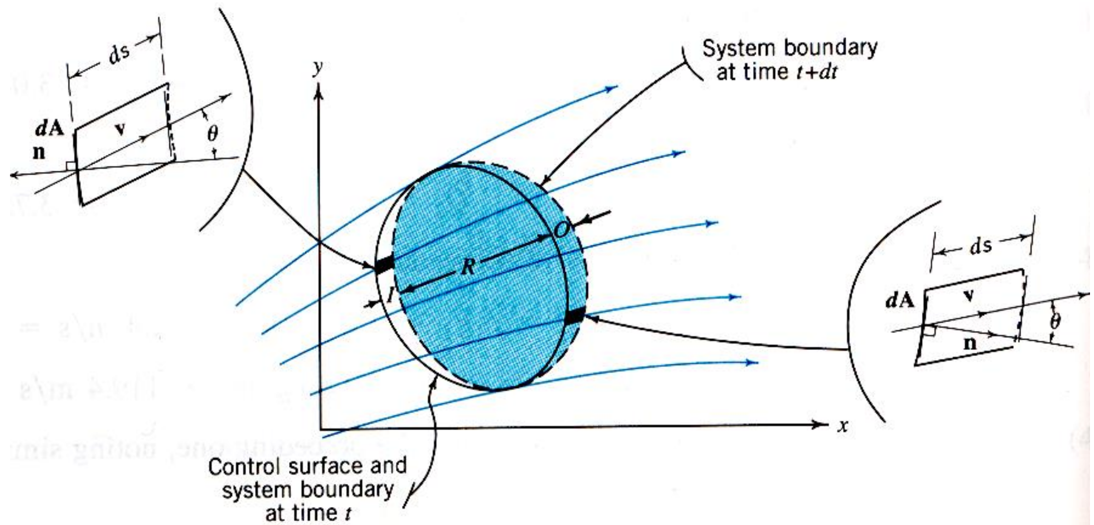
Lecture 5

Continuity, Energy, and Momentum Equations (1)



Lecture 5 Continuity, Energy, and Momentum Equations (1)

Contents

Objectives

- Apply finite control volume to get integral form of the continuity equation
- Compare integral and point form equations
- Derive the simplified equations for continuity equation

5.1 Control-Volume Concepts

- Each of the observational laws of mass, heat, and momentum transport may be formulated in the Eulerian sense of focusing attention on a fixed point in space.

- Control volume method:
  - ✓ Finite control volume - 유한검사체적
  - ✓ Differential (infinitesimal) control volume - 미소검사체적

