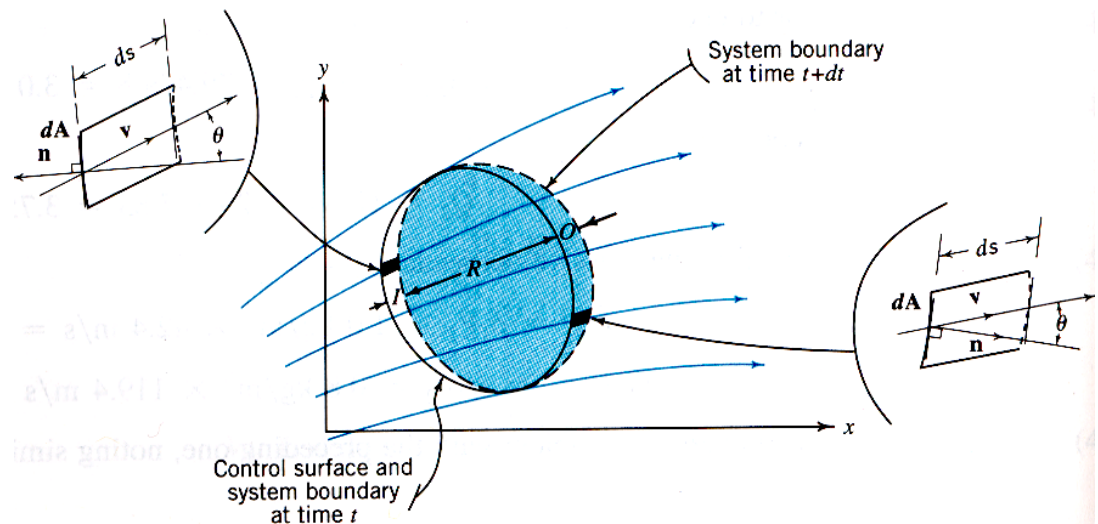


Chapter 4

Continuity, Energy, and Momentum Equations



Chapter 4 Continuity, Energy, and Momentum Equations

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4.1 Conservation of Matter in Homogeneous Fluids

4.2 The General Energy Equation

4.3 Linear Momentum Equation for Finite Control Volumes

4.4 The Moment of Momentum Equation for Finite Control Volumes

Objectives

- Apply finite control volume to get integral form of continuity, energy, and momentum equations
- Compare integral and point form equations
- Derive the simplified equations for continuity, energy, and momentum equations

Particle and Control-Volume Concepts

Infinitesimal elements and control volumes

- Each of the observational laws of mass, heat, and momentum transport may be formulated in the Eulerian sense of focusing attention on a fixed point in space.
- There are two basic method of arriving the Eulerian equation.
 - Material method (Particle approach)
 - Control volume method:
 - ✓ Finite control volume - 유한검사체적
 - ✓ Differential control volume - 미소검사체적

If fluid is considered as a continuum, end results of two methods are identical.

Particle and Control-Volume Concepts

- Material method (Particle approach)
 - Describe flow characteristics at a fixed point (x, y, z) by observing the motion of a material particle of a infinitesimal mass
 - Laws of conservation of mass, momentum, and energy can be stated in the differential form, applicable at a point.
 - Newton's 2nd law

$$d\vec{F} = dm\vec{a}$$

- Material-particle approach is used to develop a stress-strain relationship in Ch. 5.

