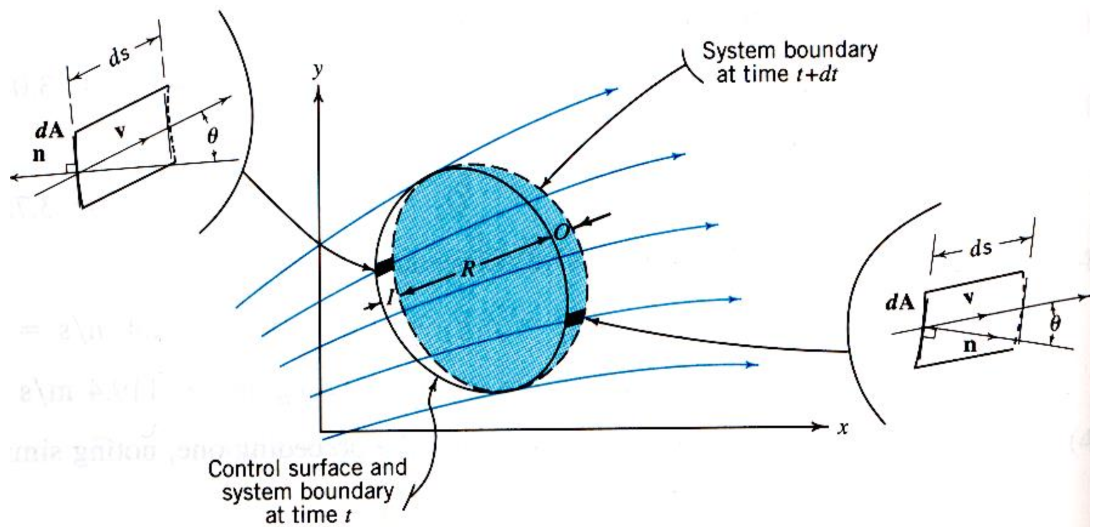
Lecture 7

Continuity, Energy, and Momentum Equations (3)



Contents

Objectives

- Derive the momentum equation by applying Newton's 2^{nd} law of motion and Reynolds Transport Theorem
- Derive the moment of the momentum equation by applying Newton's 2^{nd} law to rotating fluid masses

7.1 Linear Momentum Equation for Finite Control Volumes

7.1.1 Momentum Principle

- The momentum equation can be derived from Newton's 2nd law of motion

$$\vec{F} = m\vec{a} = m\frac{d\vec{q}}{dt} = \frac{d\left(m\vec{q}\right)}{dt} = \frac{d\vec{M}}{dt} \qquad (7.1)$$

$\vec{M}$ = linear momentum vector $= m\vec{q}$

$\vec{F}$ = external force

$$\vec{F} = \begin{cases} \text{boundary (surface) forces: } \begin{cases} \text{normal to boundary - pressure, } \vec{F}_p \\ \text{tangential to boundary - shear, } \vec{F}_s \end{cases} \\ \text{body forces - force due to gravitational or magnetic fields, } \vec{F}_b \end{cases}$$

7.1 Linear Momentum Equation for Finite Control Volumes

$$\boxed{\vec{F}_p + \vec{F}_s + \vec{F}_b = \frac{d\vec{M}}{dt}} \qquad (7.2)$$

$\vec{F}_b = \int_{CV} f_b (\rho \, dV)$, where f_b = body force per unit mass

7.1 Linear Momentum Equation for Finite Control Volumes

7.1.2 The general linear momentum equation

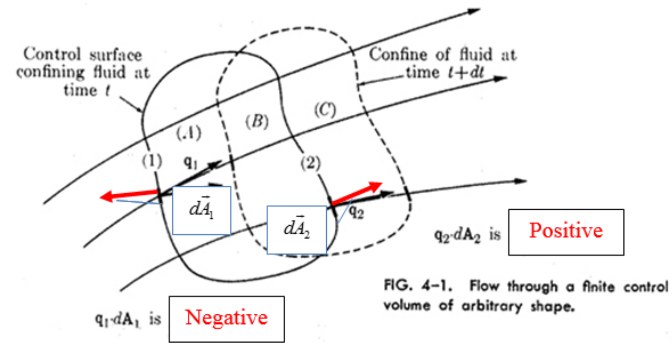


FIG. 4-1. Flow through a finite control volume of arbitrary shape.

Consider change of momentum

$\frac{d\vec{M}}{dt}$ = total rate of change of momentum

= net momentum flux across the CV boundaries

+ time rate of increase of momentum within CV

$$= \oint_{CS} \vec{q}\rho\left(\vec{q}\cdot d\vec{A}\right) + \frac{\partial}{\partial t}\int_{CV} \vec{q}\rho\, dV \qquad (7.3)$$

← *Reynolds Transport Theorem*

where $\vec{q}\rho\left(\vec{q}\cdot d\vec{A}\right)$ = momentum flux = $\text{velocity} \times \text{mass per time}$