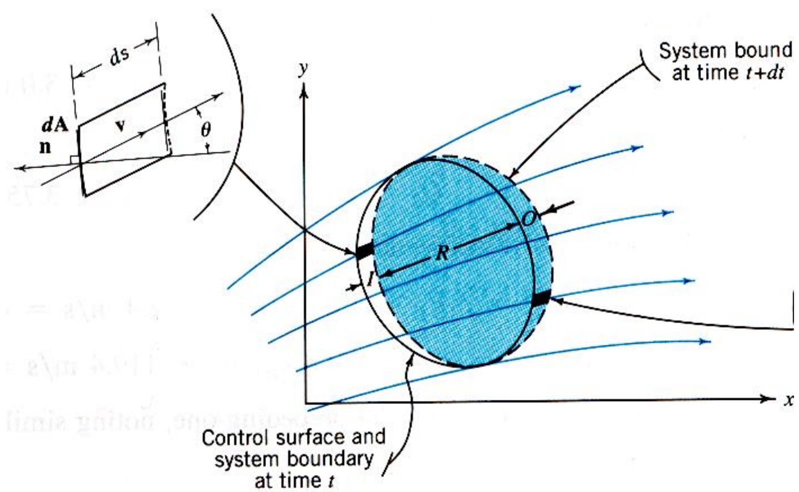
[Re] Control volume

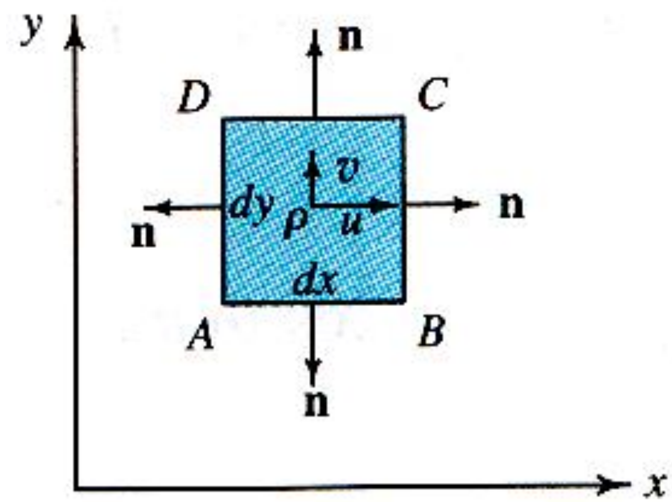
- fixed volume which consists of the same fluid particles and whose bounding surface moves with the fluid

5.1 Control-Volume Concepts

① Finite control volume – arbitrary control volume

② Differential (infinitesimal) control volume – parallelepiped control volume





5.1 Control-Volume Concepts

1) Finite control volume method

- Frequently used for 1D analysis (Ch. 4)
- Gross descriptions of flow
- Analytical formulation is easier than differential control volume method
- Integral form of equations for conservation of mass, momentum, and energy

[Ex] Continuity equation: conservation of mass

$$\int_{CV} \frac{\partial \rho}{\partial t} dV + \oint_{CS} \rho \vec{q} \cdot d\vec{A} = 0$$

5.1 Control-Volume Concepts

2) Differential control volume method

- Concerned with a fixed differential control volume (=ΔxΔyΔz) of fluid

- 2D or 3D analysis (Ch. 5~6)

$$\Delta\vec{F} = \frac{d}{dt}\left(\Delta m\vec{q}\right) = \frac{d}{dt}\left(\rho\Delta x\Delta y\Delta z\vec{q}\right)$$

– Δx, Δy, Δz become vanishingly small

- Point (differential) form of equations for conservation of mass, momentum, and energy

5.2 Conservation of Matter in Homogeneous Fluids

5.2.1 Finite control volume method-arbitrary control volume

- Consider an arbitrary control volume
- Although control volume remains fixed, mass of fluid originally enclosed regions (A+B) occupies the volume within the dashed line regions (B+C) after *dt*.
- Since mass m is conserved:

$$\left(m_A\right)_t + \left(m_B\right)_t = \left(m_B\right)_{t+dt} + \left(m_C\right)_{t+dt} \quad (5.1)$$

$$\frac{\left(m_B\right)_{t+dt} - \left(m_B\right)_t}{dt} = \frac{\left(m_A\right)_t - \left(m_C\right)_{t+dt}}{dt} \quad (5.2)$$

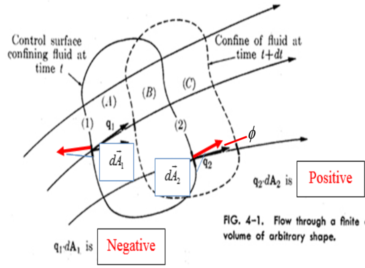


FIG. 4-1. Flow through a finite control volume of arbitrary shape.

