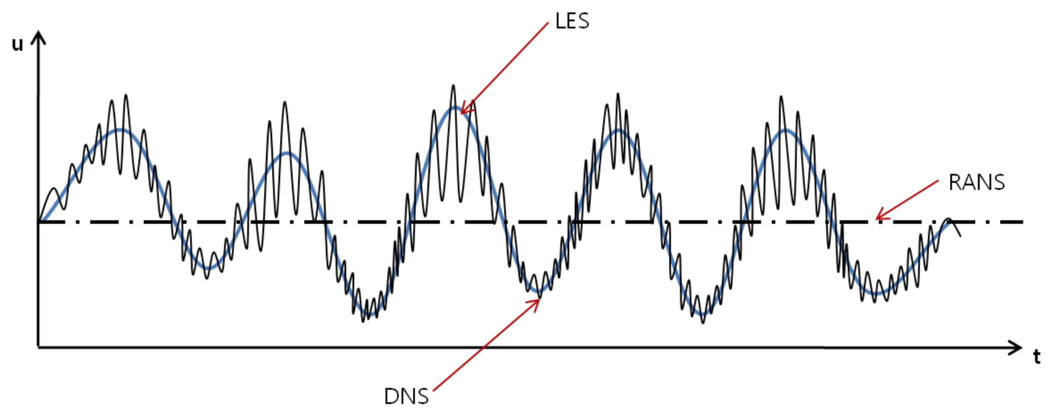
Chapter 10

Turbulence Models and Their Applications



Chapter 10 Turbulence Models and Their Applications

Contents

Objectives

- What is turbulence modeling?
- Derive specialized equations of motion in natural water bodies
- Study equations of turbulence models

10.1 Introduction

10.1.1 The Role of turbulence models

- **Why we need turbulence models?**

- Turbulent flows of practical relevance

→ highly random, unsteady, three-dimensional

→ Turbulent motion (velocity distribution), heat and mass transfer processes are extremely difficult to describe and to predict theoretically.

- **Solution for turbulent flows**

1) Navier-Stokes equation (DNS)

- Exact equations describing the turbulent motion are known.
- Numerical procedures are available to solve N-S eqs.

10.1 Introduction

- Computer simulations of the full N-S equation are usually limited to flows where periodicity of the flow can be assumed and the boundaries are simple, usually rectangular.

- Numerical grids used must be small enough to resolve the smallest significant eddy scale present in the flow, and the simulation must be carried out for a significantly long time that initial conditions have died out and significant features of the flow have evolved.

→ Storage capacity and speed of present-day computers are still not sufficient to allow a solution for any practically relevant turbulent flows.

10.1 Introduction

2) Reynolds equation (RANS)

- Average N-S equations to remove turbulent fluctuations completely

-Describe the complete effect of turbulence on the average motion by using turbulence model

3) LES

- numerical resolution of only the large eddies

10.1 Introduction

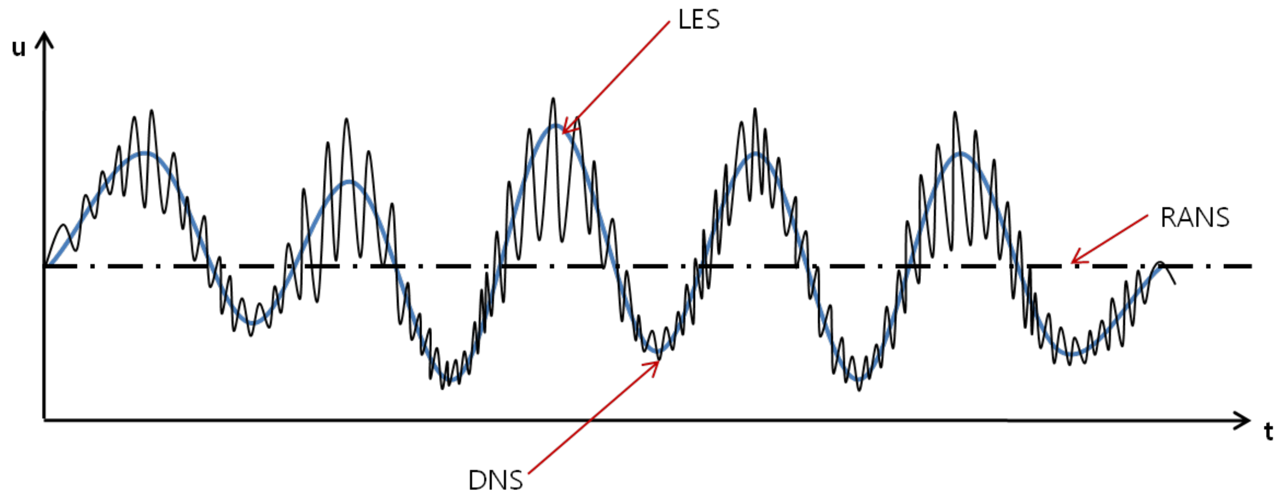
- Turbulence Modeling

DNS: direct numerical simulation of N-S eq.

LES: numerical resolution of only the large eddies

RANS: solution of Reynolds-Averaged N-S eq.

→ effects of all turbulent motions are accounted for by the turbulence model



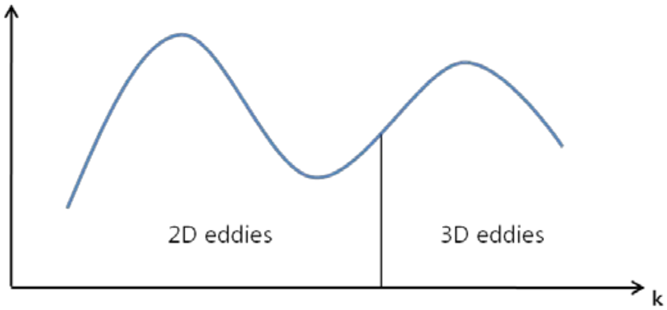
10.1 Introduction

- LES

- small-scale motion filtered out

$$\overline{u_i} = \int_{\Delta x}\int_{\Delta y}\int_{\Delta z} \frac{u_i dxdydz}{\Delta x \Delta y \Delta z}$$

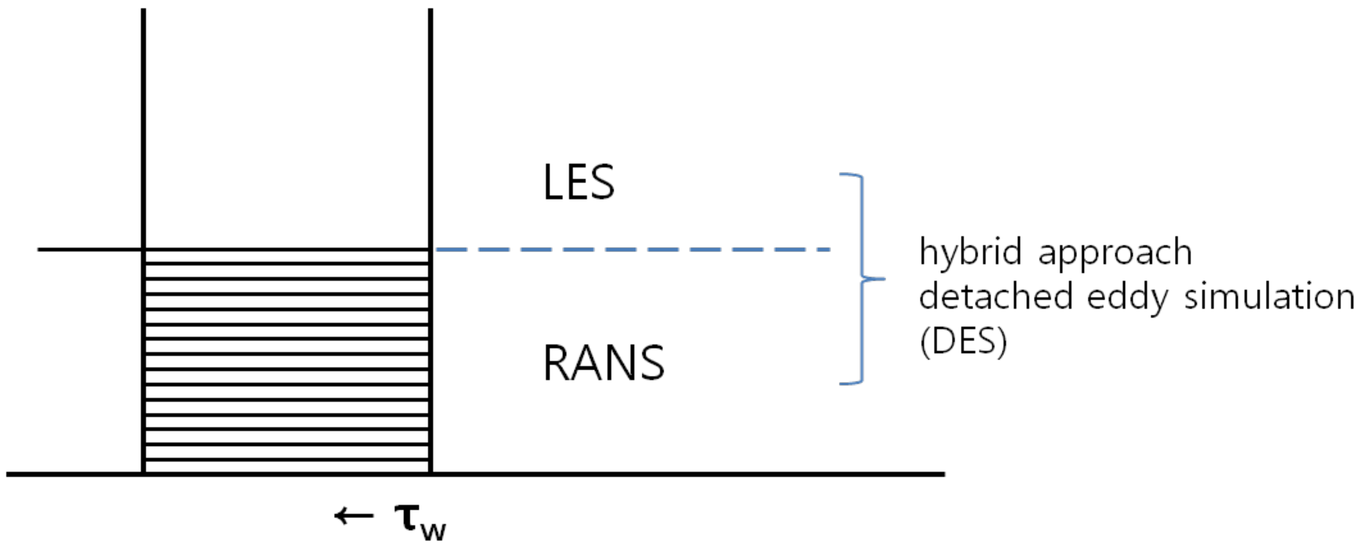
- 3D/2D LES



10.1 Introduction

- Hybrid approach

- High Re → wall model is needed



10.1 Introduction

- Turbulence

- Scale of turbulence

- eddying motion with a wide spectrum of eddy sizes and a corresponding spectrum of fluctuation frequencies

i) Large-scale eddies:

- contain much of the kinetic energy and little of the vorticity

- eddies tend to be anisotropic

- The forms of the largest eddies (low-frequency fluctuations) are determined by the boundary conditions (size of the flow domain).

- These large eddies gradually break down into smaller eddies.

10.1 Introduction

ii) Small eddies:

- have little kinetic energy but much vorticity

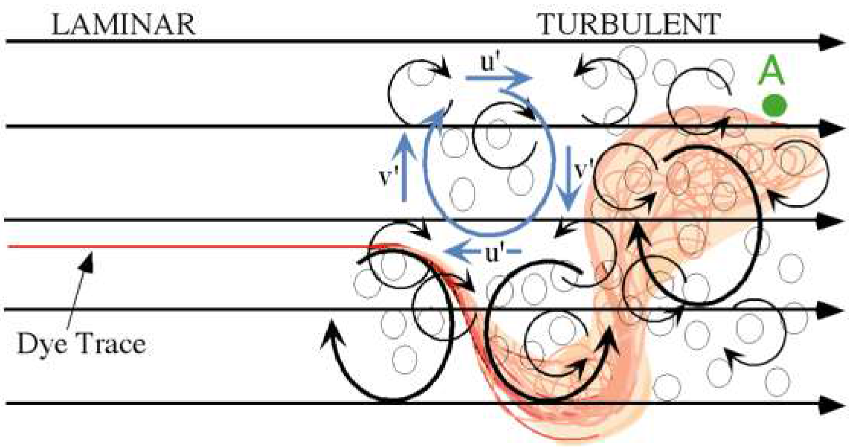
- The small eddies tend to be isotropic

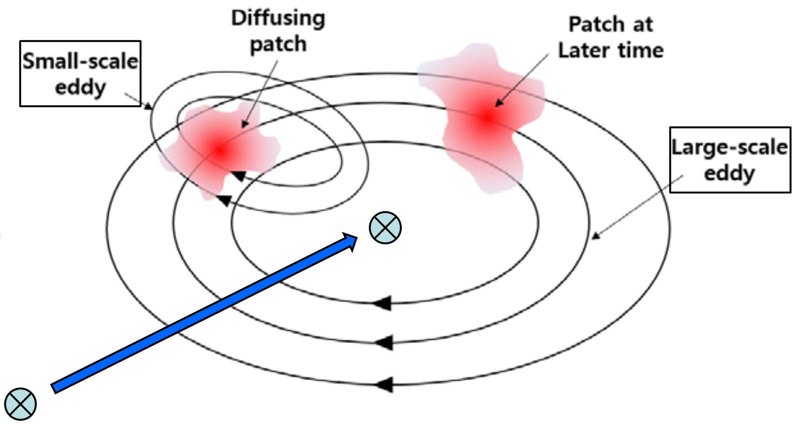
- The forms of the smallest eddies (highest-frequency fluctuations) are determined by the viscous forces.

