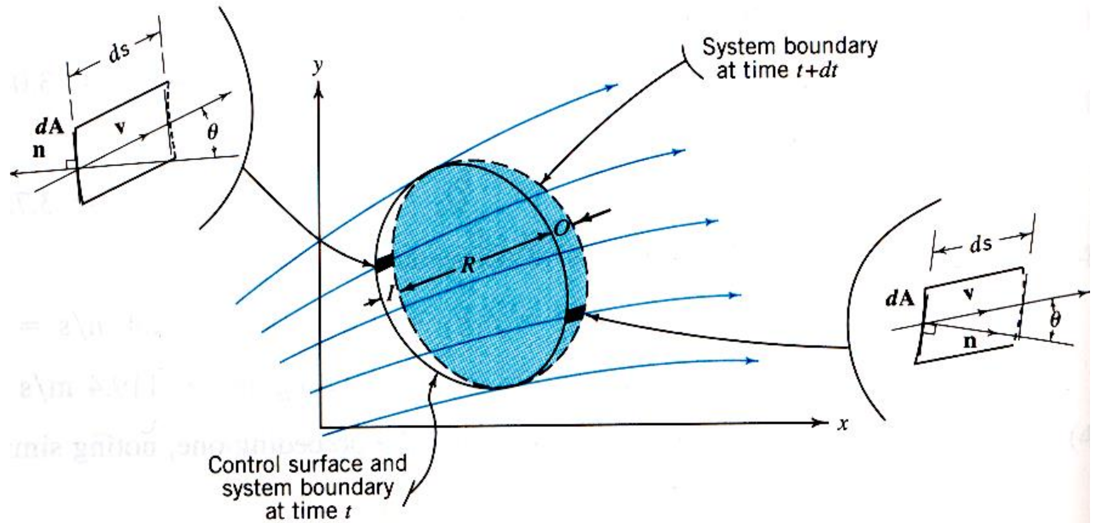
Lecture 7

Continuity, Energy, and Momentum Equations (3)



Contents

Objectives

- Derive the momentum equation by applying Newton's 2nd law of motion and Reynolds Transport Theorem
- Derive the moment of the momentum equation by applying Newton's 2nd law to rotating fluid masses

7.1 Linear Momentum Equation for Finite Control Volumes

7.1.1 Momentum Principle

- The momentum equation can be derived from Newton's 2nd law of motion

$$\vec{F} = m\vec{a} = m\frac{d\vec{q}}{dt} = \frac{d\left(m\vec{q}\right)}{dt} = \frac{d\vec{M}}{dt} \qquad (7.1)$$

$\vec{M}$ = linear momentum vector $= m\vec{q}$

$\vec{F}$ = external force

$\vec{F} =$
- boundary (surface) forces:
  - normal to boundary - pressure, $\vec{F}_p$
  - tangential to boundary - shear, $\vec{F}_s$
- body forces - force due to gravitational or magnetic fields, $\vec{F}_b$

7.1 Linear Momentum Equation for Finite Control Volumes

$$\vec{F}_p + \vec{F}_s + \vec{F}_b = \frac{d\vec{M}}{dt} \tag{7.2}$$

$\vec{F}_b = \int_{CV} f_b (\rho \, dV)$, where f_b = body force per unit mass

7.1 Linear Momentum Equation for Finite Control Volumes

7.1.2 The general linear momentum equation

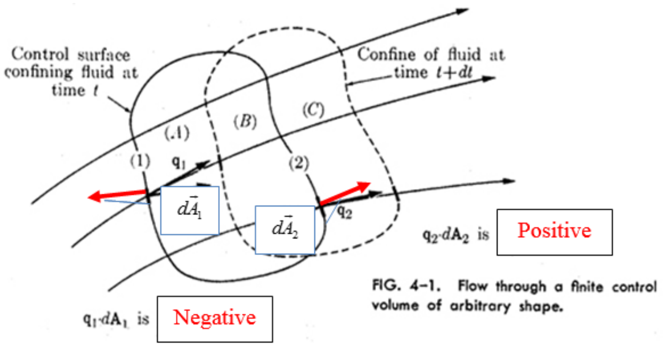


FIG. 4-1. Flow through a finite control volume of arbitrary shape.