5.2 Conservation of Matter in Homogeneous Fluids

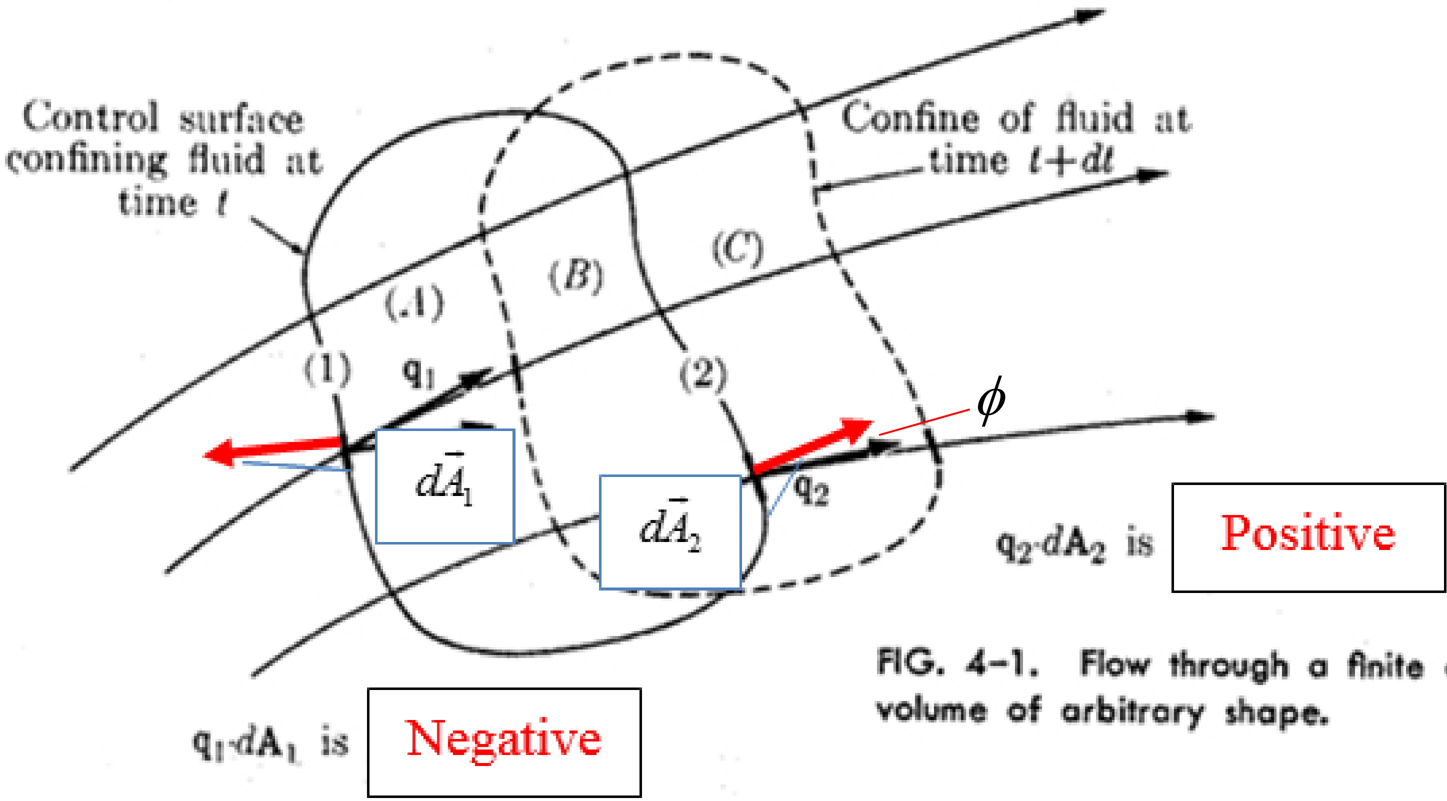


FIG. 4-1. Flow through a finite control volume of arbitrary shape.

5.2 Conservation of Matter in Homogeneous Fluids

- LHS of Eq. (5.2) = time rate of change of mass in the original control volume in the limit

$$\frac{(m_B)_{t+dt} - (m_B)_t}{dt} \approx \frac{\partial (m_B)}{\partial t} = \frac{\partial}{\partial t}\int_{CV}(\rho\, dV) \qquad (5.3)$$

where dV = volume element

- RHS of Eq. (5.2)

  = net flux of matter through the control surface

  = flux in – flux out

  $$= \int \rho q_n\, dA_1 - \int \rho q_n\, dA_2$$

5.2 Conservation of Matter in Homogeneous Fluids

where q_n = component of velocity vector normal to the surface

of $CV = |\vec{q}| \cos\phi$

$$\therefore \quad \frac{\partial}{\partial t}\int_{CV}(\rho\, dV) = \int_{CS}\rho q_n dA_1 - \int_{CS}\rho q_n dA_2 \qquad (5.4)$$

※ Flux (= mass/time) is due to velocity of the flow.