$d\vec{A}$ = vector unit area pointing outward over the control surface

7.1 Linear Momentum Equation for Finite Control Volumes

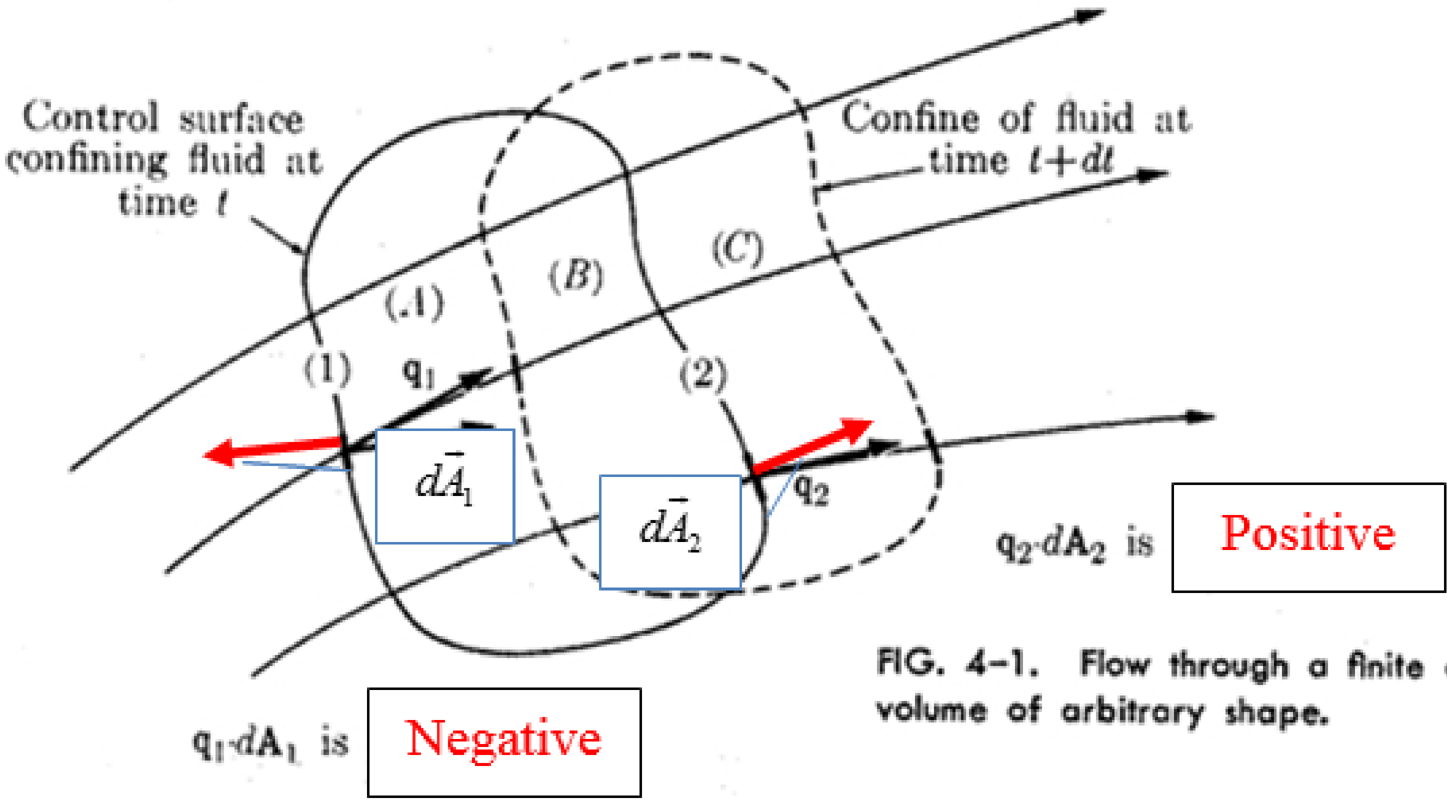


FIG. 4-1. Flow through a finite control volume of arbitrary shape.

7.1 Linear Momentum Equation for Finite Control Volumes

Substitute (7.3) into (7.2)

$$\boxed{\vec{F}_p + \vec{F}_s + \vec{F}_b = \oint_{CS} \vec{q}\rho\left(\vec{q}\cdot d\vec{A}\right) + \frac{\partial}{\partial t}\int_{CV} \vec{q}\rho\, dV} \qquad (7.4)$$

For steady flow and negligible body forces

$$\boxed{\vec{F}_p + \vec{F}_s = \oint_{CS} \vec{q}\rho\left(\vec{q}\cdot d\vec{A}\right)} \qquad (7.5)$$

- Eq. (7.4)
  - It is applicable to both ideal fluid systems and viscous fluid systems involving friction and energy dissipation.
  - It is applicable to both compressible fluid and incompressible fluid.

7.1 Linear Momentum Equation for Finite Control Volumes

- Combined effects of friction, energy loss, and heat transfer appear implicitly in the magnitude of the external forces, with corresponding effects on the local flow velocities.
- Knowledge of the internal conditions is not necessary.
- We can consider only external conditions.

7.1 Linear Momentum Equation for Finite Control Volumes

7.1.3 Inertial control volume for a generalized apparatus

- Three components of the forces

$$x-dir.:\ \vec{F}_{p_x}+\vec{F}_{s_x}+\vec{F}_{b_x}=\oint_{CS} u\rho\left(\vec{q}\cdot d\vec{A}\right)+\frac{\partial}{\partial t}\int_{CV} u\rho\, dV$$

$$y-dir.:\ \vec{F}_{p_y}+\vec{F}_{s_y}+\vec{F}_{b_y}=\oint_{CS} v\rho\left(\vec{q}\cdot d\vec{A}\right)+\frac{\partial}{\partial t}\int_{CV} v\rho\, dV$$

$$z-dir.:\ \vec{F}_{p_z}+\vec{F}_{s_z}+\vec{F}_{b_z}=\oint_{CS} w\rho\left(\vec{q}\cdot d\vec{A}\right)+\frac{\partial}{\partial t}\int_{CV} w\rho\, dV \qquad (7.6)$$