- several orders of magnitude smaller

→ In numerical solution, to resolve the small-scale turbulent motion, 10^9 to 10^{12} grid points would be necessary to cover the flow domain in three dimensions.

Chapter 10 Turbulence Models and Their Applications





10.1 Introduction

- Classification of turbulence

i) anisotropic turbulence ~ general turbulence; it varies in intensity in direction

ii) isotropic turbulence ~ smallest turbulence; independent of direction (orientation)

$$\overline{u_i u_j} = \begin{cases} 0, \ i \neq j \\ const. \ , i = j \end{cases}$$

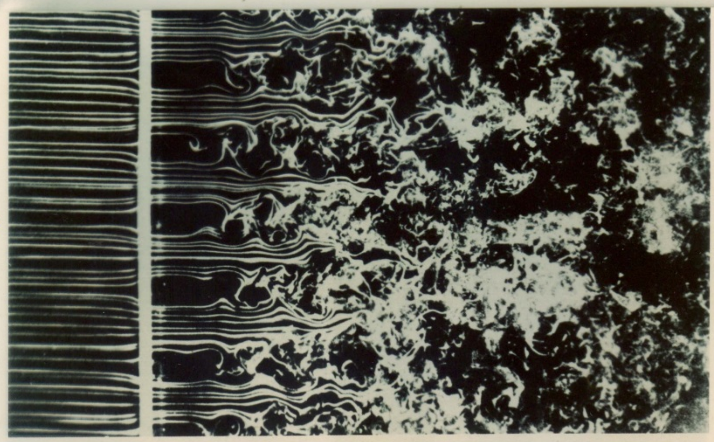
iii) nonhomogeneous turbulence

iv) homogeneous turbulence ~ statistically independent of the location

$$\overline{u_{i_a}^{\ 2}} = \overline{u_{i_b}^{\ 2}}$$

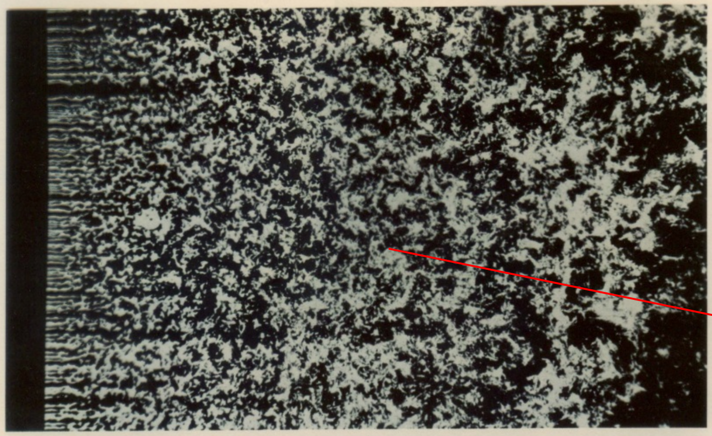
$$\overline{\left(u_i u_j\right)_a} = \overline{\left(u_i u_j\right)_b}$$

10.1 Introduction



152. Generation of turbulence by a grid. Smoke wires show a uniform laminar stream passing through a $^1/_{16}$-inch plate with ¼-inch square perforations. The Reynolds number is 1500 based on the 1-inch mesh size. Instability of the shear layers leads to turbulent flow downstream. *Photograph by Thomas Corke and Hassan Nagib*

grid turbulence



153. Homogeneous turbulence behind a grid. Behind a finer grid than above, the merging unstable wakes quickly form a homogeneous field. As it decays downstream, it provides a useful approximation to the idealization of isotropic turbulence. *Photograph by Thomas Corke and Hassan Nagib*

10.1 Introduction

- Turbulence models

~ a set of equations (algebraic or differential) which determine the turbulent transport terms in the mean-flow equations and thus close the system of equations

- Simulation of turbulence

1) Time-averaging approaches (models)

Name	No. of turbulent transport eqs.	Turbulence quantities transported
Zero equation model	0	None
One equation model	1	*k* (turbulent kinetic energy)
Two equation model	2	*k*, ε (turbulent energy, dissipation rate)
Stress/flux model	6	$\overline{u_i u_j}$ components (stress terms)
Algebraic stress model	2	*k*, ε used to calculate

10.1 Introduction

2) Space-averaged approaches

→ Large Eddy Simulation (LES)

- simulate the larger and more easily-resolvable scales of the motions while accepting the smaller scales will not be properly represented

10.1 Introduction

- References

ASCE Task Committee on Turbulence Models in Hydraulic Computations (1988). Turbulence modeling of surface water flow and transport: Part I, *J.Hydr. Eng.*, 114: 970-1073.

Rodi, W. (1993). Turbulence models and their application in hydraulics-A state of the art review, IAHR MONOGRAPH.

Graebel, W.P. (2007). Advanced Fluid Mechanics, Academic Press, Burlington, USA.

10.2 Mean Flow Equation and Closure Problem

10.2.1 Reynolds averaged basic equation

- Navier-Stokes eq.

~ Eq. of motion for turbulent motion

~ describes all the details of the turbulent fluctuating motion

~ These details cannot presently be resolved by a numerical calculation procedure.

~ Engineers are not interested in obtaining these details but interested in average **quantities.**

10.2 Mean Flow Equation and Closure Problem

- Definition of mean quantities by Reynolds

$$U_i = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \widetilde{U}_i \, dt \qquad (10.1a)$$

$$\Phi = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \widetilde{\Phi} \, dt \qquad (10.1b)$$

where $t_2 - t_1$ = averaging time Φ = scalar quantity (temperature, concentration)

- Averaging time should be long compared with the time scale of the turbulent motion but small compared with that of the mean flow in transient (unsteady) problems.

Example: in stream $t_2 - t_1 \sim 10^1 \sim 10^2$ sec

10.2 Mean Flow Equation and Closure Problem

- Decomposition of instantaneous values

$$\tilde{U}_i = U_i + u_i \qquad (10.2a)$$

$$\tilde{\Phi} = \Phi + \phi \qquad (10.2b)$$

$$\underset{\text{mean}}{\Phi \downarrow} \quad \underset{\text{fluctuations}}{\phi \searrow}$$

Substitute (10.2) into time-dependent equations of continuity and N-S eqs. and average over time as indicated by (10.1) → mean flow equations

10.2 Mean Flow Equation and Closure Problem

$\frac{\mu}{\rho}\nabla^2 U_i$ dropped

Continuity:

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}=0 \quad (10.3)$$

x-momentum:

$$\frac{\partial U}{\partial t}+\frac{\partial U^2}{\partial x}+\frac{\partial (VU)}{\partial y}+\frac{\partial (WU)}{\partial z} \quad (10.4)$$

$$=-\frac{1}{\rho}\frac{\partial P}{\partial x}+fV-\frac{\partial \overline{u^2}}{\partial x}-\frac{\partial \overline{uv}}{\partial y}-\frac{\partial \overline{uw}}{\partial z}$$

y-momentum:

$$\frac{\partial V}{\partial t}+\frac{\partial UV}{\partial x}+\frac{\partial (V^2)}{\partial y}+\frac{\partial (WV)}{\partial z} \quad (10.5)$$

$$=-\frac{1}{\rho}\frac{\partial P}{\partial y}-fU-\frac{\partial \overline{vu}}{\partial x}-\frac{\partial \overline{v^2}}{\partial y}-\frac{\partial \overline{vw}}{\partial z}$$

10.2 Mean Flow Equation and Closure Problem

z-momentum

$$\frac{\partial W}{\partial t}+\frac{\partial (UW)}{\partial x}+\frac{\partial (VW)}{\partial y}+\frac{\partial (W^2)}{\partial z} \tag{10.6}$$

$$=-\frac{1}{\rho}\frac{\partial P}{\partial z}-g-\frac{\partial \overline{wu}}{\partial x}-\frac{\partial \overline{wv}}{\partial y}-\frac{\partial \overline{w^2}}{\partial z}$$

Scalar transport:

$$\frac{\partial \Phi}{\partial t}+\frac{\partial (U\Phi)}{\partial x}+\frac{\partial (V\Phi)}{\partial y}+\frac{\partial (W\Phi)}{\partial z} \tag{10.7}$$

$$=S_\Phi-\frac{\partial \overline{u\phi}}{\partial x}-\frac{\partial \overline{v\phi}}{\partial y}-\frac{\partial \overline{w\phi}}{\partial z}$$

$D\nabla^2 U_i$ dropped

in which P = mean static pressure

f = Coriolis parameter

ρ = fluid density

S_Φ = volumetric source/sink term of scalar quantity

10.2 Mean Flow Equation and Closure Problem

Eqs. (10.3)~(10.7) do not form a closed set.

- Non-linearity of the original N-S eq. and scalar transport eq.

$$\left(\frac{\partial u^2}{\partial x}, \frac{\partial uv}{\partial y}, \frac{\partial uw}{\partial z}, \cdots ; \ \frac{\partial uc}{\partial x}, \frac{\partial vc}{\partial y}, \frac{\partial wc}{\partial z}\right)$$

→ introduce unknown correlations between fluctuating velocities and between velocity and scalar fluctuations in the averaging processes

$$(\overline{u^2}, \overline{v^2}, \overline{uv}, \cdots\cdots ; \ \overline{u\phi} \ etc.,)$$

$\overline{\rho u^2} \ \ etc.$ = rate of transport of momentum = turbulent Reynolds stresses

$\overline{\rho u\phi} \ \ etc.$ = rate of transport of heat or mass = turbulent heat or mass fluxes

10.2 Mean Flow Equation and Closure Problem

- In Eqs. (10.3)~(10.7), viscous stresses and molecular heat or mass fluxes are neglected because they are much smaller than their turbulent counterparts except in the viscous sublayer very near walls.

- Eqs. (10.3)~(10.7) can be solved for average dependent variables when the turbulence correlation can be determined in some way.

→ task of the **turbulence models**

10.2 Mean Flow Equation and Closure Problem

- Level of a turbulence model

~ depends on the relative importance of the turbulent transport terms

For the turbulent jet motion, simulation of turbulence is important.

For the horizontal motion in large shallow water bodies, refined turbulence modeling is not important because the inertial term in the momentum equations are balanced mainly by the pressure gradient and/or buoyancy terms.

→ The simulation of turbulence in heat and mass transport models is always important because the scalar transport equation does not contain any pressure gradient and/or buoyancy terms.

10.3 Specialized Model Equations

10.3.1 Three-dimensional lake circulation and transport models → Quasi-3D model

▪ In most shallow water situations and especially in calculating wind-driven lake circulation as well as continental shelf and open coast transport, vertical momentum equation can be reduced by the hydrostatic pressure approximation.