- Vector form is

$$\frac{\partial}{\partial t}\int_{CV}(\rho\, dV) = -\oint_{CS}\rho\vec{q}\cdot d\vec{A} \qquad (5.5)$$

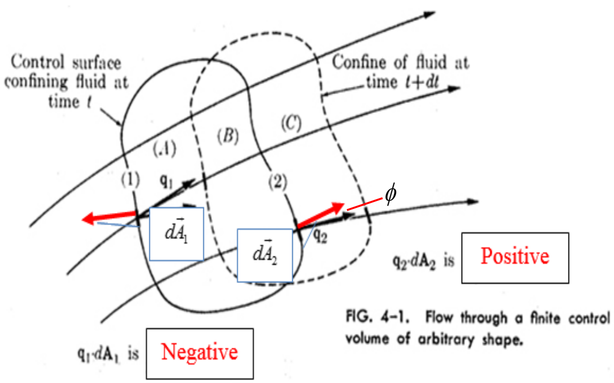
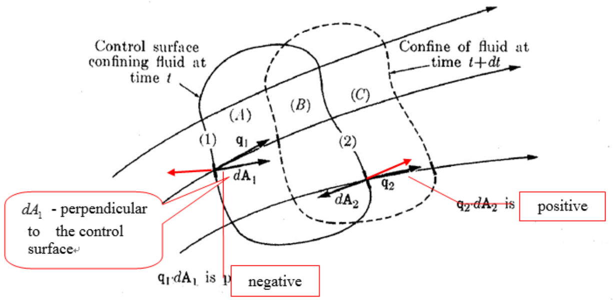


FIG. 4-1. Flow through a finite control volume of arbitrary shape.

5.2 Conservation of Matter in Homogeneous Fluids

where $d\vec{A}$ = vector differential area pointing in the outward direction over an enclosed control surface



$$\therefore \quad \vec{q} \cdot d\vec{A} = |\vec{q}|\,|d\vec{A}|\,\cos\phi$$

$$= \begin{cases} \text{positive for an outflow from cv,} \quad \phi \le 90^\circ \\ \text{negative for inflow into cv,} \quad 90^\circ \le \phi \le 180^\circ \end{cases}$$

If fluid continues to occupy the entire control volume at subsequent times

→ time independent

LHS: $$\frac{\partial}{\partial t}\int_{CV} (\rho\, dV) \Rightarrow \int_{CV} \frac{\partial \rho}{\partial t} dV \qquad (5.5a)$$

5.2 Conservation of Matter in Homogeneous Fluids

Eq. (5.4) becomes

$$\int_{CV} \frac{\partial \rho}{\partial t} dV + \oint_{CS} \rho \vec{q} \cdot d\vec{A} = 0 \qquad (5.6)$$

→ General form of continuity equation→ Integral form

[Re] Differential form

Use Gauss divergence theorem

$$\int_{V} \frac{\partial F}{\partial x_i} dV = \int_{A} F dA_i$$

5.2 Conservation of Matter in Homogeneous Fluids

Transform surface integral of Eq. (5.6) into volume integral

$$\oint_{CS} \rho \vec{q} \cdot d\vec{A} = \int_{CV} \nabla \cdot (\rho \vec{q}\,) dV$$

Then, Eq. (5.6) becomes

$$\int_{CV} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{q}\,) \right] dV = 0 \qquad (5.6a)$$

Eq. (5.6a) holds for any volume only if the integrand vanishes at every point.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{q}\,) = 0 \qquad (5.6b)$$

→ Differential (point) form

5.2 Conservation of Matter in Homogeneous Fluids

Simplified form of continuity equation

1) Steady flow of a compressible fluid $\rightarrow$ $\int_{CV} \frac{\partial \rho}{\partial t} dV = 0$

Therefore, Eq. (5.6) becomes

$$\oint_{CS} \rho \vec{q} \cdot d\vec{A} = 0 \tag{5.7}$$

2) Incompressible fluid (for both steady and unsteady conditions)

$$\rho = \text{const.} \rightarrow \frac{\partial \rho}{\partial t} = 0, \ \frac{d\rho}{dt} = 0$$

