x} dy$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

Eq. (D) becomes

$$\int_0^h v\frac{\partial u}{\partial y}dy = -U\int_0^h \frac{\partial u}{\partial x}dy + \int_0^h u\frac{\partial u}{\partial x}dy \qquad \text{(E)}$$

Then, (C) becomes

$$\int_0^h u\frac{\partial u}{\partial x}dy - U\int_0^h \frac{\partial u}{\partial x}dy + \int_0^h u\frac{\partial u}{\partial x}dy = -\frac{\tau_0}{\rho} \qquad \text{(F)}$$

For steady motion with and U = const., (F) becomes

$$\frac{\tau_0}{\rho} = U\int_0^h \frac{\partial u}{\partial x}dy - 2\int_0^h u\frac{\partial u}{\partial x}dy = \int_0^h \frac{\partial Uu}{\partial x}dy - \int_0^h \frac{\partial u^2}{\partial x}dy$$

$$= \int_0^h \frac{\partial}{\partial x}[u(U-u)]dy = \frac{\partial}{\partial x}\underbrace{\int_0^h u(U-u)\,dy}_{\theta U^2} = \frac{\partial}{\partial x}(\theta U^2)$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

where θ = momentum thickness

$$\boxed{\frac{\tau_0}{\rho} = \frac{\partial}{\partial x}(U^2\theta) = U^2\frac{\partial\theta}{\partial x}} \qquad (7.18)$$

$$\frac{\partial\theta}{\partial x} = \frac{\tau_0}{\rho U^2}$$

Introduce local surface (frictional) resistance coefficient C_f

$$C_f = \frac{D_f}{\frac{\rho}{2}u^2 A_f} = \frac{\tau_0}{\frac{\rho}{2}U^2} \qquad (7.19)$$

$$\boxed{D_f = \frac{\rho}{2}C_f A_f u^2}$$

$$\boxed{\tau_0 = \frac{\rho}{2}C_f U^2}$$

Combine (7.18) with (7.19)

$$\boxed{C_f = 2\frac{\partial\theta}{\partial x}} \qquad (7.20)$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

[Re] Integral momentum equation for <u>unsteady motion</u>

→ unsteady motion: $\dfrac{\partial ()}{\partial t} \neq 0$

→ pressure gradient, $\dfrac{\partial p}{\partial x} \neq 0$

First, simplify Eq. (7.7) <u>for external flow</u> where viscous influence is negligible.

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + v\cancel{\frac{\partial U}{\partial y}} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\cancel{\frac{\partial^2 U}{\partial y^2}}$$

$$\rho\frac{\partial U}{\partial t} + \rho U\frac{\partial U}{\partial x} = -\frac{\partial p}{\partial x} \qquad \text{(A)}$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

Substitute (A) into (7.7)

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\underbrace{\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}}_{\boxed{-\frac{1}{\rho}\frac{\partial p}{\partial x}}}+\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}$$

Integrate

$$\int_0^h \underbrace{\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}}_{\text{①}}dy=\int_0^h\left\{\underbrace{\frac{\partial u}{\partial t}-\frac{\partial U}{\partial t}}_{\text{②}}+\underbrace{u\frac{\partial u}{\partial x}-U\frac{\partial U}{\partial x}}_{\text{③}}+\underbrace{v\frac{\partial u}{\partial y}}_{\text{④}}\right\}dy \qquad \text{(B)}$$

$$\text{①}:\ \int_0^h \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}dy=-\frac{\tau_0}{\rho}$$

$$\text{②}:\ \int_0^h\left(\frac{\partial u}{\partial t}-\frac{\partial U}{\partial t}\right)dy=\int_0^h\frac{\partial}{\partial t}(u-U)dy=\frac{\partial}{\partial t}\underbrace{\int_0^h(u-U)dy}_{\boxed{-U\delta^*}}=-\frac{\partial}{\partial t}U\delta^*$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

$$③ = \underbrace{\int_0^h \left(u\frac{\partial u}{\partial x} - u\frac{\partial U}{\partial x} \right) dy}_{③\text{-}1} + \underbrace{\int \left(u\frac{\partial U}{\partial x} - U\frac{\partial U}{\partial x} \right) dy}_{③\text{-}2}$$