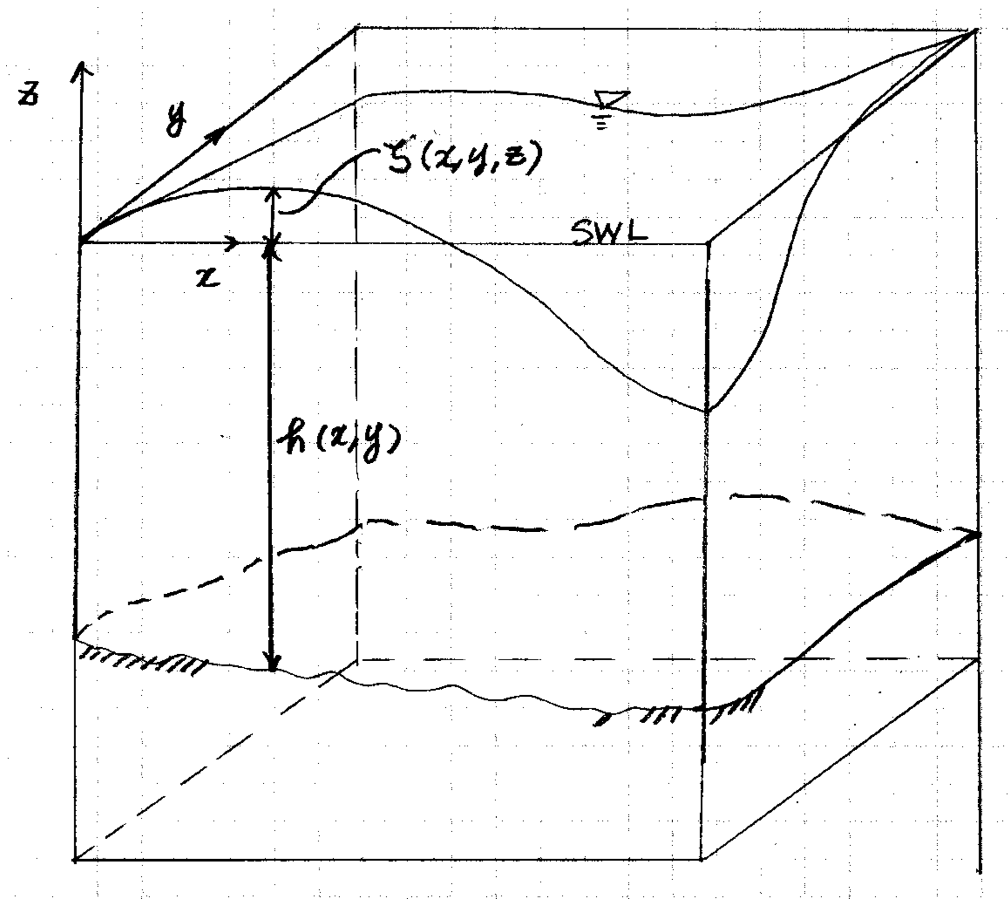
$$\frac{\partial p}{\partial z} = -\rho g \qquad \text{(a)}$$

Simplifies the calculation of the pressure field

Only horizontal two-dimensional pressure distribution must be calculated from the differential equations

The vertical variation of pressure follows Eq. (a).

10.3 Specialized Model Equations



10.3 Specialized Model Equations

▪ Two ways of determining the horizontal variation of pressure

→ Two ways of surface approximation

1) Assume atmospheric pressure at the water surface

→ calculate surface elevation ζ with kinematic boundary condition at the surface

$$\frac{\partial \zeta}{\partial t} + U \frac{\partial \zeta}{\partial x} + V \frac{\partial \zeta}{\partial y} - W = 0 \qquad (10.8)$$

With this kinematic condition, the continuity equation can be integrated over the depth H to yield an equation governing the surface elevation ζ.

10.3 Specialized Model Equations

2) Use rigid-lid approximation

- assume that the surface is covered by a frictionless lid

- allows no surface deformations but permits variations of the surface pressure

→ properly accounts for the pressure-gradient terms in the momentum equations, but an error is made in the continuity equations.

→ is valid when the relative surface elevation ζ/h is small

→ suppresses surface waves and therefore permits longer time steps in a numerical solutions

→ Bennett (1974) , J. Physical Oceanography, 4(3), 400-414

Haq and Lick (1975), J. Geophysical Res, 180, 431-437

10.3 Specialized Model Equations

10.3.2 Two-dimensional depth-averaged models

- For shallow water situations

~ vertical variation of flow quantities is small

~ horizontal distribution of vertically averaged quantities is determined

$$\bar{U} = \frac{1}{H}\int_{-h}^{\zeta} U\,dz \quad (10.9a)$$

$$\bar{\Phi} = \frac{1}{H}\int_{-h}^{\zeta} \Phi\,dz \quad (10.9b)$$

in which H = total water depth = h + ζ

h = location of bed below still water level

ζ = surface elevation

10.3 Specialized Model Equations

Average Eqs. (10.3)-(10.7) over depth

continuity: $$\frac{\partial \zeta}{\partial t} + \frac{\partial (H\bar{U})}{\partial x} + \frac{\partial (H\bar{V})}{\partial y} = 0 \qquad (10.10)$$

x-momentum: $$\frac{\partial (H\bar{U})}{\partial t} + \frac{\partial (H\bar{U}^2)}{\partial x} + \frac{\partial (H\bar{V}\bar{U})}{\partial y} = -gH\frac{\partial \zeta}{\partial x}$$
$$+\frac{1}{\rho}\frac{\partial (H\overline{\tau_{xx}})}{\partial x} + \frac{1}{\rho}\frac{\partial (H\overline{\tau_{xy}})}{\partial y} + \frac{\tau_{sx} - \tau_{bx}}{\rho}$$
$$+\frac{1}{\rho}\frac{\partial}{\partial x}\int_{-h}^{\zeta}\rho(U-\bar{U})^2 dz + \frac{1}{\rho}\frac{\partial}{\partial y}\int_{-h}^{\zeta}\rho(U-\bar{U})(V-\bar{V})dz$$

Turbulent shear stress

dispersion stress

10.3 Specialized Model Equations

y-momentum:

$$\frac{\partial(H\bar{V})}{\partial t}+\frac{\partial(H\overline{UV})}{\partial x}+\frac{\partial(H\bar{V}^2)}{\partial y}=-gH\frac{\partial\zeta}{\partial y}$$

$$+\frac{1}{\rho}\frac{\partial(H\overline{\tau_{yx}})}{\partial x}+\frac{1}{\rho}\frac{\partial(H\overline{\tau_{yy}})}{\partial y}+\frac{\tau_{sy}-\tau_{by}}{\rho}$$

$$+\frac{1}{\rho}\frac{\partial}{\partial x}\int_{-h}^{\zeta}\rho(U-\bar{U})(V-\bar{V})dz+\frac{1}{\rho}\frac{\partial}{\partial y}\int_{-h}^{\zeta}\rho(V-\bar{V})^2dz \tag{10.12}$$

Scalar transport:

Turbulent diffusion

$$\frac{\partial(H\bar{\Phi})}{\partial t}+\frac{\partial(H\bar{U}\bar{\Phi})}{\partial x}+\frac{\partial(H\bar{V}\bar{\Phi})}{\partial y}=\frac{1}{\rho}\frac{\partial(H\bar{J}_x)}{\partial x}+\frac{1}{\rho}\frac{\partial(H\bar{J}_y)}{\partial y}$$

$$+\frac{q_s}{\rho}+\frac{1}{\rho}\frac{\partial}{\partial x}\int_{-h}^{\zeta}\rho(U-\bar{U})(\Phi-\bar{\Phi})dz+\frac{1}{\rho}\frac{\partial}{\partial y}\int_{-h}^{\zeta}\rho(V-\bar{V})(\Phi-\bar{\Phi})dz$$

Shear flow dispersion (10.13)

10.3 Specialized Model Equations

where $\overline{\tau}_{ij}$= depth-averaged turbulent stress ($-\rho\overline{uv}$) acting in x_i-direction on a face perpendicular to x_j ; τ_b = bottom shear stress; τ_s = surface shear stress; $\overline{J}_i$ = depth-averaged turbulent flux of $\Phi(-\rho\overline{u\phi}\ or\ -\rho\overline{v\phi})$ in direction x_i ; q_s = heat flux through surface

①Buoyancy effects

~ cannot be represented in a depth-averaged model because the hydrodynamic model, (10.10) ~ (10.12), is not coupled to the scalar transport model, (10.13).

10.3 Specialized Model Equations

② Turbulent stresses and diffusion terms

▪ Vertical turbulent transport has been eliminated by the depth-averaging and appear only as bottom stresses, τ_b as well surface stresses, τ_s and as surface flux, q_s.

▪ Horizontal momentum transport by the turbulent motion

~ represented by $\bar{\tau}_{ij}$

~ These terms are often neglected in large water body calculations.

~ A turbulence model is needed when terms are important.

▪ Horizontal mass or heat transport by the turbulent motion

~ represented by $\bar{J}_i$

~ A turbulence model is always needed.

10.3 Specialized Model Equations

③ Dispersion terms

~ have same physical effects as turbulent terms but do not represent turbulent transport

~ due to vertical non-uniformities (variations) of various quantities (velocity, concentration)

~ consequence of the depth-averaging process

~ are very important in unsteady condition and require accurate modeling (Fischer et al., 1979)

[Re 1] **Dispersion stress model**

For open flows in which vertical variations of the velocity components are significant, such as modeling of the secondary currents in channels, models should be incorporated in order to represent the dispersion stress terms.

10.3 Specialized Model Equations

i) Moment of momentum approach

~ use additional equations of moment of momentum equations

~ should solve additional transport equations

$$\frac{\partial \hat{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\frac{q_j \hat{u}_i}{h}\right) + \hat{u}_k \frac{\partial}{\partial x_k}\left(\frac{q_i}{h}\right) = \frac{3}{2}\left[\frac{4\tau_{ij}}{h\rho}\frac{\partial z_m}{\partial x_j} - \frac{4\tau_{iz}}{h\rho} + \frac{2}{h\rho}\tau_{bi}\right]$$

where $\hat{u}_i$ = velocities at the water surface in excess of mean velocity in the x-, y-directions

10.3 Specialized Model Equations

ii) Dispersion stress approach

~ Dispersion stress terms associated with the integration of the products of the fluctuating velocity components are directly calculated by incorporating vertical profiles of both longitudinal and transverse velocities

~ For the vertical profiles of both longitudinal and transverse velocities, several equations can be adopted (Rozovskii, 1961; Kikkawa et al., 1976; de Vriend, 1977; Odgaard, 1986).

Use de Vriend equation, then, the first term (S_{11}) indicates the integration of the products of the discrepancy between the mean and the vertically varying velocity distribution in x-direction

10.3 Specialized Model Equations

$$S_{11} = \frac{1}{h}\int_{H}^{H+h}\left(u_1(z) - u_1\right)^2 dz = \int_0^1 \left(u_1(\zeta) - u_1\right)^2 d\zeta$$

$$= u_1^2\left(\frac{\sqrt{g}}{\kappa C}\right)^2 - 2hu_1U_1\frac{\sqrt{g}}{\kappa C}FF_1 + h^2U_1^2FF_2$$

where

$$FF_1 = \int_0^1 \left(1 + \ln\zeta\right) f_s\left(\zeta\right) d\zeta$$

$$FF_2 = \int_0^1 f_s^2\left(\zeta\right) d\zeta$$

$$f_s\left(\zeta\right) = 2F_1(\zeta) + \frac{\sqrt{g}}{\kappa C}F_2(\zeta) - 2\left(1 - \frac{\sqrt{g}}{\kappa C}\right) f_m(\zeta)$$

$$f_m(\zeta) = 1 + \frac{\sqrt{g}}{\kappa C}\left(1 + \ln\zeta\right)$$

$$F_1\left(\zeta\right) = \int_0^1 \frac{\ln\zeta}{\zeta - 1} d\zeta$$

$$F_2\left(\zeta\right) = \int_0^1 \frac{\ln^2\zeta}{\zeta - 1} d\zeta$$

10.3 Specialized Model Equations

The second term (S_{12}) indicates the integration of the products of the discrepancy in x-, and y-directions

$$S_{12} = S_{21} = \int_0^1 \left(u_1(\zeta) - u_1\right)\left(u_2(\zeta) - u_2\right) d\zeta$$

$$= u_1 u_2 \left(\frac{\sqrt{g}}{\kappa C}\right)^2 - h\left(u_1 U_1 + u_2 U_2\right)\frac{\sqrt{g}}{\kappa C} FF_1 + h^2 U_1 U_2 FF_2$$