Therefore, Eq. (5.6) becomes

$$\oint_{CS} \vec{q} \cdot d\vec{A} = 0 \tag{5.8}$$

5.2 Conservation of Matter in Homogeneous Fluids

[Cf] Non-homogeneous fluid mixture

- Conservation of mass equations for the individual species

→ Advection-diffusion equation

= conservation of mass equation + mass flux equation due to advection and diffusion

$$\frac{\partial c}{\partial t} + \frac{\partial q}{\partial x} = 0 \qquad q = uc - D\frac{\partial c}{\partial x}$$

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x}\left(uc - D\frac{\partial c}{\partial x}\right) = 0 \quad \rightarrow \quad \frac{\partial c}{\partial t} + \frac{\partial uc}{\partial x} = \frac{\partial}{\partial x}\left(D\frac{\partial c}{\partial x}\right)$$

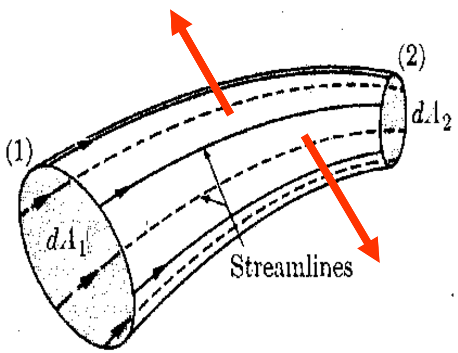
5.2 Conservation of Matter in Homogeneous Fluids

5.2.2 Stream - tube control volume analysis for steady flow

- Steady flow: There is no flow across the longitudinal boundary of the stream tube.
- Eq. (5.7) becomes

$$\oint \rho \vec{q} \cdot d\vec{A} = -\rho_1 q_1 dA_1 + \rho_2 q_2 dA_2 = 0$$

$$\rho q dA = \text{const.} \qquad (5.9)$$



5.2 Conservation of Matter in Homogeneous Fluids

- If density = const.

$$q_1 dA_1 = q_2 dA_2 = dQ \qquad (5.10)$$

where dQ = volume rate of flow

- For flow in conduit with variable density

$$V = \frac{\int q dA}{A} \rightarrow \text{average velocity}$$

$$\bar{\rho} = \frac{\int \rho dQ}{Q} \rightarrow \text{average density}$$

$$\bar{\rho}_1 V_1 A_1 = \bar{\rho}_2 V_2 A_2 \qquad (5.11)$$

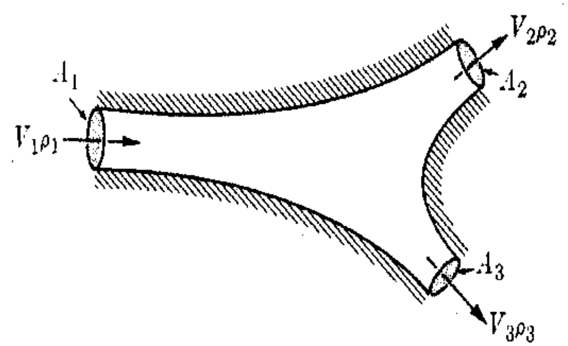
5.2 Conservation of Matter in Homogeneous Fluids

- For a branching conduit

$$\oint \rho \vec{q} \cdot d\vec{A} = 0$$

$$-\int_{A_1} \rho_1 q_1 dA_1 + \int_{A_2} \rho_2 q_2 dA_2 + \int_{A_3} \rho_3 q_3 dA_3 = 0$$

$$\bar{\rho}_1 V_1 A_1 = \bar{\rho}_2 V_2 A_2 + \bar{\rho}_3 V_3 A_3 \qquad (5.12)$$