Consider change of momentum

$\dfrac{d\vec{M}}{dt}$ = total rate of change of momentum

= net momentum flux across the CV boundaries

+ time rate of increase of momentum within CV

$$= \oint_{CS} \vec{q}\rho\left(\vec{q}\cdot d\vec{A}\right) + \frac{\partial}{\partial t}\int_{CV} \vec{q}\rho\, dV \qquad (7.3)$$ ← *Reynolds Transport Theorem*

where $\vec{q}\rho\left(\vec{q}\cdot d\vec{A}\right)$ = momentum flux = $\text{velocity} \times \text{mass per time}$

$d\vec{A}$ = vector unit area pointing outward over the control surface

7.1 Linear Momentum Equation for Finite Control Volumes

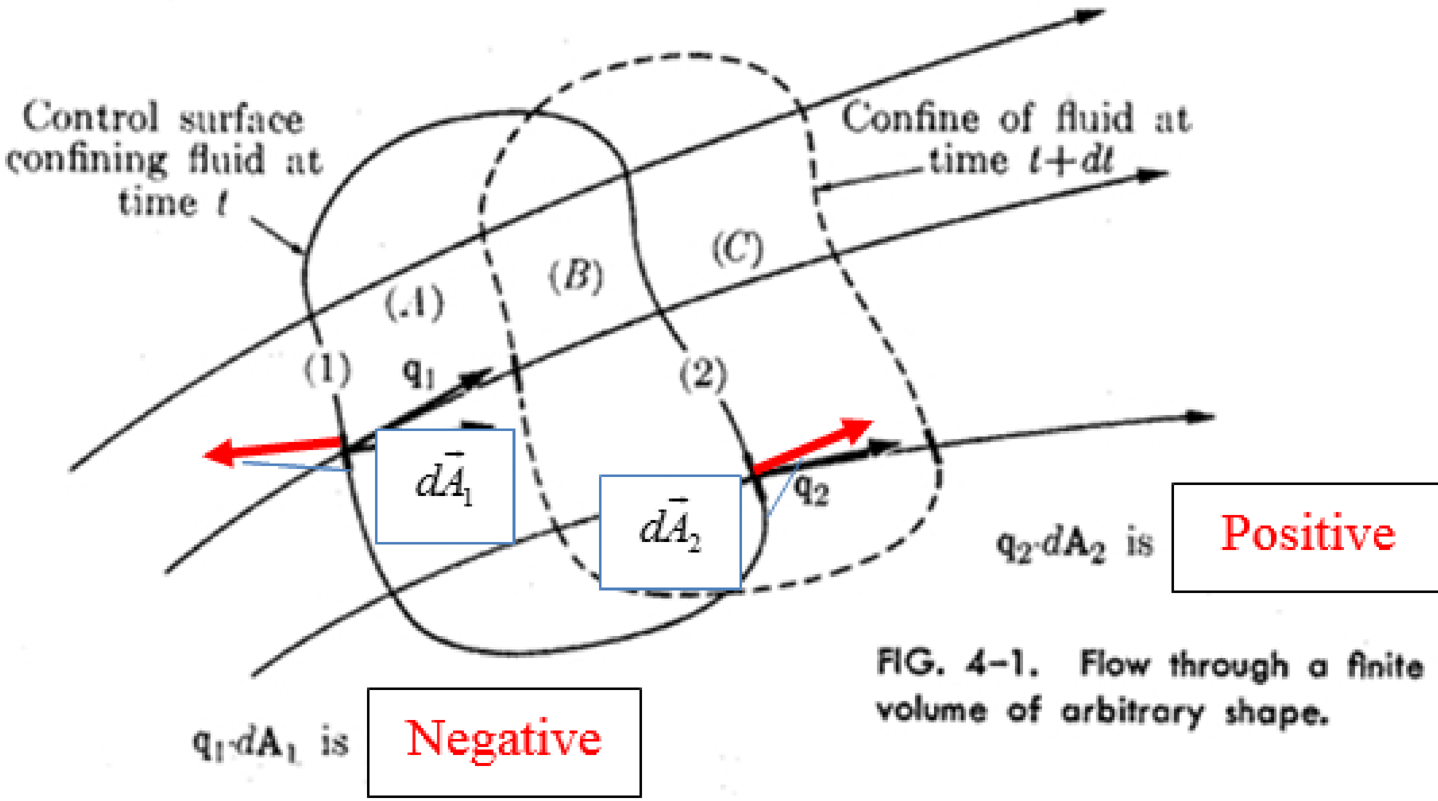


FIG. 4-1. Flow through a finite control volume of arbitrary shape.