The third term (S_{22}) indicates the integration of the products of the discrepancy y-direction

$$S_{22} = \int_0^1 \left(u_2(\zeta) - u_2\right)^2 d\zeta = u_2^2 \left(\frac{\sqrt{g}}{\kappa C}\right)^2 - 2hu_2 U_2 \frac{\sqrt{g}}{\kappa C} FF_1 + h^2 U_2^2 FF_2$$

10.3 Specialized Model Equations

iii) Gradient model → find existing theory

In analogy to eddy viscosity concept (Boussinesq, 1877), assume that the dispersion stresses are proportional to the mean velocity gradients

$$\overline{U_i'U_j'} = \frac{1}{H}\int_{-h}^{\zeta}(U_i - \bar{U}_i)(U_j - \bar{U}_j)dz = \nu_d\left(\frac{\partial \bar{U}_i}{\partial x_j} + \frac{\partial \bar{U}_j}{\partial x_i}\right)$$

where ν_d = dispersion viscosity coefficient

10.3 Specialized Model Equations

[Re 2] **Shear flow dispersion**

In direct analogy to the turbulent diffusion, mass transport by dispersion is assumed to be proportional to the gradient of the transported quantity (**Gradient model**).

$$\overline{U_i'\Phi'} = \frac{1}{H}\int_{-h}^{\zeta}(U_i - \bar{U}_i)(\Phi - \bar{\Phi})dz = \Gamma_d \frac{\partial \bar{\Phi}}{\partial x_i}$$

where Γ_d = dispersive diffusivity of heat or mass

→ dispersion mixing coefficient

10.3 Specialized Model Equations

10.3.3 Two-dimensional vertical plane and width-averaged models

Examples:

- long-wave-affected mixing of water masses with different densities

- salt wedges in seiche

- tide-affected estuaries

- separation regions behind obstacles, sizable vertical motion

Define width-averaged quantities

$$\overline{\overline{U}} = \frac{1}{B(x,z)} \int_{y_1(x,z)}^{y_2(x,z)} U \, dy \qquad (10.14a)$$

$$\overline{\overline{\Phi}} = \frac{1}{B(x,z)} \int_{y_1(x,z)}^{y_2(x,z)} \Phi \, dy \qquad (10.14b)$$

in which B = channel width (local width of the flow)

10.3 Specialized Model Equations

(1) Models for the vertical structure are obtained by width-averaging the original three dimensional eqs.

continuity:

$$\frac{\partial}{\partial x}(B\overline{\overline{U}}) + \frac{\partial}{\partial z}(B\overline{\overline{W}}) = 0 \tag{10.15}$$

x-momentum:

$$\frac{\partial}{\partial t}(B\overline{\overline{U}}) + \frac{\partial}{\partial x}(B\overline{\overline{U^2}}) + \frac{\partial}{\partial z}(B\overline{\overline{WU}}) = -gB\frac{\partial \zeta}{\partial x} - \frac{B}{\rho_0}\frac{\partial p_d}{\partial x}$$

$$+\frac{\tau_{wx}}{\rho_0} + \frac{1}{\rho_0}\frac{\partial}{\partial x}(B\overline{\overline{\tau_{xx}}}) + \frac{1}{\rho_0}\frac{\partial}{\partial z}(B\overline{\overline{\tau_{xz}}})$$

$$\underbrace{+\frac{1}{\rho_0}\frac{\partial}{\partial x}\int_{y_1}^{y_2}\rho(U-\overline{\overline{U}})^2 dy + \frac{1}{\rho_0}\frac{\partial}{\partial z}\int_{y_1}^{y_2}(U-\overline{\overline{U}})(W-\overline{\overline{W}})dy}_{dispersion\ stress}$$

(10.16)

10.3 Specialized Model Equations

z-momentum:

$$+\frac{\overline{\overline{\rho}}-\rho_0}{\rho_0}\zeta B+\frac{\tau_{wz}}{\rho_0}+\frac{1}{\rho_0}\frac{\partial}{\partial x}(B\overline{\overline{\tau}}_{xz})+\frac{1}{\rho_0}\frac{\partial}{\partial z}(B\overline{\overline{\tau}}_{zz})$$

$$\underbrace{+\frac{1}{\rho_0}\frac{\partial}{\partial x}\int_{y_1}^{y_2}\rho(U-\overline{\overline{U}})(W-\overline{\overline{W}})dy+\frac{1}{\rho_0}\frac{\partial}{\partial z}\int_{y_1}^{y_2}\rho(W-\overline{\overline{W}})^2 dy}_{dispersion\ stress} \qquad (10.17)$$

scalar transport :

$$\frac{\partial(B\overline{\overline{\Phi}})}{\partial t}+\frac{\partial(B\overline{\overline{U}}\overline{\overline{\Phi}})}{\partial x}+\frac{\partial(B\overline{\overline{W}}\overline{\overline{\Phi}})}{\partial z}$$

$$=\frac{Bq_s}{\rho_0}+\frac{1}{\rho_0}\frac{\partial(B\overline{\overline{J}}_x)}{\partial x}+\frac{1}{\rho_0}\frac{\partial(B\overline{\overline{J}}_x)}{\partial z}$$

$$\underbrace{+\frac{1}{\rho_0}\frac{\partial}{\partial x}\int_{y_1}^{y_2}\rho(U-\overline{\overline{U}})(\Phi-\overline{\overline{\Phi}})dy+\frac{1}{\rho_0}\frac{\partial}{\partial z}\int_{y_1}^{y_2}\rho(W-\overline{\overline{W}})(\Phi-\overline{\overline{\Phi}})dy}_{dispersion\ mixing} \qquad (10.18)$$

10.3 Specialized Model Equations

where ρ_0 = reference density

p_d =dynamic pressure

~ pressure due to motion and buoyancy forces

= static pressure - reference hydrostatic pressure

(2) kinematic free surface condition

$$\frac{\partial \zeta}{\partial t} + \overline{\overline{U}} \frac{\partial \zeta}{\partial x} - \overline{\overline{W}} = 0 \qquad (10.19)$$

(3) dispersion terms

~ due to lateral non-uniformities of the flow quantities

10.3 Specialized Model Equations

(4) Further simplification

Replace *z*-momentum Eq. by hydrostatic pressure assumption

$$\frac{\partial p_d}{\partial z} = (\overline{\overline{\rho}} - \rho_0) g \qquad (10.20)$$

Replace $\frac{\partial p_d}{\partial x}$ in *x*-momentum Eq. as

$$\frac{\partial p_d}{\partial x} = g \frac{\partial}{\partial x} \int_z^{\zeta} (\overline{\overline{\rho}} - \rho_0) dz \qquad (10.21)$$

Integrate continuity Eq. (10.15) over the depth and combine with Eq. (10.19)

$$\frac{\partial \zeta}{\partial t} + \frac{1}{B_s} \frac{\partial}{\partial x} \int_{-h}^{\zeta} B \overline{\overline{U}} dz = 0 \qquad (10.22)$$

10.4 Turbulence-Closure Models

○ Turbulence model

~ represent the turbulence correlations $\overline{u^2}, \overline{uv}, \overline{u\phi}$ etc. in the mean-flow equations in a way that these equations are closed by relating the turbulence correlations to the averaged dependent variables

○ **Hypotheses** must be introduced for the behavior of these correlations which are based on empirical information.

→ Turbulence models always contain empirical constants and functions.

→ Turbulence models do not describe the details of the turbulent fluctuations but only the average effects of these terms on the mean quantities.

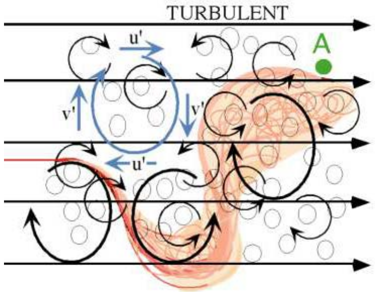
10.4 Turbulence-Closure Models

○ Parameterization of turbulence

~ core of turbulence modeling

~ local state of turbulence and turbulence correlations are assumed to be characterized by only a few parameters.

→ Two important parameters are velocity scale and length scale.



○ Three steps of parameterization

1) choose parameters

2) establish relation between turbulence correlations and chosen parameters

3) determine distribution of these parameters over the flow field.

10.4 Turbulence-Closure Models

[Re] Friction coefficient and mixing coefficient

For 1D flow models, parameterization of turbulence and its effects has been achieved by the use of friction coefficients (Chow, 1956) or mixing coefficients (Fischer et el., 1979).

→ In 1D calculations, the flow is assumed to be fully mixed by the turbulence over any cross section so that the only further effect that turbulence can have is to exert wall friction, which can be accounted for adequately by the use of friction coefficients.

But for multi-dimensional flow models, turbulence has been parameterized by constant or mixing-length-controlled eddy viscosities and diffusivities.

10.4 Turbulence-Closure Models

10.4.1 Basic concepts

(1) Eddy viscosity concept

(1) Boussinesq (1877) introduced **eddy viscosity**, ν_t assuming that, in analogy to the viscous stresses in laminar flow, the turbulent stresses are proportional to the mean velocity gradients.

$$-\overline{u_i u_j} = \nu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij} \qquad (10.23)$$

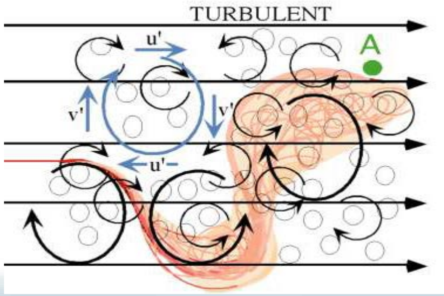
where k = turbulent kinetic energy per unit mass; δ_{ij}= Kronecker delta

δ_{ij} = 1 for $i = j$ and δ_{ij} = 0 for $i \neq j$

$$k = \frac{1}{2}(\overline{u^2} + \overline{v^2} + \overline{w^2}) \qquad (10.24)$$

10.4 Turbulence-Closure Models

- This eddy viscosity concept is based on the close analogy between laminar and turbulent stresses, and has often been criticized as physically unsound.
- This concept has often been found to work well in practice because ν_t can be determined to good approximation in many flows.
- Eq. (10.23) alone does not constitute a turbulence model.
- It provides the frame-work for constructing the turbulence model.
- The turbulence model is to determine the distribution of ν_t.



10.4 Turbulence-Closure Models

- Eddy viscosity, ν_t

~ not a fluid property, and depends on state of the turbulence

~ may vary considerably over the flow field

~ is proportional to a velocity scale $\hat{V}$, and a length scale L

$$\nu_t \propto \hat{V} L \qquad (10.25)$$

→ it is actually the distribution of the velocity and length scales that can be approximated reasonably well in many flows.

10.4 Turbulence-Closure Models

(2) Eddy diffusivity concept

In direct analogy to the turbulent momentum transport, the turbulent heat or mass transport is assumed to be proportional to the gradient of the transported quantity,

$$-\overline{u_i \phi} = \Gamma_t \frac{\partial \Phi}{\partial x_i} \qquad (10.26)$$

where Γ_t = eddy (turbulent) diffusivity of heat or mass

10.4 Turbulence-Closure Models

- Eddy diffusivity, Γ_t

~ is not a fluid property, like the eddy viscosity, and depends on state of the turbulence.