$$③\text{-}1 = \int_0^h \left\{ u\frac{\partial}{\partial x}(u-U) \right\} dy$$

$$③\text{-}2 = \int_0^h \left\{ (u-U)\frac{\partial U}{\partial x} \right\} dy = \frac{\partial U}{\partial x} \underline{\int_0^h (u-U)dy} = \frac{\partial U}{\partial x}(-U\delta^*)$$

$$④ = \int_0^h v\frac{\partial u}{\partial y}dy = -U\int_0^h \frac{\partial u}{\partial x}dy + \int u\frac{\partial u}{\partial x}dy = \int_0^h (u-U)\frac{\partial u}{\partial x}dy$$

$\boxed{Eq.(E)}$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

Combine ③-1 and ④

$$\int_0^h u\frac{\partial}{\partial x}(u-U)dy+\int_0^h (u-U)\frac{\partial u}{\partial x}dy=\int_0^h\left[u\frac{\partial}{\partial x}(u-U)+(u-U)\frac{\partial u}{\partial x}\right]dy$$

$$=\int_0^h \frac{\partial}{\partial x}\{u(u-U)\}dy=\frac{\partial}{\partial x}\int_0^h \underline{u(u-U)dy}=\frac{\partial}{\partial x}(-\theta U^2)$$

Substituting all these into (B) yields

$$-\frac{\tau_0}{\rho}=-\frac{\partial}{\partial t}(U\delta^*)-U\frac{\partial U}{\partial x}\delta^*-\frac{\partial}{\partial x}(\theta U^2)$$

$$\boxed{\frac{\tau_0}{\rho}=\frac{\partial}{\partial x}(U^2\theta)+U\frac{\partial U}{\partial x}\delta^*+\frac{\partial}{\partial t}(U\delta^*)} \qquad (7.21)$$

→ Karman's integral momentum eq.

7.3 The notion of resistance, drag, and lift

→ Study Ch.15 (D&H)

Resistance to motion = drag of a fluid on an immersed body in the direction of flow

♦ Dynamic (surface) force exerted on the rigid boundary by moving fluid are

1) Tangential force caused by shear stresses due to viscosity and velocity gradients at the boundary surfaces

2) Normal force caused by pressure intensities which vary along the surface due to dynamic effects

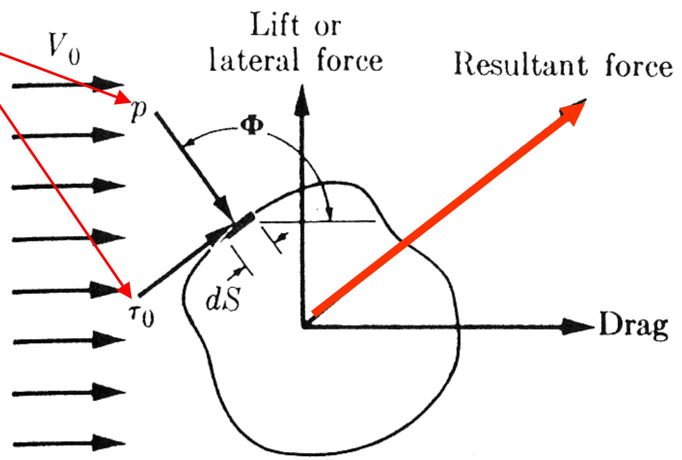


FIG. 8–7. Definition diagram for flow-induced forces.

7.3 The notion of resistance, drag, and lift

♦ Resultant force = vector sum of the normal and tangential surface forces integrated over the complete surface

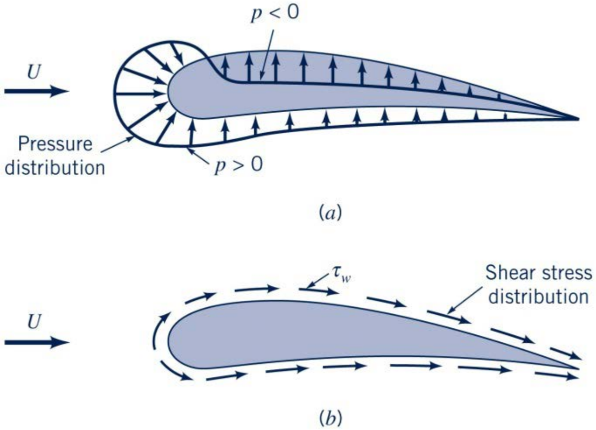
~ resultant force is divided into two components:

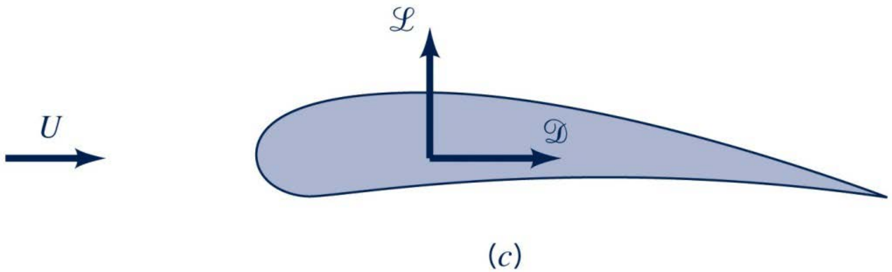
1) **Drag force** = component of the resultant force in the direction of relative velocity

2) **Lift force** = component of the resultant force normal to the relative velocity

~ Both drag and lift include frictional and pressure components.

7.3 The notion of resistance, drag, and lift





7.3 The notion of resistance, drag, and lift

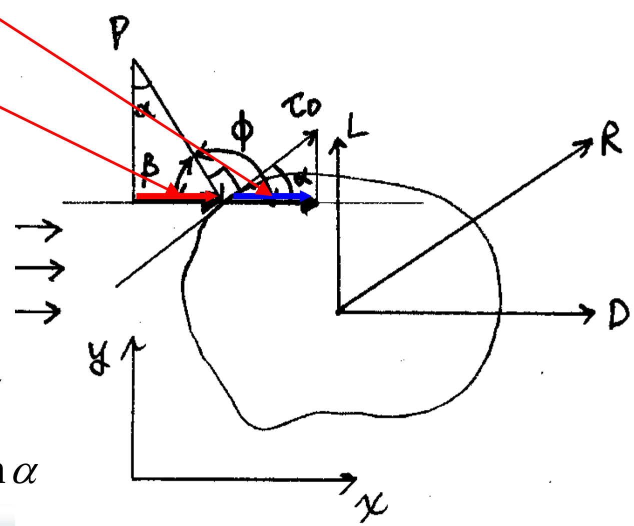
7.3.1 Drag force

♦ Total drag, *D*

$$D = D_f + D_p$$

where $D_f = \text{ frictional drag} = \int_s \tau_0 \sin\phi \, ds$

$$D_p = \text{pressure drag} = -\int_s p \cos\phi \, ds$$



$$\sin\phi = \sin(90° + \alpha) = \cos\alpha$$

$$\cos\phi = \cos(90° + \alpha) = -\sin\alpha$$

7.3 The notion of resistance, drag, and lift

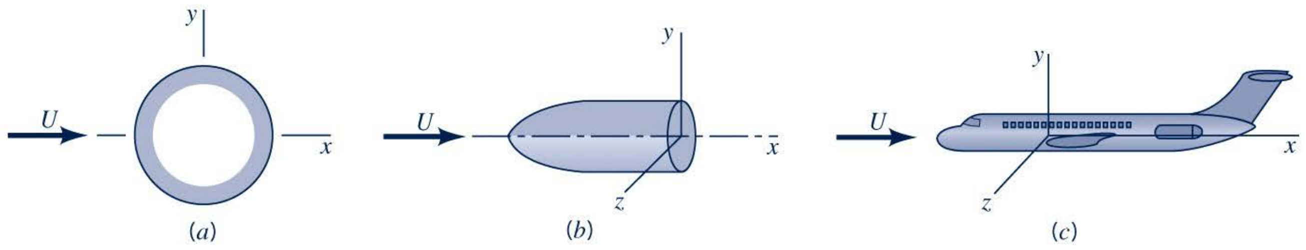
① Frictional drag = surface resistance = skin drag