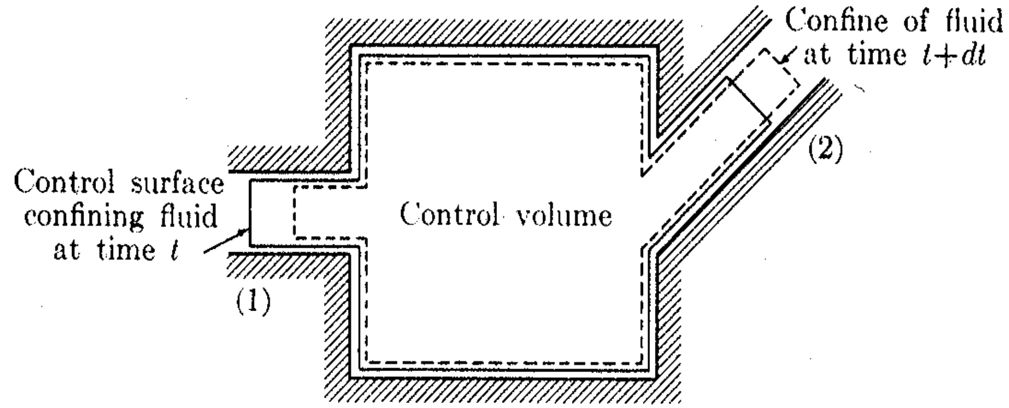
7.1 Linear Momentum Equation for Finite Control Volumes

- For flow through generalized apparatus

$$x-dir.:\ \vec{F}_{p_x}+\vec{F}_{s_x}+\vec{F}_{b_x}=\int_2 u\rho\, dQ-\int_1 u\rho\, dQ+\frac{\partial}{\partial t}\int_{CV} u\rho\, dV$$

- For 1D steady flow,

$$\frac{\partial}{\partial t}\int_{CV} q\rho\, dV=0$$



7.1 Linear Momentum Equation for Finite Control Volumes

- Velocity and density are constant normal to the flow direction.

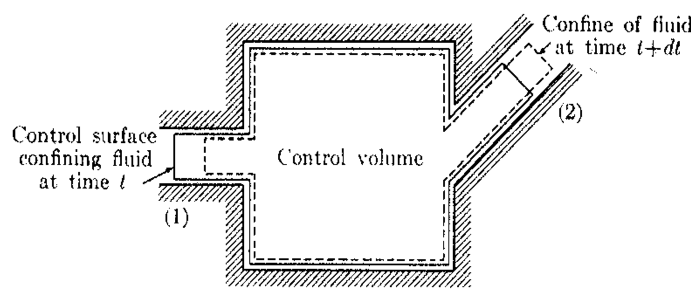
$$x-dir.: \ \vec{F}_{p_x}+\vec{F}_{s_x}+\vec{F}_{b_x}=\sum F_x=\left(V_x\rho Q\right)_2-\left(V_x\rho Q\right)_1$$

$$= V_{x_2}\rho_2 Q_2 - V_{x_1}\rho_1 Q_1 = Q\rho\left(V_{x_2}-V_{x_1}\right)= Q\rho\left(V_{x_{out}}-V_{x_{in}}\right)$$

$$\boxed{\rho_1 Q_1 = \rho_2 Q_2 = Q\rho}$$

$$y-dir.: \ \sum F_y=\left(V_y\rho Q\right)_2-\left(V_y\rho Q\right)_1$$

$$z-dir.: \ \sum F_z=\left(V_z\rho Q\right)_2-\left(V_z\rho Q\right)_1$$



where V = average velocity in flow direction

7.1 Linear Momentum Equation for Finite Control Volumes

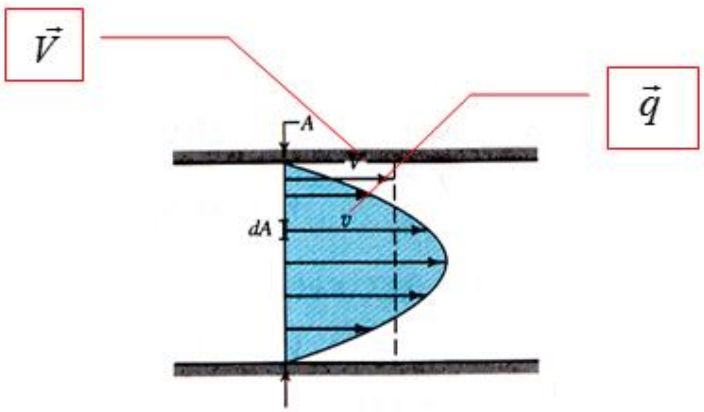
- Non-uniform velocity profile

If velocity varies over the cross section, then introduce momentum flux coefficient

$$\int \vec{q}\rho\left(\vec{q}\cdot d\vec{A}\right) = K_m \vec{V}\left(\rho VA\right)$$

$$\int \vec{q}\rho \, dQ = K_m \vec{V}\rho Q$$

$$K_m = \frac{\int \vec{q}\rho dQ}{\vec{V}\rho Q}$$



7.1 Linear Momentum Equation for Finite Control Volumes

where

V = magnitude of average velocity over cross section = Q/A

$\vec{V}$ = average velocity vector

K_m = momentum flux coefficient ≥ 1

= 1.33 for laminar flow (pipe flow)
  1.03-1.04 for turbulent flow (smooth pipe)

$$\sum F_x = \left(K_m V_x \rho Q\right)_2 - \left(K_m V_x \rho Q\right)_1$$

$$\sum F_y = \left(K_m V_y \rho Q\right)_2 - \left(K_m V_y \rho Q\right)_1$$

$$\sum F_z = \left(K_m V_z \rho Q\right)_2 - \left(K_m V_z \rho Q\right)_1$$

7.1 Linear Momentum Equation for Finite Control Volumes

[Cf] Energy correction coefficient

$$K_e = \frac{\int \frac{\rho}{2} q^2 dQ}{\frac{\rho}{2} \vec{V}^2 Q}$$

7.1 Linear Momentum Equation for Finite Control Volumes

[Example 7-1] Continuity, energy, and linear momentum with unsteady flow

A large tank mounted on rollers is filled with water to a depth of 16 ft above a discharge port. At time t = 0, the fast-acting valve on the discharge nozzle is opened. Determine depth h, discharge rate Q, and force F necessary to keep the tank stationary at t = 50 *sec*.