~ depends in general on the direction of the heat or mass flux → anisotropic

- Relation between eddy viscosity and eddy diffusivity

→ use turbulent Prandtl (heat) or Schmidt number (mass), σ_t

$$\Gamma_t = \frac{\nu_t}{\sigma_t} \tag{10.27}$$

where σ_t ~ is assumed to be constant, is usually less than unity

$$\Gamma_t \geq \nu_t$$

10.4 Turbulence-Closure Models

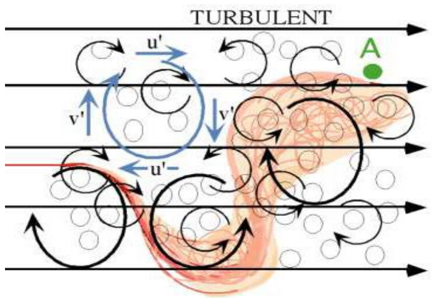
10.4.2 Types of turbulence models

1) Classification based on the use of eddy viscosity concept

Classification of turbulence model would be according to whether the models use the eddy viscosity concept.

① Eddy viscosity model

② Non- eddy viscosity model: Bradshaw et al.'s Model, Reynolds-stress equations



2) Classification based on the use of transport equations

① No transport model

• These models do not involve transport equations for turbulence quantities

• These models assume that the turbulence is dissipated by viscous action at the point where it is generated by shear

→ There is no transport of turbulence over the flow field.

10.4 Turbulence-Closure Models

② Transport model

- These models employ transport equations for quantities characterizing the turbulence in order to account for the transport of turbulence in space and time.
- These models are adequate in cases where the status of turbulence at a point is influenced by the turbulence generation somewhere else in the flow or by the generation at previous times (history effects).
- These equations, similar to the mass/heat transport equation, contain terms representing both advective transport by mean motion and the diffusive transport by the turbulent motion

10.4 Turbulence-Closure Models

3) Classification based on the number of transport equations

It is customary to classify turbulence models according to the number of transport equations used for turbulence parameters.

① Zero-Equation Models

- Constant eddy viscosity (diffusivity) model
- Mixing-length model
- Free-shear-layer model

② One-Equation Models

- k equation model
- Bradshaw et al.'s model: non-eddy viscosity model

10.4 Turbulence-Closure Models

③ Two-Equation Models

- k-ε model

- k-l model

③ Turbulent Stress/Flux-Equation Models

- Reynolds-stress equations

- Algebraic stress/flux models

~ employ transport equations for the individual stresses $\overline{u_i u_j}$

~ non-eddy viscosity model

10.4 Turbulence-Closure Models

10.4.3 Zero-equation models

~ do not involve transport equations for turbulence quantities

~ assume that the turbulence is dissipated by viscous action at the point where it is generated by shear

~ there is no transport of turbulence over the flow field

~ employ the eddy viscosity concept

~ specify the eddy viscosity from experiments, by trial and error, through empirical formulae, by relating it to the mean-velocity distribution

10.4 Turbulence-Closure Models

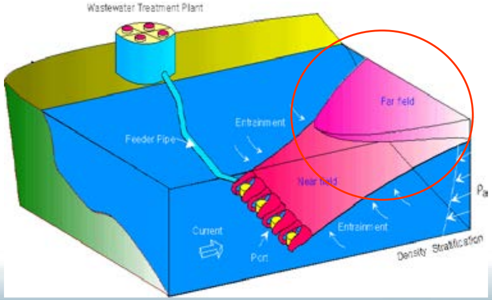
(1) Eddy viscosity (diffusivity) model

~ the simplest turbulence model

~ used for large water bodies in which the turbulence terms in the momentum equations are unimportant

~ use constant eddy viscosity (diffusivity) over the whole flow field

~ The constant eddy diffusivity model is appropriate only for far-field situations where the turbulence is governed by the natural water body and not by local man-made disturbances such as water intake or discharges.



10.4 Turbulence-Closure Models

- **Depth-variable viscosity/diffusivity**

· Open channel flow: ν_t has a nearly parabolic distribution with depth

$$\nu_t = \kappa d u^* (\frac{z}{d}) \left[1 - \frac{z}{d} \right]$$

· Plane jet: ν_t increases with the one-half power of the distance from the origin

- **Depth-averaged viscosity/diffusivity**

- Constant eddy viscosity (diffusivity) concept has its greatest importance in depth average calculation where only horizontal transport is considered.

10.4 Turbulence-Closure Models

- Constant eddy viscosity model is used in the depth-averaged model in which vertical momentum transport is not important, and heat and mass transfer cannot be separated from dispersion effect due to vertical non-uniformity

[Re] The vertical transport of momentum is represented by the bottom shear.

▪ When turbulences are mainly bed-generated, as in the channel flow, the depth-mean diffusivity for the horizontal transport is given as

$$\bar{\Gamma} = C\,hu^{*}$$

where h = water depth; u^* = friction velocity;

C = empirical constant ~ 0.135 for wide laboratory channels

10.4 Turbulence-Closure Models

[Re] Mixing coefficients for 3D transport model

$$\frac{\partial c}{\partial t}+u\frac{\partial c}{\partial x}+v\frac{\partial c}{\partial y}+w\frac{\partial c}{\partial z}=\frac{\partial}{\partial x}(\varepsilon_x\frac{\partial c}{\partial x})+\frac{\partial}{\partial y}(\varepsilon_y\frac{\partial c}{\partial y})+\frac{\partial}{\partial z}(\varepsilon_z\frac{\partial c}{\partial z})$$

Turbulent diffusion coefficients

$$\varepsilon_x=\varepsilon_l=0.15du^*$$

$$\varepsilon_y=\varepsilon_t=0.15du^*$$

$$\varepsilon_z=\varepsilon_v=0.067du^*$$

10.4 Turbulence-Closure Models

- Mixing coefficients for 2D model

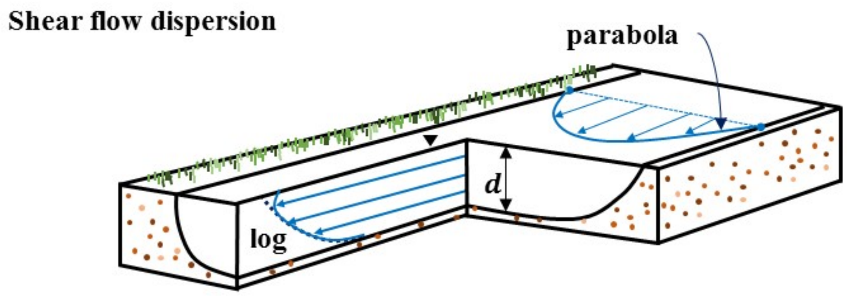
Depth-averaged 2D transport model is

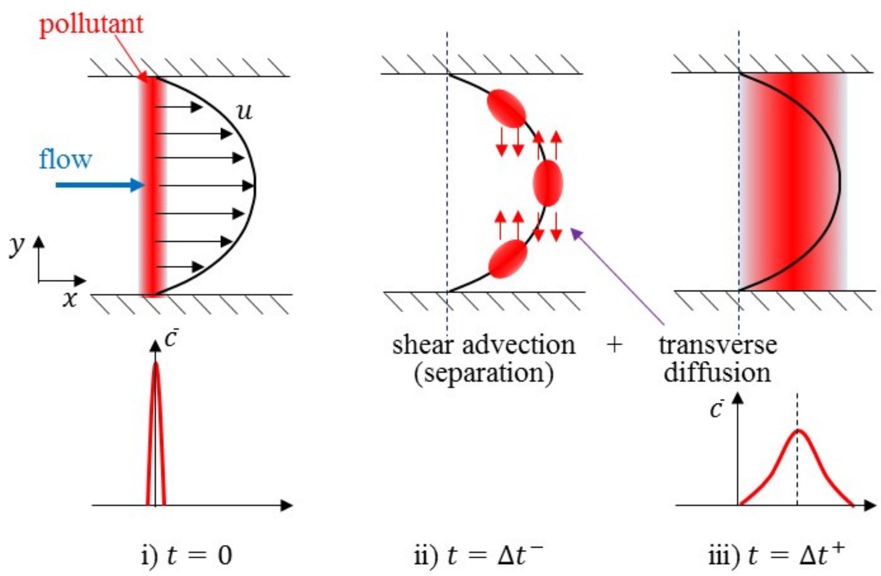
$$\frac{\partial \overline{c}}{\partial t} + \overline{u}\frac{\partial \overline{c}}{\partial x} + \overline{v}\frac{\partial \overline{c}}{\partial z} = D_L \frac{\partial^2 \overline{c}}{\partial x^2} + D_T \frac{\partial^2 \overline{c}}{\partial z^2}$$

The depth-mean diffusivities account for both turbulent transport and the dispersive transport due to vertical non-uniformities of velocity.

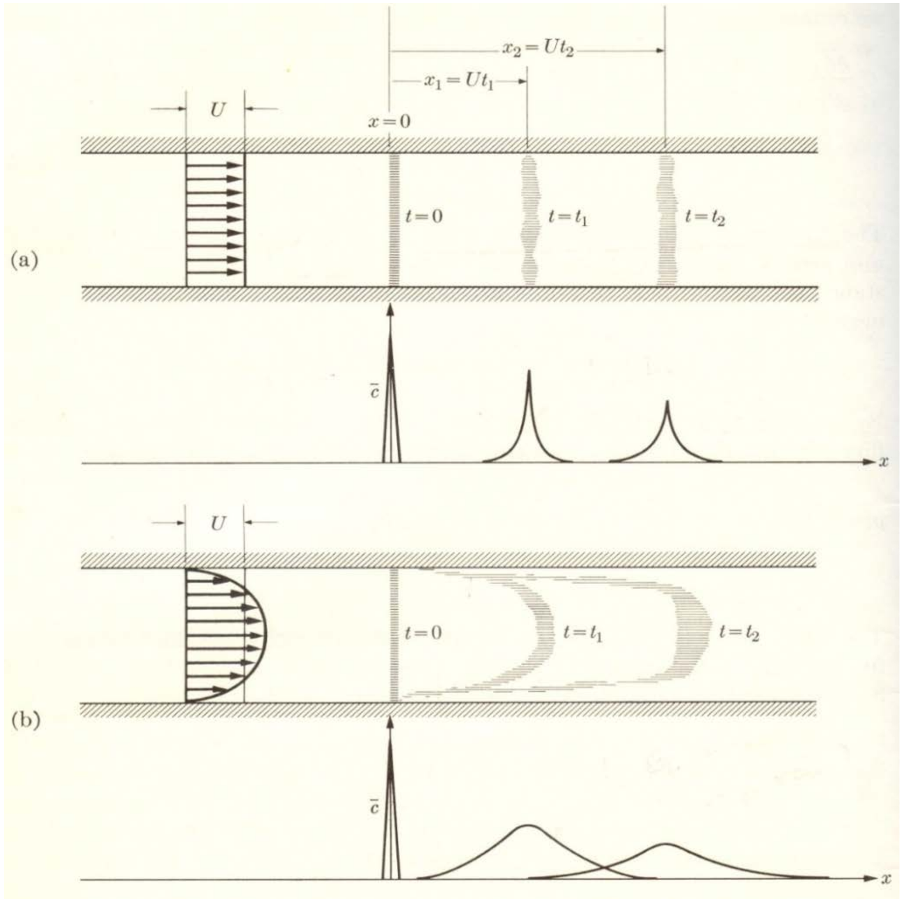
→ Mixing coefficients = dispersion coefficient + turbulent diffusion coefficients

10.4 Turbulence-Closure Models





10.4 Turbulence-Closure Models



Turbulent diffusion in uniform velocity flow vs.
Shear dispersion due to non-uniform velocity distribution (Daily and Harlemann, 1966)

10.4 Turbulence-Closure Models

$$D_L = D_l + \varepsilon_\ell$$

$$\varepsilon_l = 0.15du^*$$

$$D_l = 60du^* \approx 400\varepsilon_l$$

[Re] Elder's formula based on logarithmic velocity distribution (1959)

$$D_l = 5.93du^* \approx 40\varepsilon_l$$

$$D_T = D_t + \varepsilon_t$$

$$\varepsilon_t = 0.15du^*$$

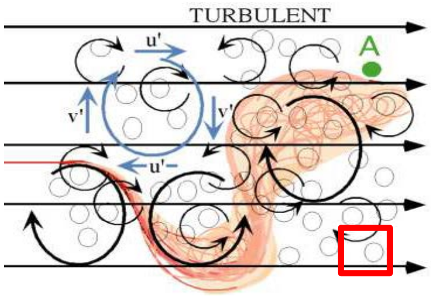
$$\frac{D_t}{du^*} = 0.029\left(\frac{\overline{u}}{u^*}\right)^{0.463}\left(\frac{W}{d}\right)^{0.299} S_n^{\ 0.733}$$

$$\frac{D_t}{du^*} = 0.3 \sim 3.0 = (2 \sim 20)\varepsilon_t$$

10.4 Turbulence-Closure Models

▪ **Mixing coefficients in numerical model**

· In numerical calculations of large water bodies, additional processes are represented by the diffusivity.