7.1 Linear Momentum Equation for Finite Control Volumes

Substitute (7.3) into (7.2)

$$\vec{F}_p + \vec{F}_s + \vec{F}_b = \oint_{CS} \vec{q}\rho\left(\vec{q}\cdot d\vec{A}\right) + \frac{\partial}{\partial t}\int_{CV} \vec{q}\rho\, dV \qquad (7.4)$$

For steady flow and negligible body forces

$$\vec{F}_p + \vec{F}_s = \oint_{CS} \vec{q}\rho\left(\vec{q}\cdot d\vec{A}\right) \qquad (7.5)$$

- Eq. (7.4)
  - It is applicable to both ideal fluid systems and viscous fluid systems involving friction and energy dissipation.
  - It is applicable to both compressible fluid and incompressible fluid.

7.1 Linear Momentum Equation for Finite Control Volumes

- Combined effects of friction, energy loss, and heat transfer appear implicitly in the magnitude of the external forces, with corresponding effects on the local flow velocities.
- Knowledge of the internal conditions is not necessary.
- We can consider only external conditions.

7.1 Linear Momentum Equation for Finite Control Volumes

7.1.3 Inertial control volume for a generalized apparatus

- Three components of the forces

$$x-dir.:\ \vec{F}_{p_x}+\vec{F}_{s_x}+\vec{F}_{b_x}=\oint_{CS}u\rho\left(\vec{q}\cdot d\vec{A}\right)+\frac{\partial}{\partial t}\int_{CV}u\rho\,dV$$

$$y-dir.:\ \vec{F}_{p_y}+\vec{F}_{s_y}+\vec{F}_{b_y}=\oint_{CS}v\rho\left(\vec{q}\cdot d\vec{A}\right)+\frac{\partial}{\partial t}\int_{CV}v\rho\,dV$$

$$z-dir.:\ \vec{F}_{p_z}+\vec{F}_{s_z}+\vec{F}_{b_z}=\oint_{CS}w\rho\left(\vec{q}\cdot d\vec{A}\right)+\frac{\partial}{\partial t}\int_{CV}w\rho\,dV \qquad (7.6)$$

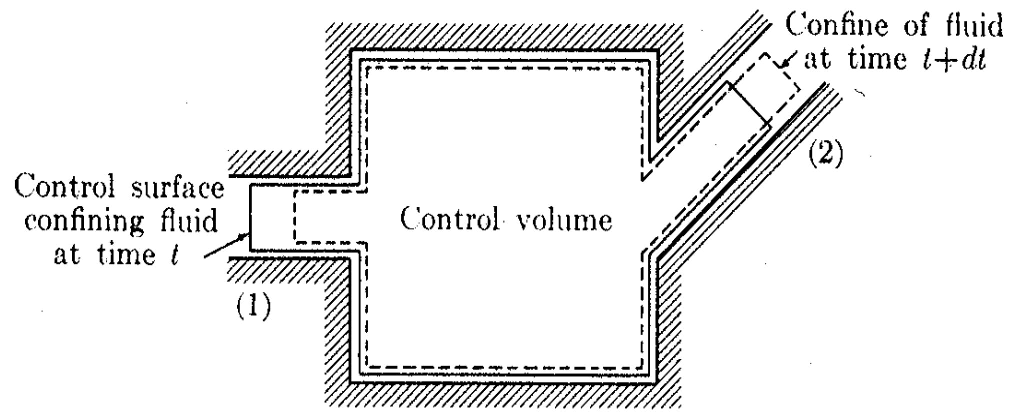
7.1 Linear Momentum Equation for Finite Control Volumes

- For flow through generalized apparatus

$$x-dir.:\ \vec{F}_{p_x}+\vec{F}_{s_x}+\vec{F}_{b_x}=\int_2 u\rho\, dQ-\int_1 u\rho\, dQ+\frac{\partial}{\partial t}\int_{CV} u\rho\, dV$$

- For 1D steady flow,

$$\frac{\partial}{\partial t}\int_{CV} q\rho\, dV=0$$



7.1 Linear Momentum Equation for Finite Control Volumes

- Velocity and density are constant normal to the flow direction.