Particle and Control-Volume Concepts

- **Control volume method**

- ① Finite control volume – arbitrary control volume

- ② Differential (infinitesimal) control volume – parallelepiped control volume

[Re] Control volume

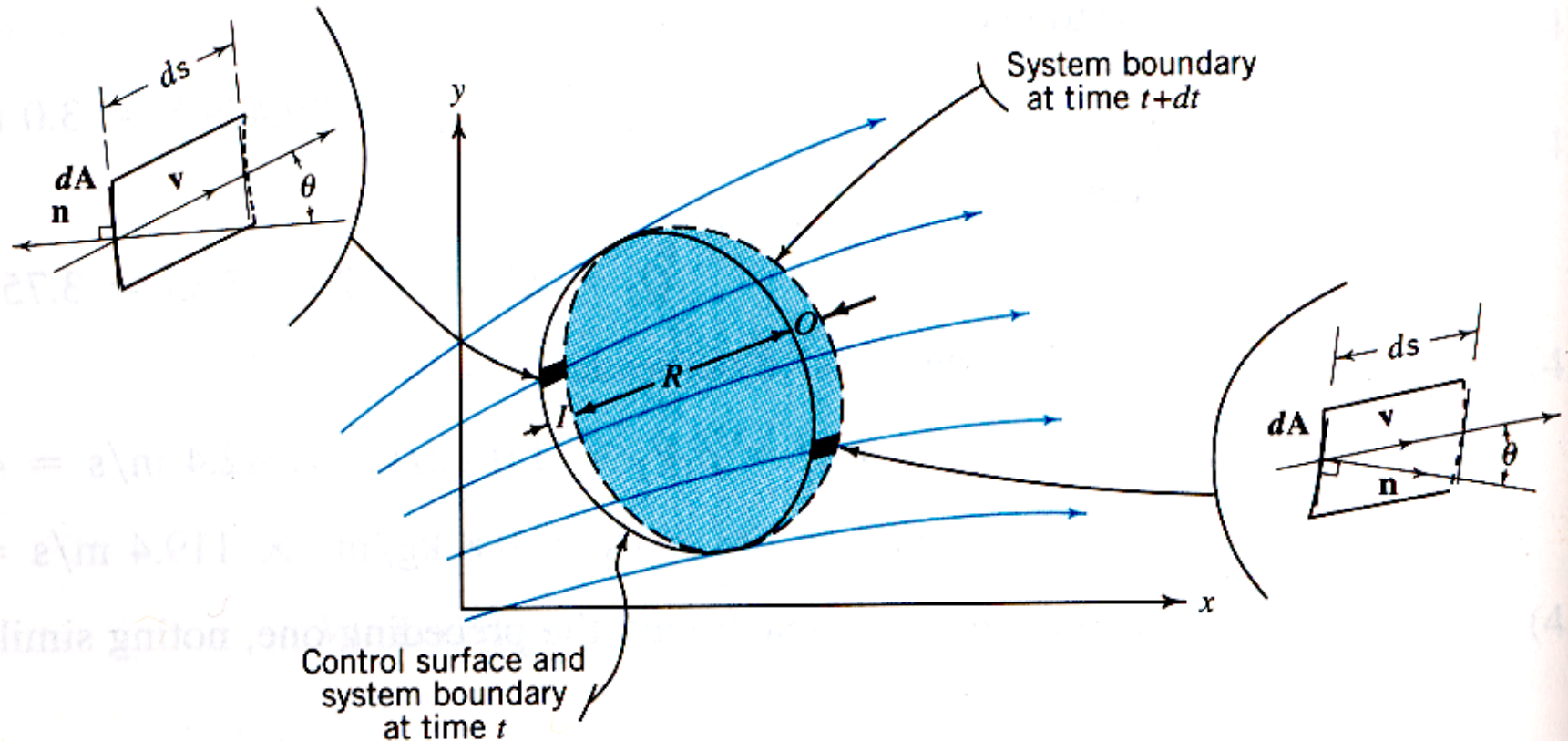
- fixed volume which consists of the same fluid particles and whose bounding surface moves with the fluid

Particle and Control-Volume Concepts

- Finite control volume method
 - Frequently used for 1D analysis (Ch. 4)
 - Gross descriptions of flow
 - Analytical formulation is easier than differential control volume method
 - **Integral form** of equations for conservation of mass, momentum, and energy
 - Continuity equation: conservation of mass

$$\int_{cv} \frac{\partial \rho}{\partial t} dV + \oint_{cs} \rho \vec{q} \cdot d\vec{A} = 0$$