1) Sub-grid advection

Owing to computer limitations, the numerical grid of the numerical calculations cannot be made so fine as to obtain grid-independent solutions.

→ All advective motions smaller than the mesh size, such as in small recirculation zones, cannot be resolved. Thus, their contribution to the transport must be accounted for by the diffusivity.

10.4 Turbulence-Closure Models

2) Numerical diffusion (Truncation error)

The approximation of the differential equations by difference equations introduces errors which act to smooth out variations of the dependent variables and thus effectively increase the diffusivity.

→ This numerical diffusion is larger for coarser grids.

· An effective diffusivity accounts for turbulent transport, numerical diffusion, sub-grid scale motions, and dispersion (in the case of depth-average calculations).

→ The choice of a suitable mixing coefficient (D_{MT}) is usually not a turbulence model problem but a matter of numerical model calibration.

For 2D model,

$$D_{MT} = D_t + \varepsilon_t + \varepsilon_{sgm} - \varepsilon_{nd}$$

10.4 Turbulence-Closure Models

(2) Mixing-length model

- Application:

For near-field problems involving discharge jets, wakes, and the vicinity of banks and structures, assumption of a constant eddy viscosity is not sufficient.

→ distribution of ν_t over the flow field should be determined

- Prandtl's mixing-length hypothesis (Prandtl, 1925)

Prandtl assumed that eddy viscosity ν_t is proportional to a mean representation of the fluctuating velocity $\hat{V}$ and a mixing-length l_m.

$$\nu_t \propto \hat{V} l_m \qquad \text{(A)}$$

10.4 Turbulence-Closure Models

Considering shear layers with only one significant turbulent stress ($\overline{uv}$) and velocity gradient , $\partial U / \partial z$ he postulated

$$\hat{V} = l_m \frac{\partial U}{\partial z} \qquad \text{(B)}$$

Combine (A) and (B)

$$\nu_t = l_m^{\ 2} \left| \frac{\partial U}{\partial z} \right| \qquad (10.28)$$

→ The eddy viscosity is related directly to the local mean velocity gradient.

→ Therefore, the mixing length hypothesis involves a single parameter that needs empirical specification; the mixing length l_m .

(A)

10.4 Turbulence-Closure Models

- Mixing length

i) Boundary-layer flows along walls:

① Near-wall region

$$l_m = \kappa z$$

where κ = von Karman constant (0.4)

② Outer region

$$l_m \propto \delta$$

where δ = boundary layer thickness

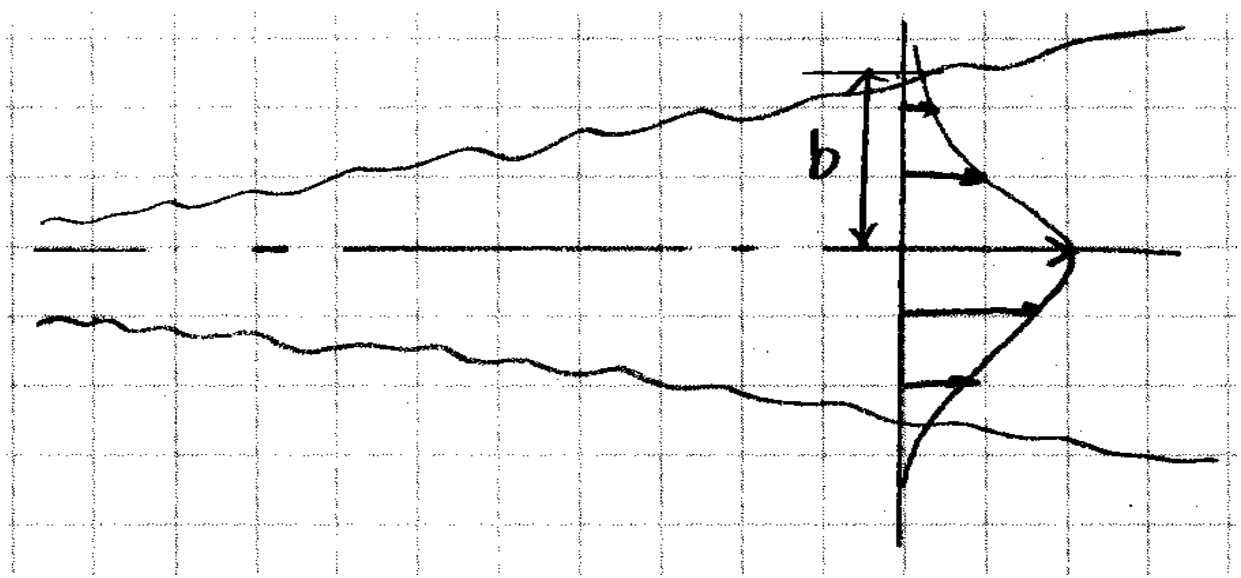
ii) Free shear flows: mixing layers, jets, wakes

$$l_m \propto b$$

where b = local shear-layer width

10.4 Turbulence-Closure Models

	Plane mixing layer	Plane jet	Round jet	Radial jet	Plane wake
$\frac{l_m}{b}$	0.07	0.09	0.075	0.125	0.16



10.4 Turbulence-Closure Models

- Effect of Buoyancy

~ Buoyancy forces acting on stratified fluid layers have a strong effect on the vertical turbulent transport of momentum and heat or mass

→ eddy viscosity relations for vertical transport must be modified by introducing a Richardson number correction

Munk-Anderson (1948) relation

$$\nu_{tz} = (\nu_{tz})_0 (1 + 10R_i)^{-0.5} \qquad (10.29a)$$

$$\Gamma_{tz} = \left(\Gamma_{tz}\right)_0 \left(1 + 3.3R_i\right)^{-1.5} \qquad (10.29b)$$

10.4 Turbulence-Closure Models

$$\frac{l_m}{l_{m_0}} = 1 - \beta_1 R_i \ , \ R_i > 0 \text{ (stable stratification)} \qquad (10.30a)$$

$$l_m / l_{m_0} = \left(1 - \beta_2 R_i\right)^{-1/4} \ , \ R_i < 0 \text{ (unstable stratification)} \qquad (10.30b)$$

where $\beta_1 \approx 7, \quad \beta_2 \approx 14$

Subscript 0 refers to values during unstratified conditions ($R_i = 0$)

Define gradient local Richardson number R_i as

$$R_i = -\frac{g}{\rho} \frac{\partial \rho / \partial z}{\left(\frac{\partial U}{\partial z}\right)^2} \qquad (10.31)$$

~ ratio of gravity to inertial forces

10.4 Turbulence-Closure Models

- Limitation of mixing length model

1) The mixing length model has been applied mainly to two-dimensional shear-flows with only one significant velocity gradient.

2)Mixing-length distribution is empirical and rather problem-dependent.

→ model lacks universality

3) Close link of eddy viscosity (diffusivity) with velocity gradient, i.e. ν_t = 0 when $\frac{\partial U_i}{\partial x_i} = 0$, implies that this model is based on the assumption of local equilibrium of turbulence.

10.4 Turbulence-Closure Models

[Re] Local equilibrium of turbulence

~ Turbulence is locally dissipated by viscous action at the same rate as it is produced by shear.

→ Transport and history effects are neglected (turbulence generation at previous times).

→ This model is not suitable when these effects are important as is the case in rapidly developing flows, recirculating flows and also in unsteady flows.

- Mixing length model for general flows

$$\nu_t = l_m^{\,2} \left[\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \frac{\partial U_i}{\partial x_j} \right]^{\frac{1}{2}} \tag{10.32}$$

~ very difficult to specify the distribution of l_m in complex flow

10.4 Turbulence-Closure Models

- l_m in general duct flows (Buleev, 1962)

$$l_m = \kappa \frac{1}{\pi} \int_D \frac{1}{\delta} d\Omega$$

where δ = distance of the point at which l_m is to be determined from wall along direction Ω; D = integration domain (= cross section of the duct)

- Heat and mass transfer

The mixing-length hypothesis is also used in heat and mass transfer calculations.

$$\Gamma_t = \frac{\nu_t}{\sigma_t} = \frac{1}{\sigma_t} l_m^{\,2} \left| \frac{\partial U}{\partial z} \right| \qquad (10.33)$$

10.4 Turbulence-Closure Models

where σ_t = turbulent Prandtl (Schmidt) number

$$\begin{cases} 0.9 \text{ in near-wall flows} \\ 0.5 \text{ in plane jets and mixing layers} \\ 0.7 \text{ in round jets} \end{cases}$$

• Buoyancy effect on σ_t

→ Munk-Anderson formula

$$\frac{\sigma_t}{\sigma_{t_0}} = \frac{(1 + 3.33R_i)^{1.5}}{(1 + 10R_i)^{0.5}}$$

10.4 Turbulence-Closure Models

• Shortcomings of mixing-length model for heat and mass transport

i) ν_t and Γ_t vanish whenever the velocity gradient is zero.

[Ex] For pipes and channels,

In reality, ν_t ⓐ centerline $\approx 0.8(\nu_t)_{\max}$

However, $\dfrac{\partial U}{\partial z} = 0$ ⓐ centerline $\rightarrow \nu_t = \Gamma_t = 0$

ii) The mixing-length model implies that turbulence is in a state of local equilibrium.

→ Thus, this model is unable to account for transport of turbulence quantities.

10.4 Turbulence-Closure Models

(3) Prandtl's free-shear-layer model

Prandtl (1942) proposed a simpler model applicable only to free shear layers (mixing layers, jets, wakes).

$$l_m \propto \delta$$

$$\hat{V} \propto \left| U_{\max} - U_{\min} \right|$$

$$\nu_t = C\delta \left| U_{\max} - U_{\min} \right| \quad (10.34)$$

Table 10.1 Values of empirical constant *C*

Plane mixing layers	Plane jet	Round jet	Radial jet	Plane wake
0.01	0.014	0.01	0.019	0.026

10.4 Turbulence-Closure Models

10.4.4 One-equation models

- This model accounts for transport or history effects (time-rate change) of turbulence quantities by solving differential transport equations.

- One-equation models determine the fluctuating velocity scale from a transport equation rather than the direct link between this scale and the mean velocity gradients.