5.2 Conservation of Matter in Homogeneous Fluids

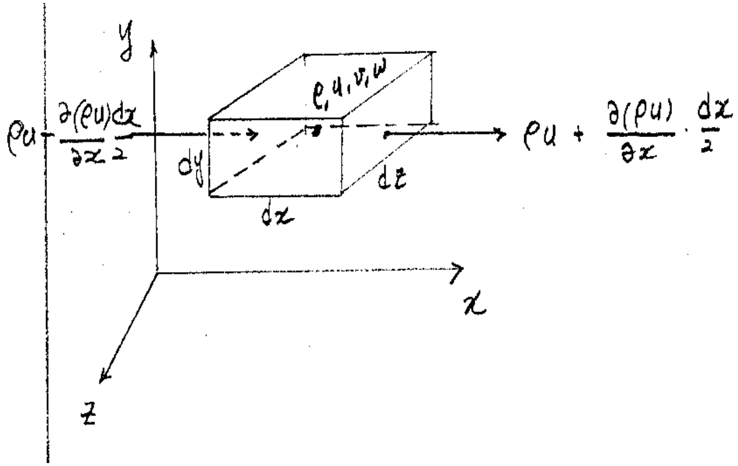
◆ Equation of Continuity

Use Infinitesimal (differential) control volume method

- At the centroid of the control volume,

  $\rho, \ u, \ v, \ w$

- rate of mass flux across the surface perpendicular to x is



5.2 Conservation of Matter in Homogeneous Fluids

$$\text{flux in} = \left\{ \rho u - \frac{\partial(\rho u)}{\partial x}\frac{dx}{2} \right\} dydz$$

$$\text{flux out} = \left\{ \rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2} \right\} dydz$$

$$\text{net flux} = \text{flux in} - \text{flux out} = -\frac{\partial(\rho u)}{\partial x} dxdydz$$

net mass flux across the surface perpendicular to $y = -\frac{\partial(\rho v)}{\partial y} dydxdz$

net mass flux across the surface perpendicular to $z = -\frac{\partial(\rho w)}{\partial z} dzdxdy$

Time rate of change of mass inside the c.v. $= \frac{\partial(\rho dxdydz)}{\partial t}$

5.2 Conservation of Matter in Homogeneous Fluids

Time rate of change of mass inside = sum of three net rates

$$\frac{\partial(\rho dxdydz)}{\partial t} = -\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] dxdydz$$

By taking limit *dV = dxdydz*

$$-\frac{\partial \rho}{\partial t} = \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = \nabla \cdot \rho \vec{q} = div\,(\rho \vec{q})$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{q} = 0} \qquad \text{(A1)}$$

→ point (differential) form of Continuity Equation (the same as Eq. 5.6b)

5.2 Conservation of Matter in Homogeneous Fluids

[Re] $$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = \nabla \cdot \rho\vec{q} = div\,(\rho\vec{q})$$

By the way,

$$\nabla \cdot \rho\vec{q} = \vec{q} \cdot \nabla\rho + \rho\nabla \cdot \vec{q}$$

Thus, (A1) becomes

$$\frac{\partial\rho}{\partial t} + \vec{q} \cdot \nabla\rho + \rho\nabla \cdot \vec{q} = 0 \qquad \text{(A2)}$$

5.2 Conservation of Matter in Homogeneous Fluids

1) For incompressible fluid

$$\frac{d\rho}{dt} = 0 \ (\rho = \text{const.})$$

$$\rightarrow \quad \frac{\partial \rho}{\partial t} + \vec{q} \cdot \nabla \rho = \frac{d\rho}{dt} = 0$$

Therefore Eq. (A2) becomes

$$\rho \nabla \cdot \vec{q} = 0 \quad \rightarrow \quad \boxed{\nabla \cdot \vec{q} = 0} \qquad \text{(A3)}$$

In scalar form,

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0} \qquad \text{(A4)}$$

→ Continuity Eq. for 3D incompressible fluid

5.2 Conservation of Matter in Homogeneous Fluids

For 2D incompressible fluid,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

2) For steady flow,

$$\frac{\partial \rho}{\partial t} = 0$$

Thus, (A1) becomes

$$\nabla \cdot \rho \vec{q} = \vec{q} \cdot \nabla \rho + \rho \nabla \cdot \vec{q} = 0 \qquad (5.13)$$