Particle and Control-Volume Concepts



Particle and Control-Volume Concepts

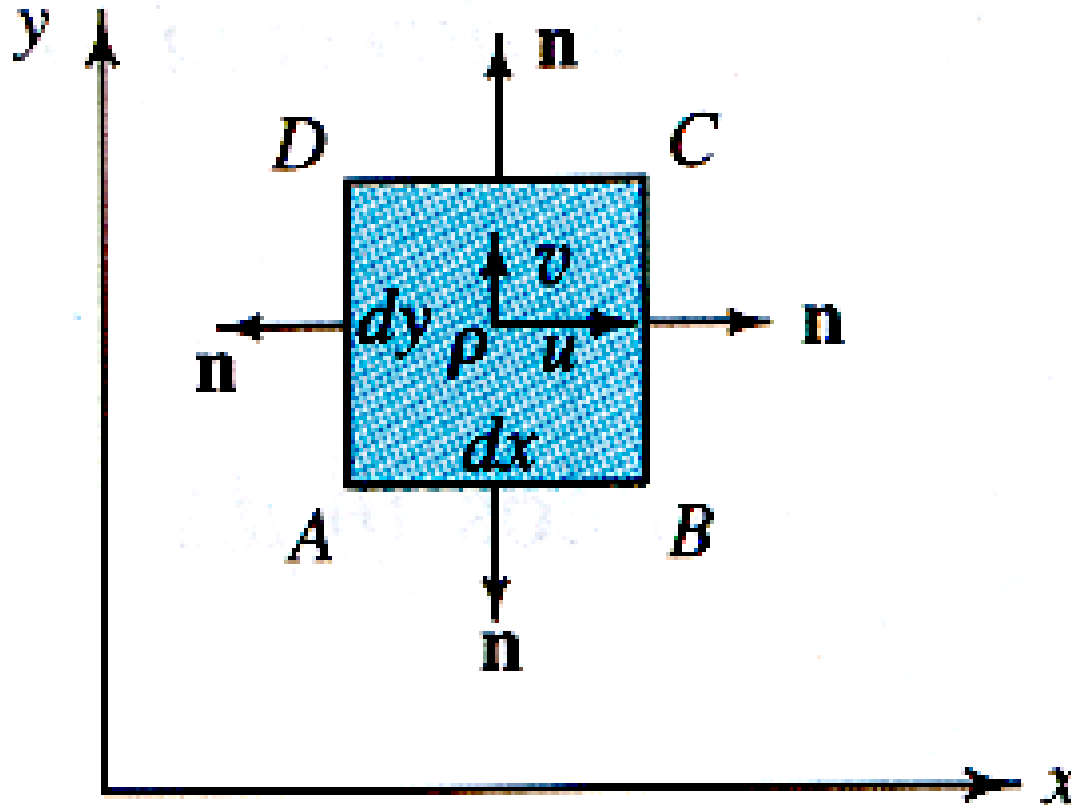
- Differential control volume method
 - Concerned with a fixed differential control volume ($=\Delta x\Delta y\Delta z$) of fluid
 - 2D or 3D analysis (Ch. 6)

$$\Delta \vec{F} = \frac{d}{dt}(\Delta m \vec{q}) = \frac{d}{dt}(\rho \Delta x \Delta y \Delta z \vec{q})$$