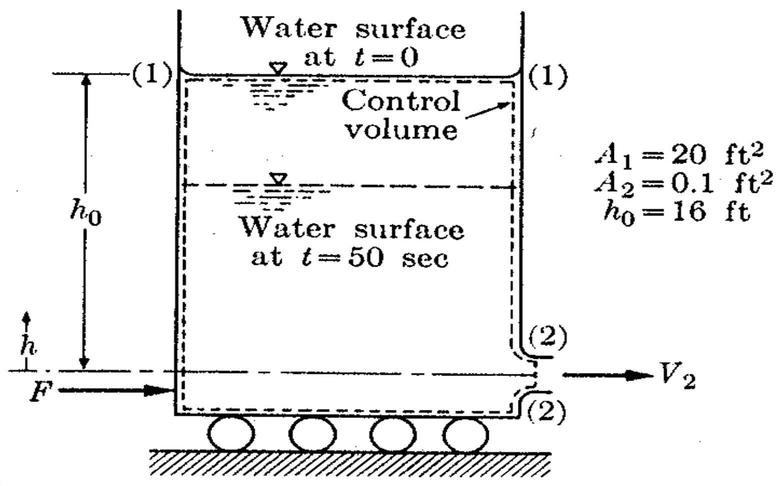
Assumptions:

$V_1 \approx 0$

$\rho = \text{const.}$

$p_2 = p_{atm} = 0$

$h_2 = 0 \text{ (datum)}$



7.1 Linear Momentum Equation for Finite Control Volumes

Apply continuity, energy, and linear momentum equations

(7.7)

$$\int_{CV} \frac{\partial \rho}{\partial t} dV + \oint_{CS} \rho \vec{q} \cdot d\vec{A} = 0$$

(7.8)

$$\frac{\delta Q}{dt} - \frac{\delta W_{shaft}}{dt} - \frac{\delta W_{shear}}{dt}$$

$$= \oint_{CS} \left(\frac{p}{\rho} + u + gh + \frac{q^2}{2} \right) \rho \left(\vec{q} \cdot d\vec{A} \right) + \frac{\partial}{\partial t} \int_{CV} e \rho \, dV$$

(7.9)

$$\vec{F}_p + \vec{F}_s + \vec{F}_b = \oint_{CS} \vec{q} \rho \left(\vec{q} \cdot d\vec{A} \right) + \frac{\partial}{\partial t} \int_{CV} \vec{q} \rho \, dV$$

7.1 Linear Momentum Equation for Finite Control Volumes

i) Use integral form of continuity equation, Eq. (7.7)

$$\frac{\partial}{\partial t}\int_{CV}\rho\,dV = \int \rho q_n dA_1 - \int \rho q_n dA_2$$

$$dV = A_1 dh\,, \quad \rho q_n dA_1 = 0$$ (because no inflow across the Section 1)

$$\therefore \quad \rho A_1 \frac{\partial}{\partial t}\int_0^h dh = -\rho V_2 A_2$$

$$A_1 \frac{dh}{dt} = -V_2 A_2 \qquad \text{(A)}$$

ii) Energy equation, Eq. (7.8)

~ no shaft work

~ heat transfer and temperature changes <u>due to friction are negligible</u>

7.1 Linear Momentum Equation for Finite Control Volumes

$$\cancel{\frac{\delta Q}{dt}} - \cancel{\frac{\delta W_{shaft}}{dt}} - \cancel{\frac{\delta W_{shear}}{dt}}$$

$$= \underbrace{\oint_{CS}\left(\frac{p}{\rho} + u + gh + \frac{q^2}{2}\right)\rho\left(\vec{q}\cdot d\vec{A}\right)}_{\boxed{\mathrm{I}}} + \underbrace{\frac{\partial}{\partial t}\int_{CV} e\rho\, dV}_{\boxed{\mathrm{II}}}$$

e = energy per unit mass = $u + gh + \dfrac{q^2}{2}$

$$\mathrm{I} = \oint_{CS}\left(u + \frac{p}{\rho} + gh + \frac{q^2}{2}\right)\rho\left(\vec{q}\cdot d\vec{A}\right)$$

$$= \left(u + \frac{p}{\rho} + gh + \frac{q^2}{2}\right)_2 \rho V_2 A_2 - \left(u + \frac{p}{\rho} + gh + \frac{q^2}{2}\right)_1 \rho V_1 A_1$$

7.1 Linear Momentum Equation for Finite Control Volumes

$$= \left(u + \frac{p}{\rho} + gh + \frac{q^2}{2} \right)_2 \rho V_2 A_2 \qquad (V_1 \approx 0)$$

$$\mathrm{II} = \frac{\partial}{\partial t} \int_{CV} e \rho \, dV = \frac{\partial}{\partial t} \int_{CV} \left(u + gh + \cancel{\frac{q^2}{2}} \right) \rho dV$$

$A_1 dh$

∵ nearly constant in the tank

except near the nozzle

$$= A_1 \rho \frac{\partial}{\partial t} \int_0^h (u + gh) dh$$

$$\therefore \ 0 = \left(u + \frac{p}{\rho} + gh + \frac{q^2}{2} \right)_2 \rho V_2 A_2 + A_1 \rho \frac{\partial}{\partial t} \int_0^h (u + gh) dh$$