$$x-dir.:\ \vec{F}_{p_x}+\vec{F}_{s_x}+\vec{F}_{b_x}=\sum F_x=\left(V_x\rho Q\right)_2-\left(V_x\rho Q\right)_1$$

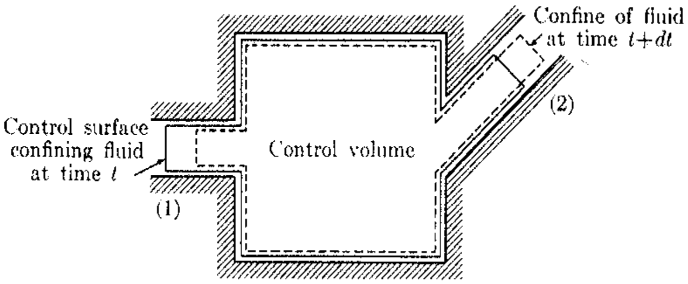
$$=V_{x_2}\rho_2 Q_2 - V_{x_1}\rho_1 Q_1 = Q\rho\left(V_{x_2}-V_{x_1}\right)=Q\rho\left(V_{x_{out}}-V_{x_{in}}\right)$$

$$\rho_1 Q_1 = \rho_2 Q_2 = Q\rho$$

$$y-dir.:\ \sum F_y=\left(V_y\rho Q\right)_2-\left(V_y\rho Q\right)_1$$

$$z-dir.:\ \sum F_z=\left(V_z\rho Q\right)_2-\left(V_z\rho Q\right)_1$$

where V = average velocity in flow direction



7.1 Linear Momentum Equation for Finite Control Volumes

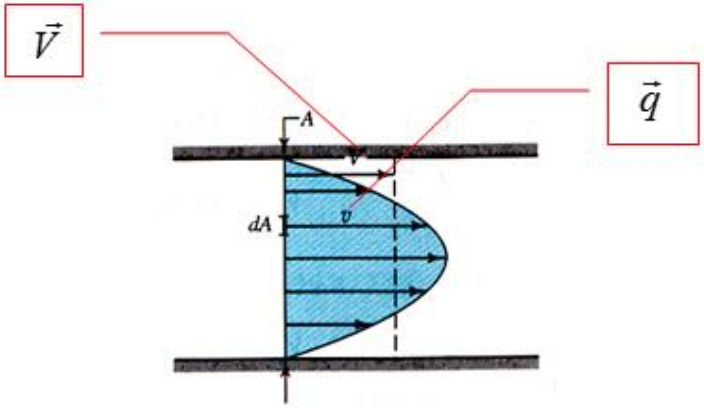
- Non-uniform velocity profile

If velocity varies over the cross section, then introduce momentum flux coefficient

$$\int \vec{q}\rho\left(\vec{q}\cdot d\vec{A}\right) = K_m\vec{V}\left(\rho VA\right)$$

$$\int \vec{q}\rho\, dQ = K_m\vec{V}\rho Q$$

$$K_m = \frac{\int \vec{q}\rho dQ}{\vec{V}\rho Q}$$



7.1 Linear Momentum Equation for Finite Control Volumes

where

V = magnitude of average velocity over cross section = Q/A

$\vec{V}$ = average velocity vector

K_m = momentum flux coefficient ≥ 1

= 1.33 for laminar flow (pipe flow)
1.03-1.04 for turbulent flow (smooth pipe)

$$\sum F_x = \left(K_m V_x \rho Q\right)_2 - \left(K_m V_x \rho Q\right)_1$$

$$\sum F_y = \left(K_m V_y \rho Q\right)_2 - \left(K_m V_y \rho Q\right)_1$$

$$\sum F_z = \left(K_m V_z \rho Q\right)_2 - \left(K_m V_z \rho Q\right)_1$$

7.1 Linear Momentum Equation for Finite Control Volumes

[Cf] Energy correction coefficient

$$K_e = \frac{\int \frac{\rho}{2} q^2 dQ}{\frac{\rho}{2} \vec{V}^2 Q}$$

7.1 Linear Momentum Equation for Finite Control Volumes

[Example 7-1] Continuity, energy, and linear momentum with unsteady flow

A large tank mounted on rollers is filled with water to a depth of 16 ft above a discharge port. At time $t =$ 0, the fast-acting valve on the discharge nozzle is opened. Determine depth h, discharge rate Q, and force F necessary to keep the tank stationary at $t = 50$ *sec*.