– Δx , Δy , Δz become vanishingly small

- Point form of equations for conservation of mass, momentum, and energy

Particle and Control-Volume Concepts



4.1 Conservation of Matter in Homogeneous Fluids

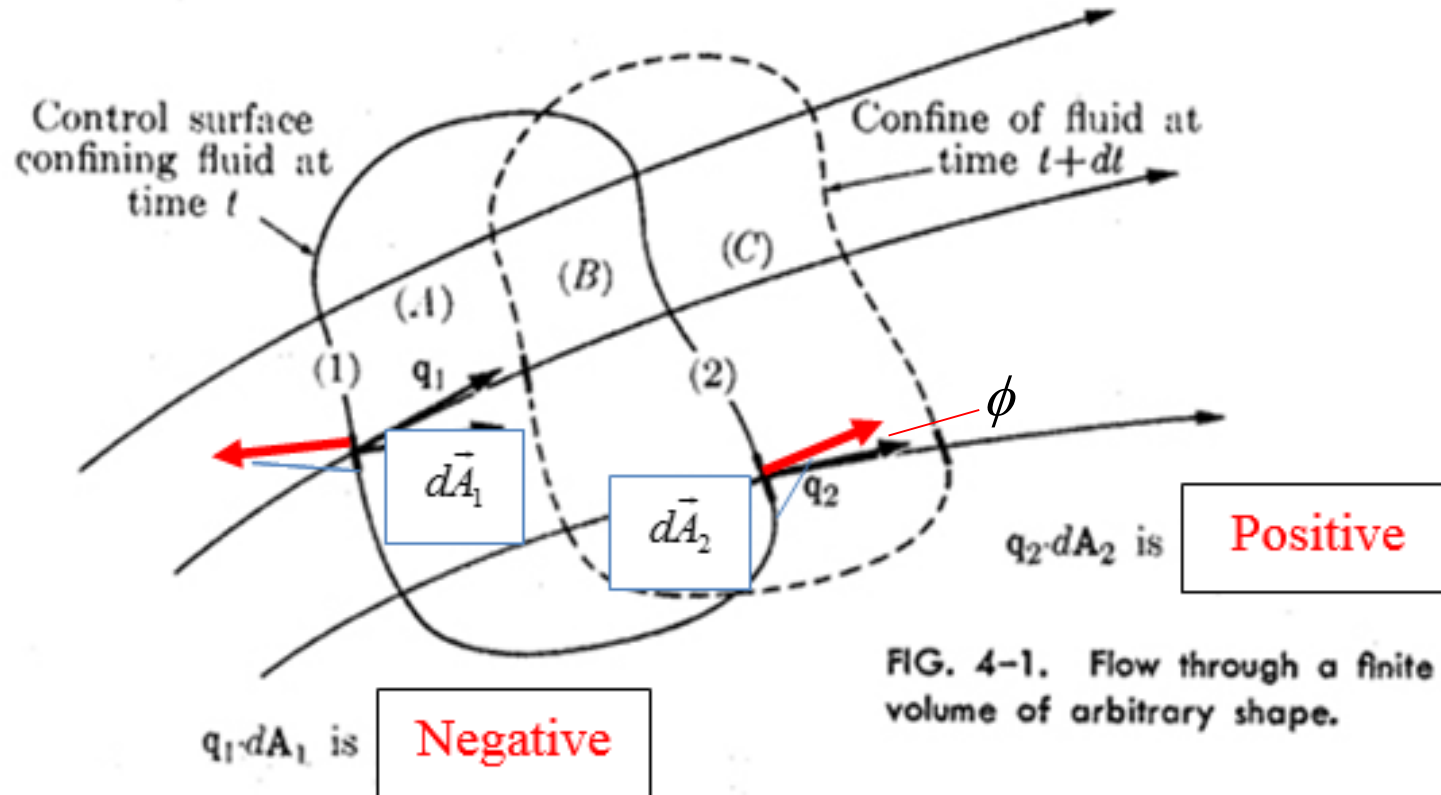


FIG. 4-1. Flow through a finite control volume of arbitrary shape.

4.1 Conservation of Matter in Homogeneous Fluids

4.1.1 Finite control volume method-arbitrary control volume

- Consider an arbitrary control volume
- Although control volume remains fixed, mass of fluid originally enclosed regions (A+B) occupies the volume within the dashed line regions (B+C) after dt .
- Since mass m is conserved:

$$(m_A)_t + (m_B)_t = (m_B)_{t+dt} + (m_C)_{t+dt} \quad (4.1)$$

$$\frac{(m_B)_{t+dt} - (m_B)_t}{dt} = \frac{(m_A)_t - (m_C)_{t+dt}}{dt} \quad (4.2)$$

4.1 Conservation of Matter in Homogeneous Fluids

- LHS of Eq. (4.2) = time rate of change of mass in the original control volume in the limit

$$\frac{(m_B)_{t+dt} - (m_B)_t}{dt} \approx \frac{\partial(m_B)}{\partial t} = \frac{\partial}{\partial t} \int_{CV} (\rho dV) \quad (4.3)$$

where dV = volume element

- RHS of Eq. (4.2)
 - = net flux of matter through the control surface
 - = flux in – flux out
 - = $\int \rho q_n dA_1 - \int \rho q_n dA_2$