7.1 Linear Momentum Equation for Finite Control Volumes

Assume $\rho = \text{const.}\ ,\quad p_2 = p_{atm} = 0\ ,\quad h_2 = 0\ (\text{datum})$

$$0 = uV_2A_2 + \frac{V_2^2}{2}V_2A_2 + uA_1\frac{dh}{dt} + A_1gh\frac{dh}{dt} \qquad \text{(B)}$$

$$\boxed{A_1\frac{dh}{dt} = -V_2A_2}$$

Substitute (A) into (B)

$$0 = \cancel{uV_2A_2} + \frac{V_2^2}{2}V_2A_2 + \cancel{u\left(-V_2A_2\right)} + gh\left(-V_2A_2\right)$$

$$\therefore \quad \frac{V_2^2}{2}V_2A_2 = ghV_2A_2$$

$$V_2 = \sqrt{2gh} \qquad \text{(C)}$$

7.1 Linear Momentum Equation for Finite Control Volumes

Substitute (C) into (A)

$$A_2\sqrt{2gh} = -A_1\frac{dh}{dt}$$

$$\frac{dh}{\sqrt{h}} = -\frac{A_2}{A_1}\sqrt{2g}\,dt$$

Integrate

$$\int_{h_0}^{h}\frac{dh}{\sqrt{h}} = \int_0^t -\frac{A_2}{A_1}\sqrt{2g}\,dt$$

$$\left\{\int_{h_0}^{h} h^{-\frac{1}{2}}dh = \left[2h^{\frac{1}{2}}\right]_{h_0}^{h}\right\}$$

$$h = \left(h_0^{\frac{1}{2}} - \frac{A_2}{A_1}\frac{\sqrt{2g}}{2}t\right)^2$$

7.1 Linear Momentum Equation for Finite Control Volumes

$$h = \left(\sqrt{16} - \frac{0.1}{20} \frac{\sqrt{2(32.2)}}{2} t \right)^2$$

$$= (4 - 0.0201t)^2$$

At $t = 50\,\mathrm{sec}$, $h = (4 - 0.0201 \times 50)^2 = 8.98\,ft$

$$V_2 = \sqrt{2gh} = \sqrt{2(32.2)(8.98)} = 24.05 \text{ fps}$$

$$Q_2 = (VA)_2 = 24.05(0.1) = 2.405 \text{ cfs}$$

7.1 Linear Momentum Equation for Finite Control Volumes

iii) Momentum equation, Eq. (7.4)

$$\vec{F}_p + \cancel{\vec{F}_s} + \cancel{\vec{F}_b} = \underbrace{\oint_{CS} \vec{q}\rho\left(\vec{q}\cdot d\vec{A}\right)}_{\mathrm{I}} + \underbrace{\cancel{\frac{\partial}{\partial t}\int_{CV} \vec{q}\rho\, dV}}_{\mathrm{II}}$$

II = Time rate of change of momentum inside CV is negligible

if tank area (A_1) is large compared to the nozzle area (A_2).

$$\mathrm{I} = \oint_{CS} \vec{q}\rho\left(\vec{q}\cdot d\vec{A}\right) = \int q_n \rho q_n dA_2 - \cancel{\int q_n \rho q_n dA_1} = V_2 \rho V_2 A_2$$

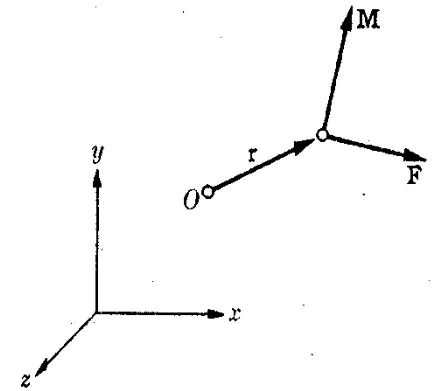
$$\therefore F_{px} = V_2 \rho V_2 A_2 = V_2 \rho Q_2$$