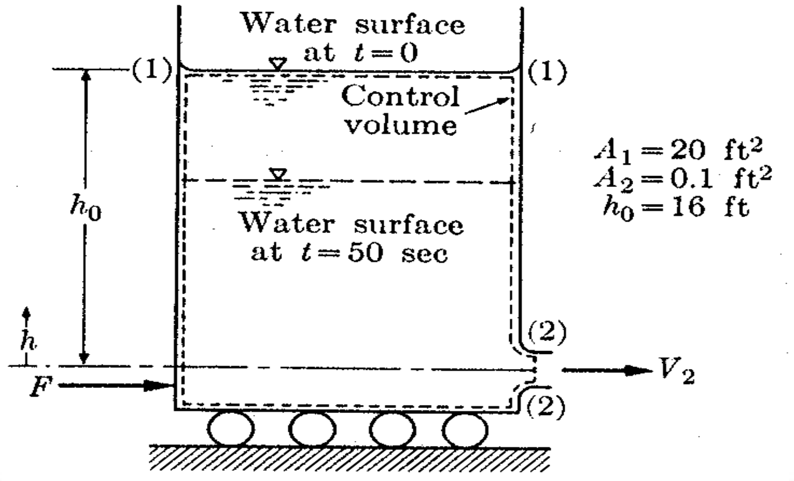
Assumptions:

$V_1 \approx 0$

$\rho = \text{const.}$

$p_2 = p_{atm} = 0$

$h_2 = 0$ (datum)



7.1 Linear Momentum Equation for Finite Control Volumes

Apply continuity, energy, and linear momentum equations

(7.7)

$$\int_{CV} \frac{\partial \rho}{\partial t} dV + \oint_{CS} \rho \vec{q} \cdot d\vec{A} = 0$$

(7.8)

$$\frac{\delta Q}{dt} - \frac{\delta W_{shaft}}{dt} - \frac{\delta W_{shear}}{dt} = \oint_{CS} \left(\frac{p}{\rho} + u + gh + \frac{q^2}{2} \right) \rho \left(\vec{q} \cdot d\vec{A} \right) + \frac{\partial}{\partial t} \int_{CV} e \rho \, dV$$

(7.9)

$$\vec{F}_p + \vec{F}_s + \vec{F}_b = \oint_{CS} \vec{q} \rho \left(\vec{q} \cdot d\vec{A} \right) + \frac{\partial}{\partial t} \int_{CV} \vec{q} \rho \, dV$$

7.1 Linear Momentum Equation for Finite Control Volumes

i) Use integral form of continuity equation, Eq. (7.7)

$$\frac{\partial}{\partial t}\int_{CV} \rho \, dV = \int \rho q_n dA_1 - \int \rho q_n dA_2$$

$dV = A_1 dh$, $\rho q_n dA_1 = 0$ (because no inflow across the Section 1)

$$\therefore \quad \rho A_1 \frac{\partial}{\partial t}\int_0^h dh = -\rho V_2 A_2$$

$$A_1 \frac{dh}{dt} = -V_2 A_2 \qquad \text{(A)}$$

ii) Energy equation, Eq. (7.8)

~ no shaft work

~ heat transfer and temperature changes due to friction are negligible

7.1 Linear Momentum Equation for Finite Control Volumes

$$\frac{\delta Q}{dt} - \frac{\delta W_{shaft}}{dt} - \frac{\delta W_{shear}}{dt}$$

$$= \underbrace{\oint_{CS} \left(\frac{p}{\rho} + u + gh + \frac{q^2}{2} \right) \rho \left(\vec{q} \cdot d\vec{A} \right)}_{\text{I}} + \underbrace{\frac{\partial}{\partial t} \int_{CV} e \rho \, dV}_{\text{II}}$$

e = energy per unit mass = $u + gh + \dfrac{q^2}{2}$

$$\text{I} = \oint_{CS} \left(u + \frac{p}{\rho} + gh + \frac{q^2}{2} \right) \rho \left(\vec{q} \cdot d\vec{A} \right)$$

$$= \left(u + \frac{p}{\rho} + gh + \frac{q^2}{2} \right)_2 \rho V_2 A_2 - \left(u + \frac{p}{\rho} + gh + \frac{q^2}{2} \right)_1 \rho V_1 A_1$$