4.1 Conservation of Matter in Homogeneous Fluids

where q_n = component of velocity vector **normal** to the surface

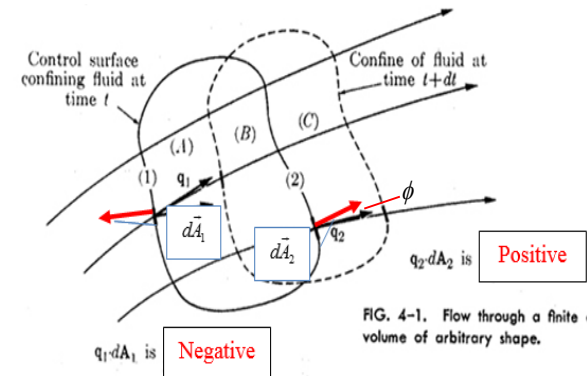
of $CV = |\vec{q}| \cos \phi$

$$\therefore \frac{\partial}{\partial t} \int_{CV} (\rho dV) = \int_{CS} \rho q_n dA_1 - \int_{CS} \rho q_n dA_2 \quad (4.4)$$

※ Flux (= mass/time) is due to velocity of the flow.

• Vector form is

$$\frac{\partial}{\partial t} \int_{CV} (\rho dV) = - \oint_{CS} \rho \vec{q} \cdot d\vec{A} \quad (4.5)$$

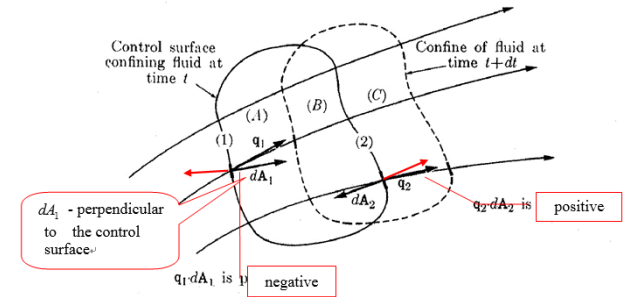


4.1 Conservation of Matter in Homogeneous Fluids

where $d\vec{A}$ = vector differential area pointing in the **outward** direction over an enclosed control surface

$$\therefore \vec{q} \cdot d\vec{A} = |\vec{q}| |d\vec{A}| \cos \phi$$

$$= \begin{cases} \text{positive for an outflow from cv, } \phi \leq 90^\circ \\ \text{negative for inflow into cv, } 90^\circ \leq \phi \leq 180^\circ \end{cases}$$



If fluid continues to occupy the entire control volume at subsequent times

→ time independent

$$\text{LHS: } \frac{\partial}{\partial t} \int_{cv} (\rho dV) \Rightarrow \int_{cv} \frac{\partial \rho}{\partial t} dV \quad (4.5a)$$

4.1 Conservation of Matter in Homogeneous Fluids

Eq. (4.4) becomes

$$\int_{CV} \frac{\partial \rho}{\partial t} dV + \oint_{CS} \rho \vec{q} \cdot d\vec{A} = 0 \quad (4.6)$$

→ General form of continuity equation → Integral form

[Re] Differential form

Use Gauss divergence theorem

$$\int_V \frac{\partial F}{\partial x_i} dV = \int_A F dA_i$$

4.1 Conservation of Matter in Homogeneous Fluids

Transform surface integral of Eq. (4.6) into volume integral

$$\oint_{CS} \rho \vec{q} \cdot d\vec{A} = \int_{CV} \nabla \cdot (\rho \vec{q}) dV$$

Then, Eq. (4.6) becomes

$$\int_{CV} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{q}) \right] dV = 0 \quad (4.6a)$$

Eq. (4.6a) holds for any volume only if the integrand vanishes at every point.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{q}) = 0 \quad (4.6b)$$

→ **Differential (point) form**

4.1 Conservation of Matter in Homogeneous Fluids

Simplified form of continuity equation

- Steady flow of a compressible fluid

$$\int_{cv} \frac{\partial \rho}{\partial t} dV = 0$$

Therefore, Eq. (4.6) becomes

$$\oint_{cs} \rho \vec{q} \cdot d\vec{A} = 0 \quad (4.7)$$

- Incompressible fluid (for both steady and unsteady conditions)

$$\rho = \text{const.} \rightarrow \frac{\partial \rho}{\partial t} = 0, \frac{d\rho}{dt} = 0$$