$$F_{px} = (24.05)(1.94)(2.405) = 112 \text{ lb}$$

7.2 The Moment of Momentum Equation for Finite Control Volumes

7.2.1 The Moment of momentum principle for inertial reference systems

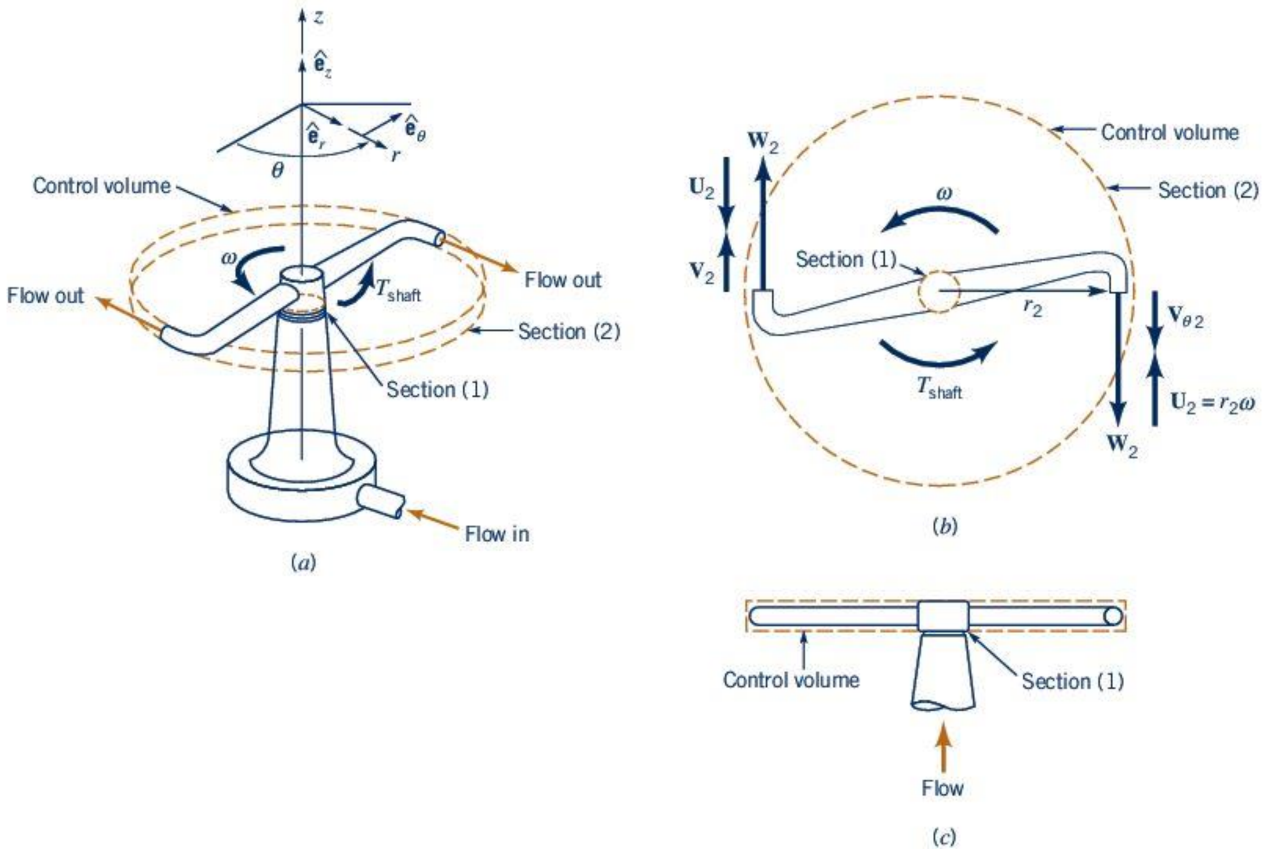
Apply Newton's 2nd law to rotating fluid masses



→ The vector sum of all the external moments acting on a fluid mass $\left(\vec{r} \times \vec{F}\right)$ equals the time rate of change of the moment of momentum (angular momentum) vector $\left(\vec{r} \times \vec{M}\right)$ of the fluid mass.

Example: rotary lawn sprinklers, ceiling fans, wind turbines

7.2 The Moment of Momentum Equation for Finite Control Volumes



7.2 The Moment of Momentum Equation for Finite Control Volumes

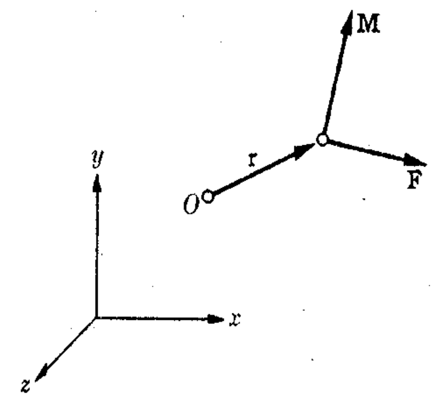
$$\boxed{T = \vec{r} \times \vec{F} = \frac{d}{dt}\left(\vec{r} \times \vec{M}\right)} \qquad (7.10)$$

where

T = torque

$\vec{r}$ = position vector of a mass in an arbitrary curvilinear motion

$\vec{M}$ = linear momentum



7.2 The Moment of Momentum Equation for Finite Control Volumes

[Re] Derivation of (7.10)

Eq. (7.1): $\vec{F} = \dfrac{d\vec{M}}{dt}$

Take the vector cross product of $\vec{r}$

$$\vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{M}}{dt}$$

By the way,

$$\frac{d}{dt}\left(\vec{r} \times \vec{M}\right) = \underbrace{\frac{d\vec{r}}{dt} \times \vec{M}}_{\boxed{\mathrm{I}}} + \vec{r} \times \frac{d\vec{M}}{dt}$$

7.2 The Moment of Momentum Equation for Finite Control Volumes

$$I = \frac{d\vec{r}}{dt} \times \vec{M} = \vec{q} \times m\,\vec{q} = 0 \quad \left(\because \frac{d\vec{r}}{dt} = \vec{q} \right)$$

$$\left(\therefore \ \vec{q} \times \vec{q} = |\vec{q}|\,|\vec{q}| \sin 0^\circ = 0 \right)$$

$$\therefore \quad \left(\vec{r} \times \frac{d\vec{M}}{dt} \right) = \frac{d}{dt}\left(\vec{r} \times \vec{M} \right)$$

$$\therefore \therefore \quad \boxed{\vec{r} \times \vec{F} = \frac{d}{dt}\left(\vec{r} \times \vec{M} \right)}$$

where $\vec{r} \times \vec{M}$ = angular momentum (moment of momentum)