(1) k-equation Model

Velocity fluctuations are to be characterized by $\sqrt{k}$ where k is the **turbulent kinetic energy per unit mass** defined as

$$k = \frac{1}{2}(\overline{u^2} + \overline{v^2} + \overline{w^2})$$

10.4 Turbulence-Closure Models

- Eddy viscosity ν_t

$$\nu_t \propto \hat{V} L$$

$$\nu_t = c_\mu' \sqrt{k} L$$ → Kolmogorov-Prandtl equation (10.35)

in which c_μ' = empirical constant.

- Turbulent Kinetic Energy (TKE) equation

~ Exact form can be derived from the Navier-Stokes equation.

~ Exact equation contains certain higher-order correlations which must be approximated by models in order to achieve a closure of the equations.

10.4 Turbulence-Closure Models

For high Reynolds number, this equation reads

$$\frac{\partial k}{\partial t} + U_i \frac{\partial k}{\partial x_i} = -\frac{\partial}{\partial x_i}\left[\overline{u_i\left(\frac{u_j u_j}{2}+\frac{p}{\rho}\right)}\right] - \overline{u_i u_j}\frac{\partial U_i}{\partial x_j}$$

new unknown correlations

rate of change of k	advective transport due to mean motion	diffusive transport due to velocity and pressure fluctuations	production by turbulent shear stress = P

Mean velocity

$$-\beta g_i \overline{u_i \phi} - \nu \overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}} \quad (10.36)$$

buoyant production / destruction due to buoyancy force = G	**viscous dissipation** into heat = ε

10.4 Turbulence-Closure Models

P = transfer of kinetic energy from the mean motion to the turbulent motion (large scale eddies)

G = exchange between the turbulent kinetic energy k and potential energy

~ negative for stable stratification (k is reduced, turbulence is damped while potential energy of the system increases)

~ positive for unstable stratification (k is produced at the expense of the potential energy)

ε = transfers kinetic energy into internal energy of the fluid ~ negative (sink)

10.4 Turbulence-Closure Models

• Concepts used for *k*-equation model

①Energy cascade

~ Kinetic energy extracted from mean motion is first fed into large scale turbulent motion.

→ This energy is then passed on to smaller and smaller eddies by vortex stretching (vortex trail, vortex street) until viscous force become active and dissipate the energy.

②Anisotropy vs. Isotropy

~ Large-scale turbulences are anisotropic, whereas small-scale turbulences are isotropic.

Because of interaction between large-scale turbulent motion and mean flow, the large-scale turbulent motion depends strongly on the boundary conditions.

10.4 Turbulence-Closure Models

During the energy cascade process, energy is passed on to smaller eddies by vortex stretching.

→ The direction sensitivity is diminished.

→ small-scale turbulence tend to be isotropic

▪ Modeled form of the k-equation

~ The exact k-equation contains new unknown correlations.

→ To obtain a closed set of equations, model assumptions must be introduced for these terms.

i) Diffusion term

~ In analogy to the diffusion expression for the scalar quantity ϕ, the diffusion flux of k is assumed proportional to the gradient k.

10.4 Turbulence-Closure Models

$$-\overline{u_i\left(\frac{u_j u_j}{2}+\frac{p}{\rho}\right)}=\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i}$$

in which σ_k = empirical diffusion constant.

ii) Reynolds stress

$$-\overline{u_i u_j}=\nu_t\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)$$

iii) heat (mass) flux

$$-\overline{u_i\phi}=\frac{\nu_t}{\sigma_t}\frac{\partial \Phi}{\partial x_i}$$

in which σ_t = turbulent Prandtl or Schmidt number

10.4 Turbulence-Closure Models

iv) viscous dissipation

$$\varepsilon = c_D \frac{k^{3/2}}{L}$$

in which c_D= empirical constant.

Substituting i) ~ iv) into exact k-equation yields

$$\frac{\partial k}{\partial t} + U_i \frac{\partial k}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i}\right) + \nu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} + \beta g_i \frac{\nu_t}{\sigma_t}\frac{\partial \Phi}{\partial x_i} - c_D \frac{k^{3/2}}{L} \qquad (10.37)$$

~ This model is restricted to high Reynolds number flows; $c_\mu{}' c_D \approx 0.08 \; and \; \sigma_k \approx 1$

~ For low Reynolds number flows, a viscous diffusion term should be accounted for and empirical constants are functions of the turbulent Reynolds number, $\mathrm{Re}_f = \sqrt{k}L / \nu$

10.4 Turbulence-Closure Models

- Special case of **local equilibrium**

~ Turbulence is said to be in local equilibrium when the rate of change, advection and diffusion terms in Eq. (10.37) are zero.

→ Then, production of k is equal to dissipation.

For non-buoyant shear layers,

$$\nu_t \left(\frac{\partial U}{\partial z}\right)^2 = c_D \frac{k^{3/2}}{L}$$

$$k^{1/2} = \left[\nu_t \left(\frac{\partial U}{\partial z}\right)^2 \frac{L}{c_D}\right]^{1/3} \quad (1)$$

10.4 Turbulence-Closure Models

Substitute (1) into Kolmogorov-Prandtl expression [$\nu_t = c_u{}' \sqrt{k} L$]

$$\nu_t = \left(\frac{c_\mu{}'^3}{c_D} \right)^{\frac{1}{2}} L^2 \left| \frac{\partial U}{\partial z} \right|$$

Set l_m = mixing length = $\left(\frac{c'\mu^3}{c_D} \right)^{\frac{1}{4}} L$,

Then

$$\nu_t = l_m{}^2 \left| \frac{\partial U}{\partial z} \right|$$

→ mixing-length model

10.4 Turbulence-Closure Models

- Length-scale determination

~ Because the length scale L appears both in Kolmogorov-Prandtl equation and in dissipation term of the k-equation, this must be specified empirically.

~ In most models, L is determined from simple empirical relations similar to those for the mixing length, l_m.

→ Launder and Spalding (1972) for estuary; Smith and Takhar (1977) for open-channel

- Bobyleva et al. (1965) 's length scale formula

~ similar to von Kaman's formula

$$L = \kappa \frac{\Psi}{\partial \Psi / \partial z}$$

10.4 Turbulence-Closure Models

where κ = von Karman's const.

$$\Psi = \frac{k^{\frac{1}{2}}}{L}$$ = turbulence parameter

~ applicable to flows where turbulent transport is mainly in vertical direction

When the turbulence is in local equilibrium in the shear layer,

$$\frac{k^{\frac{1}{2}}}{L} \propto \frac{\partial U}{\partial z}$$

$$L = l_m = \kappa \left| \frac{\partial U / \partial z}{\partial^2 U / \partial z^2} \right|$$ → von Karman's formula

10.4 Turbulence-Closure Models

(2) Bradshaw et al.'s Model

→ $\overline{uv}$ -equation model

~ This model does not employ the eddy viscosity concept.

~ It solves a transport equation for the shear stress $\overline{uv}$.

For 2D wall boundary layers, relation between k (normal stress) and $\overline{uv}$ (shear stress) is given as

$$\frac{\overline{uv}}{k} = a_1 \approx const \approx 0.3$$ (experiment)

Convert k-equation to $\overline{uv}$-equation for steady flows

10.4 Turbulence-Closure Models

$$\underbrace{U\frac{\partial\frac{\overline{uv}}{a_1}}{\partial x}+V\frac{\partial\frac{\overline{uv}}{a_1}}{\partial y}}_{\text{advection}} = \underbrace{-\frac{\partial}{\partial y}\left[G\overline{uv}\left(\overline{uv}_{\max}\right)^{\frac{1}{2}}\right]}_{\text{diffusion}} - \underbrace{\overline{uv}\frac{\partial U}{\partial y}}_{\text{production}} - \underbrace{\frac{\overline{uv}^{3/2}}{L}}_{\text{dissipation}} \qquad (10.38)$$

in which $G=\left(\frac{\overline{uv}_{\max}}{U_\infty^2}\right)^{\frac{1}{2}} f_1\left(\frac{y}{\delta}\right)$

$L=f_2\left(\frac{y}{\delta}\right)\delta$ → empirical

→ For the diffusion flux of $\overline{uv}$, the gradient-diffusion concept is not employed. Instead the diffusion flux is assumed to proportional to a bulk velocity, $\left(\overline{uv}_{\max}\right)^{\frac{1}{2}}$.

10.4 Turbulence-Closure Models

- Transport of heat and mass

- Find eddy viscosity (ν_t) or the shear stress ($\overline{u_i u_j}$) using one-equation model.

$$\Gamma_t = \frac{\nu_t}{\sigma_t}$$

- Use gradient-diffusion concept to calculate heat and mass transfer by turbulence

$$-\overline{u_i \phi} = \Gamma_t \frac{\partial \Phi}{\partial x_i}$$

- Solve scalar transport equation

10.4 Turbulence-Closure Models

• Advantage

① One-equation models can account for advective and diffusive transport and for history effects on the turbulent velocity scale.

→ One-equation model is superior to the mixing-length model when these effects are important

Examples: non-equilibrium shear layers with rapidly changing free stream conditions, abrupt changes in the boundary conditions, shear layers in estuary with velocity reversal, heat and mass exchange in area with vanishing velocity gradients

② Buoyancy term appears automatically in the *k*-equation model.

10.4 Turbulence-Closure Models

• Disadvantage

① The application is restricted to shear-layer situation not applicable to more complex flows.

②The empirical formulas for calculating length scale in general flows so far been tested insufficiently.

→ For calculating general flows, the trend has been to move on to two-equation models which determine the length scale from a transport equation.

10.4 Turbulence-Closure Models

10.4.5 Two-equation models

(1) Types of two-equation models

Length scale L characterizing the size of the large, energy-containing eddies is also subject to transport processes in a similar manner to the kinetic energy k.

① Eddies generated by a grid are advected downstream so that their size at any station depends on their initial size. → history effect

② Dissipation destroys the small eddies and thus effectively increases the eddy size.

③ Vortex stretching connected with the energy cascade reduces the eddy size.

→ The balance of all these processes can be expressed in a transport model for L.

10.4 Turbulence-Closure Models

- Length scale equations

Length scale transport equation

The general form is given as

$$Z = k^m l^n$$

Length scale transport equation of which the exact form can be derived from Navier-Stokes eq. is given as

$$\underbrace{\frac{\partial Z}{\partial t}}_{\text{rate of change}} + \underbrace{U_i \frac{\partial Z}{\partial x_i}}_{\text{advection}} = \underbrace{\frac{\partial}{\partial x_i}\left(\frac{\sqrt{k}L}{\sigma_z}\frac{\partial Z}{\partial x_i}\right)}_{\text{diffusion}} + \underbrace{c_{z1}\frac{Z}{k}P}_{\text{production}} - \underbrace{c_{z2} Z \frac{\sqrt{k}}{L}}_{\text{destruction}} + S \tag{10.39}$$

where σ_z , c_{z1} , c_{z2} = empirical constants

10.4 Turbulence-Closure Models

P = production of kinetic energy $\left(= -\overline{u_i u_j} \dfrac{\partial u_i}{\partial x_j} \right)$

S = secondary source term which is important near walls

• Mellor-Yamada (1982) model

Velocity scale – q

Length scale - L

(ii) Energy dissipation rate

Use energy dissipation rate as a combination of length scale and energy

$\varepsilon \propto \dfrac{k^{3/2}}{L}$ Chou (1945), Davidov (1961), Jones & Launder (1972)

10.4 Turbulence-Closure Models

~ The use of the ε–equation has been criticized because the process of dissipation is associated with the small-scale turbulence while it is the length scale L characterizing the large-scale, energy-containing eddies that needs to be determined.

~ However, the amount of energy dissipated is controlled by the energy fed from the large-scale motion through the spectrum to the small-scale motion.

→ ε may be considered a parameter characterizing the large-scale motion.