4.1 Conservation of Matter in Homogeneous Fluids

Therefore, Eq. (4.6) becomes

$$\oint_{CS} \vec{q} \cdot d\vec{A} = 0 \quad (4.8)$$

4.1 Conservation of Matter in Homogeneous Fluids

[Cf] Non-homogeneous fluid mixture

- Conservation of mass equations for the individual species

→ Advection-diffusion equation

= conservation of mass equation + mass flux equation due to advection and diffusion

$$\frac{\partial c}{\partial t} + \frac{\partial q}{\partial x} = 0$$

$$q = uc - D \frac{\partial c}{\partial x}$$

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x} \left(uc - D \frac{\partial c}{\partial x} \right) = 0$$

$$\rightarrow \frac{\partial c}{\partial t} + \frac{\partial uc}{\partial x} = \frac{\partial}{\partial x} \left(D \frac{\partial c}{\partial x} \right)$$

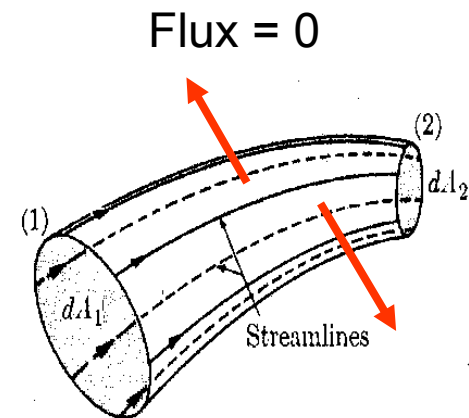
4.1 Conservation of Matter in Homogeneous Fluids

4.1.2 Stream - tube control volume analysis for steady flow

- Steady flow: There is no flow across the longitudinal boundary of the stream tube.
- Eq. (4.7) becomes

$$\oint \rho \vec{q} \cdot d\vec{A} = -\rho_1 q_1 dA_1 + \rho_2 q_2 dA_2 = 0$$

$$\rho q dA = \text{const.} \quad (4.9)$$



4.1 Conservation of Matter in Homogeneous Fluids

If density = const.

$$q_1 dA_1 = q_2 dA_2 = dQ \quad (4.10)$$

where dQ = volume rate of flow

- For flow in conduit with variable density

$$V = \frac{\int q dA}{A} \rightarrow \text{average velocity}$$

$$\bar{\rho} = \frac{\int \rho dQ}{Q} \rightarrow \text{average density}$$

$$\bar{\rho}_1 V_1 A_1 = \bar{\rho}_2 V_2 A_2 \quad (4.11)$$

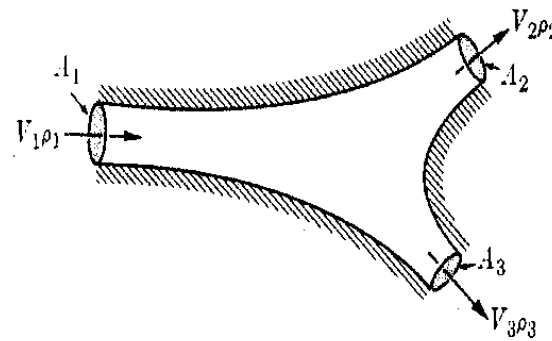
4.1 Conservation of Matter in Homogeneous Fluids

- For a branching conduit

$$\oint \rho \vec{q} \cdot d\vec{A} = 0$$

$$-\int_{A_1} \rho_1 q_1 dA_1 + \int_{A_2} \rho_2 q_2 dA_2 + \int_{A_3} \rho_3 q_3 dA_3 = 0$$

$$\bar{\rho}_1 V_1 A_1 = \bar{\rho}_2 V_2 A_2 + \bar{\rho}_3 V_3 A_3 \quad (4.12)$$