7.2 The Moment of Momentum Equation for Finite Control Volumes

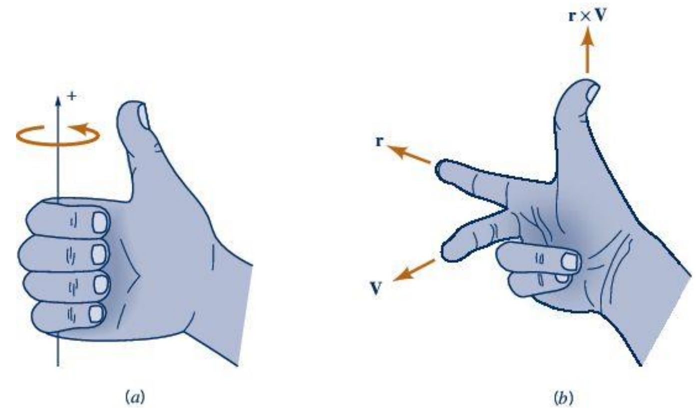
[Re] Torque $\vec{T} = \vec{r} \times \vec{F}$

- translational motion $\rightarrow$

Force – linear acceleration

- rotational motion $\rightarrow$

Torque – angular acceleration



[Re] Vector Product

$$\vec{V} = \vec{a} \times \vec{b}$$

Magnitude = $\left|\vec{V}\right| = \left|\vec{a}\right| \times \left|\vec{b}\right| \sin\gamma$ = area of parallelogram

direction = perpendicular to plane of $\vec{a}$ and $\vec{b}$ $\rightarrow$ right-handed triple

7.2 The Moment of Momentum Equation for Finite Control Volumes

$$\vec{b} \times \vec{a} = -\left(\vec{a} \times \vec{b}\right)$$

$$\left(k\vec{a}\right) \times \vec{b} = k\left(\vec{a} \times \vec{b}\right)$$

$$\vec{a} \times \left(\vec{b} + \vec{c}\right) = \left(\vec{a} \times \vec{b}\right) + \left(\vec{a} \times \vec{b}\right)$$

- External moments arise from external forces

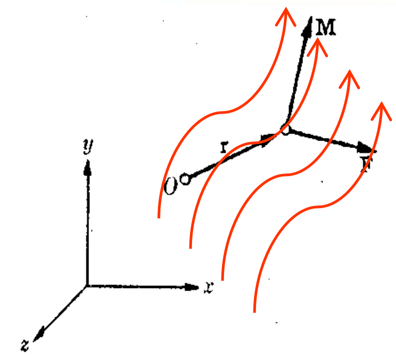
$$\underbrace{\left(\vec{r} \times \vec{F}_p\right)}_{\boxed{\vec{T}_p}} + \underbrace{\left(\vec{r} \times \vec{F}_s\right)}_{\boxed{\vec{T}_s}} + \underbrace{\left(\vec{r} \times \vec{F}_b\right)}_{\boxed{\vec{T}_b}} = \frac{d}{dt}\left(\vec{r} \times \vec{M}\right)$$

$$\vec{T}_p + \vec{T}_s + \vec{T}_b = \frac{d}{dt}\left(\vec{r} \times \vec{M}\right) \qquad (7.11)$$

where $\vec{T}_p$, $\vec{T}_s$, $\vec{T}_b$ = external torque

7.2 The Moment of Momentum Equation for Finite Control Volumes

7.2.2 The general moment of momentum equation



(7.3): $$\frac{d\vec{M}}{dt} = \oint_{CS} \vec{q}\rho\left(\vec{q}\cdot d\vec{A}\right) + \frac{\partial}{\partial t}\int_{CV} \vec{q}\rho\, dV$$

$$\therefore \quad \frac{d}{dt}\left(\vec{r}\times\vec{M}\right) = \oint_{CS}\left(\vec{r}\times\vec{q}\right)\rho\left(\vec{q}\cdot d\vec{A}\right) + \frac{\partial}{\partial t}\int_{CV}\left(\vec{r}\times\vec{q}\right)\rho\, dV$$

$$\boxed{\vec{T}_p + \vec{T}_s + \vec{T}_b = \oint_{CS}\left(\vec{r}\times\vec{q}\right)\rho\left(\vec{q}\cdot d\vec{A}\right) + \frac{\partial}{\partial t}\int_{CV}\left(\vec{r}\times\vec{q}\right)\rho\, dV \qquad (7.12)}$$

$$x-dir.: \quad \left|\left(\vec{r}\times\vec{q}\right)_{yz}\right| = r_{yz}\, q_{yz}\sin\left(\frac{\pi}{2}-\alpha_{yz}\right) = \left(rq\cos\alpha\right)$$

angle between q_{yz} and r_{yz}

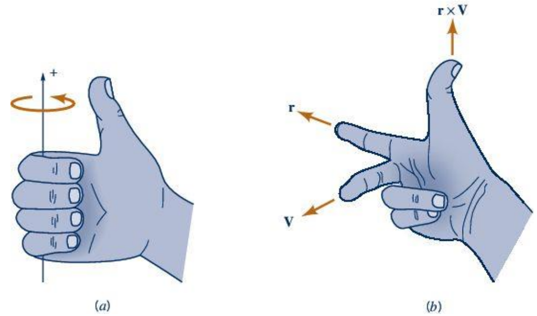
7.2 The Moment of Momentum Equation for Finite Control Volumes

$$x-dir.: \quad \vec{T}_{px} + \vec{T}_{sx} + \vec{T}_{bx} = \oint_{CS} (rq\cos\alpha)_{yz}\, \rho(\vec{q}\cdot d\vec{A}) + \frac{\partial}{\partial t}\int_{CV} (rq\cos\alpha)_{yz}\, \rho\, dV$$

$$y-dir.: \quad \vec{T}_{py} + \vec{T}_{sy} + \vec{T}_{by} = \oint_{CS} (rq\cos\alpha)_{zx}\, \rho(\vec{q}\cdot d\vec{A}) + \frac{\partial}{\partial t}\int_{CV} (rq\cos\alpha)_{zx}\, \rho\, dV$$

$$z-dir.: \quad \vec{T}_{pz} + \vec{T}_{sz} + \vec{T}_{bz} = \oint_{CS} (rq\cos\alpha)_{xy}\, \rho(\vec{q}\cdot d\vec{A}) + \frac{\partial}{\partial t}\int_{CV} (rq\cos\alpha)_{xy}\, \rho\, dV$$