② Pressure drag = form drag

~ depends largely on shape or form of the body

For airfoil, hydrofoil, and slim ships: surface drag > form drag

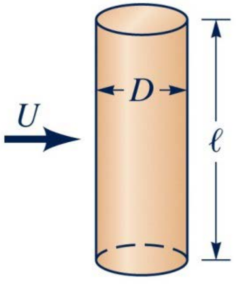
For bluff objects like spheres, bridge piers: surface drag < form drag

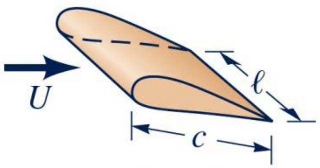


7.3 The notion of resistance, drag, and lift

♦ Drag coefficients, C_{D_f}, C_{D_p}

$$\boxed{\begin{aligned} D_f &= C_{D_f}\rho\frac{V_0^2}{2}A_f \\ D_p &= C_{D_p}\rho\frac{V_0^2}{2}A_P \end{aligned}}$$





Where A_f = <u>actual area over which shear stresses act</u> to produce D_f

A_p = <u>frontal area normal to the velocity</u> V_0

7.3 The notion of resistance, drag, and lift

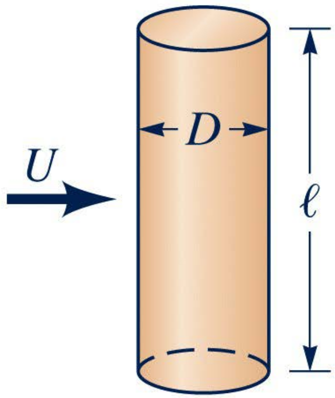
♦ Total drag coefficient C_D

$$D = C_D \rho \frac{V_0^2}{2} A$$

where A = frontal area normal to V_O

$$C_D = C_{D_f} + C_{D_p}$$

$$C_D = C_D(geometry, \mathrm{Re}) \rightarrow \text{Ch. 15}$$



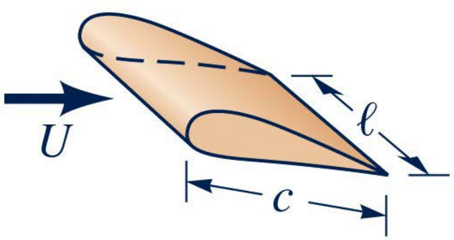
$A = D\ell$

[Re] Dimensional Analysis

$$D = f_1(\rho, \mu, V, L)$$

$$\frac{D}{\rho L^2 V^2} = f_2\left(\frac{\rho V L}{\mu}\right) = f_2(\mathrm{Re}) = C_D$$

$$\therefore D = C_D \frac{\rho}{2} A V^2$$

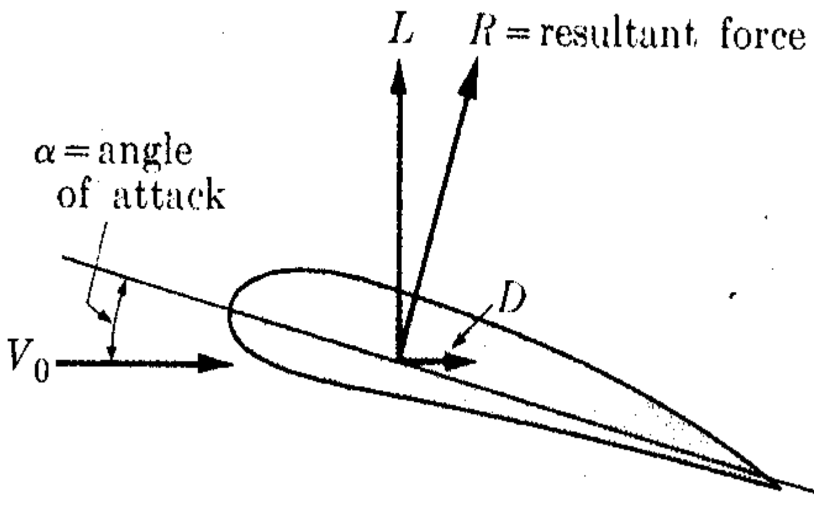


$A = c\ell$

7.3 The notion of resistance, drag, and lift

7.3.1 Lift force

For lift forces, it is not customary to separate the frictional and pressure components.



♦ Total lift, L

$$L = C_L \rho \frac{V_0^2}{2} A$$

where C_L = lift coefficient; A = largest projected area of the body