4.1 Conservation of Matter in Homogeneous Fluids

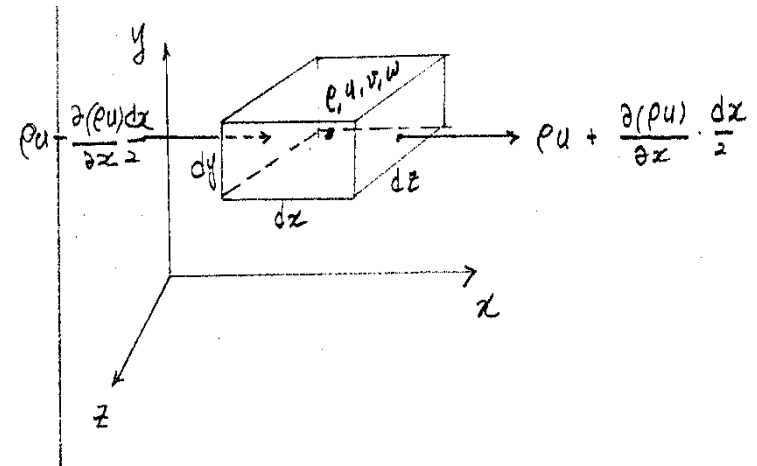
◆ Equation of Continuity

Use Infinitesimal (differential) control volume method

- At the centroid of the control volume,

$$\rho, u, v, w$$

- rate of mass flux across the surface perpendicular to x is



4.1 Conservation of Matter in Homogeneous Fluids

$$\text{flux in} = \left\{ \rho u - \frac{\partial(\rho u)}{\partial x} \frac{dx}{2} \right\} dydz$$

$$\text{flux out} = \left\{ \rho u + \frac{\partial(\rho u)}{\partial x} \frac{dx}{2} \right\} dydz$$

$$\text{net flux} = \text{flux in} - \text{flux out} = -\frac{\partial(\rho u)}{\partial x} dx dy dz$$

$$\text{net mass flux across the surface perpendicular to } y = -\frac{\partial(\rho v)}{\partial y} dy dx dz$$

$$\text{net mass flux across the surface perpendicular to } z = -\frac{\partial(\rho w)}{\partial z} dz dx dy$$

$$\text{Time rate of change of mass inside the c.v.} = \frac{\partial(\rho dx dy dz)}{\partial t}$$

4.1 Conservation of Matter in Homogeneous Fluids

Time rate of change of mass inside = sum of three net rates

$$\frac{\partial(\rho dx dy dz)}{\partial t} = - \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right] dx dy dz$$

By taking limit $dV = dx dy dz$

$$-\frac{\partial \rho}{\partial t} = \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = \nabla \cdot \rho \vec{q} = \text{div}(\rho \vec{q})$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{q} = 0} \quad (\text{A1})$$

→ point (differential) form of Continuity Equation (the same as Eq. 4.6b)

4.1 Conservation of Matter in Homogeneous Fluids

$$[\text{Re}] \quad \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = \nabla \cdot \rho \vec{q} = \text{div}(\rho \vec{q})$$

By the way,

$$\nabla \cdot \rho \vec{q} = \vec{q} \cdot \nabla \rho + \rho \nabla \cdot \vec{q}$$

Thus, (A1) becomes

$$\frac{\partial \rho}{\partial t} + \vec{q} \cdot \nabla \rho + \rho \nabla \cdot \vec{q} = 0 \quad (\text{A2})$$

4.1 Conservation of Matter in Homogeneous Fluids

1) For incompressible fluid

$$\frac{d\rho}{dt} = 0 \quad (\rho = \text{const.})$$

$$\rightarrow \frac{\partial \rho}{\partial t} + \vec{q} \cdot \nabla \rho = \frac{d\rho}{dt} = 0$$

Therefore Eq. (A2) becomes

$$\rho \nabla \cdot \vec{q} = 0 \quad \rightarrow \quad \nabla \cdot \vec{q} = 0 \quad (\text{A3})$$

In scalar form,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (\text{A4})$$

→ Continuity Eq. for 3D incompressible fluid

4.1 Conservation of Matter in Homogeneous Fluids

For 2D incompressible fluid,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

2) For steady flow,

$$\frac{\partial \rho}{\partial t} = 0$$

Thus, (A1) becomes

$$\nabla \cdot \rho \vec{q} = \vec{q} \cdot \nabla \rho + \rho \nabla \cdot \vec{q} = 0 \quad (4.13)$$