(7.13)



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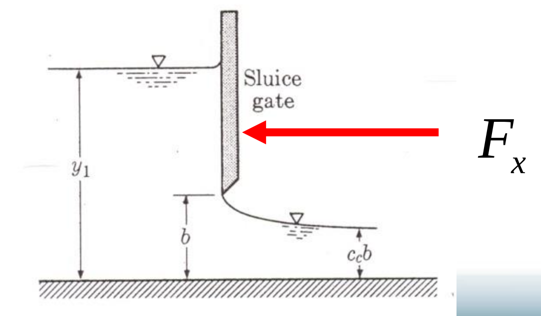
Problems

Homework Assignment # 3

Due: 1 week from today

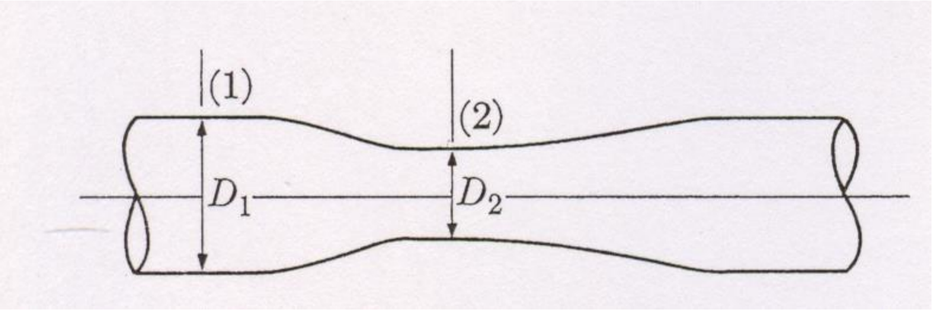
7-1. Derive the equation for the volume rate of flow per unit width for the sluice gate shown in Fig. 4-20 in terms of the geometric variable b, y_1, and C_C. Assume the pressure in hydrostatic at y_1 and $c_c b$ and the velocity is constant over the depth at each of these sections.

7-2. Derive the expression for the total force per unit width exerted by the sluice gate on the fluid in terms of vertical distances shown in Fig. 4-20.



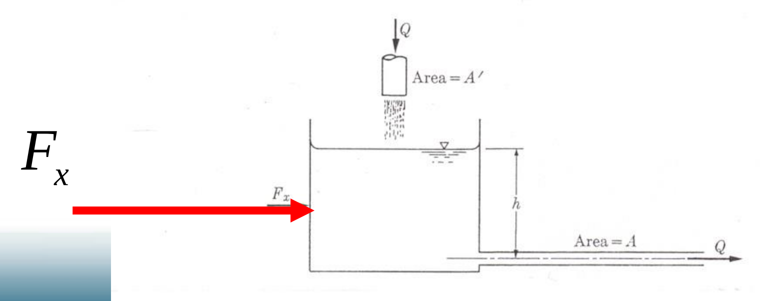
Problems

7-3. Consider the flow of an incompressible fluid through the Venturi meter shown in Fig. 4-22. Assuming uniform flow at sections (1) and (2) neglecting all losses, find the pressure difference between these sections as a function of the flow rate *Q,* the diameters of the sections, and the density of the fluid, ρ. Note that for a given configuration, *Q* is a function of only the pressure drop and fluid density.



Problems

7-4. Water flows into a tank from a supply line and out of the tank through a horizontal pipe as shown in Fig. 4-23. The rates of <u>inflow and outflow are the same</u>, and the water surface in the tank remains a distance *h* above the discharge pipe centerline. All velocities in the tank are negligible compared to those in the pipe. The head loss between the tank and the pipe exit is H_L (a) Find the <u>discharge</u> Q in terms of *h, A,* and $H_{L.}$ (b) What is the <u>horizontal force</u>, F_X required to keep the tank from moving? (c) If the supply line has an area A', what is the vertical force exerted on the water in the tank by the vertical jet?



Problems

7-5. Derive the one-dimensional continuity equation for the unsteady, non-uniform flow of an incompressible liquid in a horizontal open channel as shown in Fig. 4-29. The channel has a rectangular cross section of a constant width, b. Both the depth, y_0 and the mean velocity, V are functions of x and t.

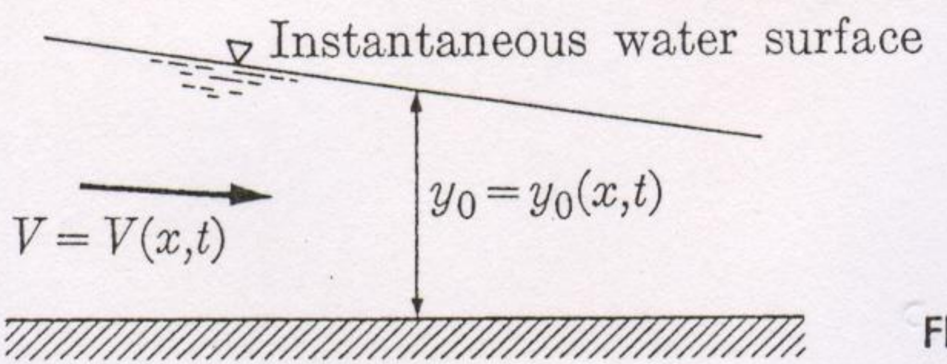


FIGURE 4–29