[Re] Other scales

Dissipation rate: kL Rotta (1968)

Frequency: $k^{\frac{1}{2}}L$ Kolmogorov (1941)

Turbulence vorticity: $\omega = k / L^2$ Spalding (1971), Saffman (1970)

10.4 Turbulence-Closure Models

(2) Standard *k* - ε model

The basic assumption is that the local state of turbulence is characterized by the two parameters k and ε.

ε model works better near walls than other equations.

The ε-equation does not require a near-wall correction term *S*.

The model employs the eddy viscosity/diffusivity concept.

$$\nu_t = c_\mu \frac{k^2}{\varepsilon} \qquad (10.40)$$

$$\Gamma_t = \frac{\nu_t}{\sigma_t} \qquad (10.41)$$

10.4 Turbulence-Closure Models

[Re] Derivation

$$\nu_t = c_\mu ' \sqrt{k} L \qquad \text{(a)}$$

$$\varepsilon = c_D \frac{k^{3/2}}{L} \rightarrow L = c_D \frac{k^{3/2}}{\varepsilon} \qquad \text{(b)}$$

Substitute (b) into (a)

$$\nu_t = c_\mu ' \sqrt{k} \frac{c_D k^{3/2}}{\varepsilon} = c_\mu ' c_D \frac{k^2}{\varepsilon}$$

Set $c_\mu = c_\mu ' c_D$

Then

$$\nu_t = c_\mu \frac{k^2}{\varepsilon}$$

10.4 Turbulence-Closure Models

Exact ε - equation can be derived from N-S equations for fluctuating vorticity.

→ rate of change + advection = diffusion + generation of vorticity due to vortex stretching + viscous destruction of vorticity

→ need model assumptions for diffusion, generation, and destruction terms (diffusion is modeled with gradient assumption).

- Modeled ε-equation

$$\underbrace{\frac{\partial \varepsilon}{\partial t}}_{\text{rate of change}} + \underbrace{U_i \frac{\partial \varepsilon}{\partial x_i}}_{\text{advection}} = \underbrace{\frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_i}\right)}_{\text{diffusion}} + \underbrace{c_{1\varepsilon}\frac{\varepsilon}{k}(P + c_{3\varepsilon}G) - c_{2\varepsilon}\frac{\varepsilon^2}{k}}_{\text{generation-destruction}} \quad (10.42)$$

10.4 Turbulence-Closure Models

where P = stress production of kinetic energy k;

G = buoyancy production of kinetic energy k

- Complete k - ε model

$$\frac{\partial k}{\partial t} + U_i \frac{\partial k}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i}\right) + \nu_t\left(\frac{\partial U_i}{\partial x_i} + \frac{\partial U_j}{\partial x_j}\right)\frac{\partial U_i}{\partial x_j} + \beta g_i \frac{\nu_t}{\sigma_t}\frac{\partial \Phi}{\partial x_i} - \varepsilon \qquad (10.43)$$

$$\frac{\partial \varepsilon}{\partial t} + U_i \frac{\partial \varepsilon}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_i}\right) + c_{1\varepsilon}\frac{\varepsilon}{k}\left(P + c_{3\varepsilon} G\right) - c_{2\varepsilon}\frac{\varepsilon^2}{k} \qquad (10.44)$$

Table 10.3 Values of constants in the model

c_μ	$c_{1\varepsilon}$	$c_{2\varepsilon}$	σ_k	σ_ε	$c_{3\varepsilon}$
0.09	1.44	1.92	1.0	1.3	1 (G>0), 0-0.2 (G<0)

10.4 Turbulence-Closure Models

y-momentum:

$$\frac{\partial(H\bar{V})}{\partial t}+\frac{\partial(H\overline{\bar{U}\bar{V}})}{\partial x}+\frac{\partial(H\bar{V}^2)}{\partial y}=-gH\frac{\partial\zeta}{\partial y}+\frac{1}{\rho}\frac{\partial(H\overline{\tau_{yx}})}{\partial x}+\frac{1}{\rho}\frac{\partial(H\overline{\tau_{yy}})}{\partial y}+\frac{\tau_{sy}-\tau_{by}}{\rho}$$

$$+\frac{1}{\rho}\frac{\partial}{\partial x}\int_{-h}^{\zeta}\rho(U-\bar{U})(V-\bar{V})dz+\frac{1}{\rho}\frac{\partial}{\partial y}\int_{-h}^{\zeta}\rho(V-\bar{V})^2dz \qquad (10.47)$$

Scalar transport:

$$\frac{\partial(H\bar{\Phi})}{\partial t}+\frac{\partial(H\bar{U}\bar{\Phi})}{\partial x}+\frac{\partial(H\bar{V}\bar{\Phi})}{\partial y}=\frac{1}{\rho}\frac{\partial(H\bar{J}_x)}{\partial x}+\frac{1}{\rho}\frac{\partial(H\bar{J}_y)}{\partial y}+\frac{q_s}{\rho}$$

$$+\frac{1}{\rho}\frac{\partial}{\partial x}\int_{-h}^{\zeta}\rho(U-\bar{U})(\Phi-\bar{\Phi})dz+\frac{1}{\rho}\frac{\partial}{\partial y}\int_{-h}^{\zeta}\rho(V-\bar{V})(\Phi-\bar{\Phi})dz \qquad (10.48)$$

10.4 Turbulence-Closure Models

- Turbulence model

Assumptions;

$\overline{\tau}_{ij}$ ~ depth-averaged turbulent stress ($-\rho\overline{uv}$) acting in x_i -direction on a face perpendicular to x_j

$$\frac{\overline{\tau}_{ij}}{\rho} = \tilde{\nu}_t \left(\frac{\partial \overline{U}_i}{\partial x_j} + \frac{\partial \overline{U}_j}{\partial x_i} \right) - \frac{2}{3} \tilde{k} \delta_{ij} \qquad (10.49)$$

$\overline{J}_i$ ~ depth-averaged flux of $\Phi(-\rho\overline{u\phi} \; or \; -\rho\overline{v\phi})$ in direction x_i

$$\frac{\overline{J}_i}{\rho} = \tilde{\Gamma}_t \frac{\partial \overline{\Phi}}{\partial x_i} \qquad (10.50)$$

10.4 Turbulence-Closure Models

with

$$\tilde{\nu}_t = c_\mu \frac{\tilde{k}^2}{\tilde{\varepsilon}} \qquad (10.51)$$

$$\tilde{\Gamma}_t = \frac{\tilde{\nu}_t}{\sigma_t} \qquad (10.52)$$

where $\tilde{k}, \tilde{\varepsilon}$ are depth-averaged values

The variation of $\tilde{k}, \tilde{\varepsilon}$ is determined from the following transport equations

$$\bar{U}\frac{\partial \tilde{k}}{\partial x} + \bar{V}\frac{\partial \tilde{k}}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\tilde{\nu}_t}{\sigma_k}\frac{\partial \tilde{k}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\tilde{\nu}_t}{\sigma_k}\frac{\partial \tilde{k}}{\partial y}\right) + P_h + P_{kV} - \tilde{\varepsilon} \qquad (10.53)$$

$$\bar{U}\frac{\partial \tilde{\varepsilon}}{\partial x} + \bar{V}\frac{\partial \tilde{\varepsilon}}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\tilde{\nu}_t}{\sigma_\varepsilon}\frac{\partial \tilde{\varepsilon}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\tilde{\nu}_t}{\sigma_\varepsilon}\frac{\partial \tilde{\varepsilon}}{\partial y}\right) + c_{1\varepsilon}\frac{\tilde{\varepsilon}}{\tilde{k}}P_h + P_{\varepsilon V} - c_{2\varepsilon}\frac{\tilde{\varepsilon}^2}{\tilde{k}} \qquad (10.54)$$

10.4 Turbulence-Closure Models

where

$$P_h = \tilde{\nu}_t \left[2\left(\frac{\partial \bar{U}}{\partial x}\right)^2 + 2\left(\frac{\partial \bar{V}}{\partial y}\right)^2 + \left(\frac{\partial \bar{U}}{\partial y} + \frac{\partial \bar{V}}{\partial x}\right)^2 \right] \qquad (10.55)$$

→ production of due to interaction of the turbulent stresses with the horizontal mean velocity gradients

$$P_{kV} = c_k \frac{U_*^3}{h} \qquad (10.56)$$

$$P_{\varepsilon V} = c_\varepsilon \frac{U_*^4}{h^2} \qquad (10.57)$$

$$U_* = \sqrt{c_f \left(\bar{U}^2 + \bar{V}^2\right)} \qquad (10.58)$$

$$c_k = \frac{1}{\sqrt{c_f}} \qquad (10.59)$$

$$c_\varepsilon = \frac{1}{\sqrt{e^* \sigma_t}} \frac{c_{2\varepsilon}}{c_f^{3/4}} \sqrt{c_\mu} = 3.6 \frac{c_{2\varepsilon}}{c_f^{3/4}} \sqrt{c_\mu} \qquad (10.60)$$

c_f = friction coefficient

10.4 Turbulence-Closure Models

- P_{kV}, $P_{\varepsilon V}$

~ All terms originating from nonuniformity of vertical profiles are assumed to be absorbed by the additional source terms, P_{kV}, $P_{\varepsilon V}$.

~ stems from the significant vertical velocity gradients near the bottom

~ relatively large turbulence shear stresses in the near-bottom region produce turbulence

~ depends strongly on the bottom roughness, via the friction velocity, U_*

- e^*

$$e^* = \frac{\bar{\Gamma}_t}{U_* h}$$

For wide laboratory flume, $e^* = 0.15$

10.4 Turbulence-Closure Models

For real rivers, $e^* = 0.60$ (Fischer et al., 1979)

→ The diffusivity obtained in real rivers accounts not only for turbulent transport but also for dispersive transport due to vertical non-uniformities of scalar quantities and velocity components.

When significant secondary motions in cross-sectional planes are present, relatively small non-uniformities of temperature or concentration may cause relatively large dispersion contribution to the Φ -Eq.

In natural rivers, secondary motions may arise from large-scale irregularities in the river bed and from rivers bends.

$\sigma_t = 0.5$

10.4 Turbulence-Closure Models

- Assessment of two-equation models

i) Advantage

①Two-equation models account for the transport of the turbulent velocity scale and the length scale.

②Two-equation models are the simplest ones that promise success for those flows for which the length scale cannot be prescribed empirically in an easy way.

Examples: separated flows, complex shear layers

③With efficient solution procedure, the additional solution of the length scale equation is not computationally expensive.

10.4 Turbulence-Closure Models

④ The model is one of the most widely tested and successfully applied turbulence models.

⑤ The depth-averaged version has been applied with success in a number of different calculations of the flow field and pollutant transport processes.

→ This model was found to be particularly suitable for situations involving the interaction of turbulence generated both at the river bed and by the shear layers of discharging jets.

10.4 Turbulence-Closure Models

ii) Limitation of model

① k - ε model uses the same eddy viscosity/diffusivity for all Reynolds stress and heat or mass flux components.

→ The standard $k-\varepsilon$ model assumes an <u>isotropic eddy viscosity/diffusivity</u> and hence constructs a direct relation to one velocity scale $\sqrt{k}$.

But in certain flow situations the assumption of an isotropic eddy viscosity is too crude.

10.4 Turbulence-Closure Models

② It cannot produce the turbulence driven secondary motions in straight open channels.

→ It does not allow any directional influences on the stresses and fluxes, for example, those due to buoyancy forces.

③ In order to allow for the non-isotropic nature of the eddy viscosity/diffusivity, the $k-\varepsilon$ model should be refined by introducing a so called algebraic stress/flux model to replace the simple combined relations, Eq. (10.23) and Eq. (10.41).