7.1 Linear Momentum Equation for Finite Control Volumes

$$= \left(u + \frac{p}{\rho} + gh + \frac{q^2}{2}\right)_2 \rho V_2 A_2 \qquad (V_1 \approx 0)$$

$$\text{II} = \frac{\partial}{\partial t}\int_{CV} e\rho\, dV = \frac{\partial}{\partial t}\int_{CV}\left(u + gh + \cancel{\frac{q^2}{2}}\right)\rho dV$$

$A_1 dh$

∵ nearly constant in the tank except near the nozzle

$$= A_1\rho\frac{\partial}{\partial t}\int_0^h (u + gh)\,dh$$

$$\therefore\ 0 = \left(u + \frac{p}{\rho} + gh + \frac{q^2}{2}\right)_2 \rho V_2 A_2 + A_1\rho\frac{\partial}{\partial t}\int_0^h (u + gh)\,dh$$

7.1 Linear Momentum Equation for Finite Control Volumes

Assume $\rho = \text{const.}\ ,\quad p_2 = p_{atm} = 0\ ,\quad h_2 = 0\ (\text{datum})$

$$0 = uV_2A_2 + \frac{V_2^2}{2}V_2A_2 + uA_1\frac{dh}{dt} + A_1gh\frac{dh}{dt} \qquad \text{(B)}$$

$$A_1\frac{dh}{dt} = -V_2A_2$$

Substitute (A) into (B)

$$0 = \cancel{uV_2A_2} + \frac{V_2^2}{2}V_2A_2 + \cancel{u\left(-V_2A_2\right)} + gh\left(-V_2A_2\right)$$

$$\therefore \quad \frac{V_2^2}{2}V_2A_2 = ghV_2A_2$$

$$V_2 = \sqrt{2gh} \qquad \text{(C)}$$

7.1 Linear Momentum Equation for Finite Control Volumes

Substitute (C) into (A)

$$A_2\sqrt{2gh} = -A_1\frac{dh}{dt}$$

$$\frac{dh}{\sqrt{h}} = -\frac{A_2}{A_1}\sqrt{2g}\,dt$$

Integrate

$$\int_{h_0}^{h}\frac{dh}{\sqrt{h}} = \int_0^t -\frac{A_2}{A_1}\sqrt{2g}\,dt$$

$$\left\{\int_{h_0}^{h} h^{-\frac{1}{2}}dh = \left[2h^{\frac{1}{2}}\right]_{h_0}^{h}\right\}$$

$$h = \left(h_0^{\frac{1}{2}} - \frac{A_2}{A_1}\frac{\sqrt{2g}}{2}t\right)^2$$

7.1 Linear Momentum Equation for Finite Control Volumes

$$h = \left(\sqrt{16} - \frac{0.1}{20} \frac{\sqrt{2(32.2)}}{2} t \right)^2$$

$$= (4 - 0.0201t)^2$$

At $t = 50\sec$, $h = (4 - 0.0201 \times 50)^2 = 8.98\, ft$

$$V_2 = \sqrt{2gh} = \sqrt{2(32.2)(8.98)} = 24.05 \text{ fps}$$

$$Q_2 = (VA)_2 = 24.05(0.1) = 2.405 \text{ cfs}$$

7.1 Linear Momentum Equation for Finite Control Volumes

iii) Momentum equation, Eq. (7.4)

I II

$$\vec{F}_p + \vec{F}_s + \vec{F}_b = \oint_{CS} \vec{q}\rho\left(\vec{q}\cdot d\vec{A}\right) + \frac{\partial}{\partial t}\int_{CV} \vec{q}\rho\, dV$$

II = Time rate of change of momentum inside CV is negligible

if tank area (A_1) is large compared to the nozzle area (A_2).

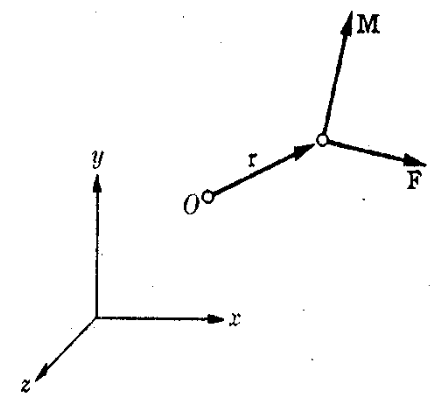
$$\text{I} = \oint_{CS} \vec{q}\rho\left(\vec{q}\cdot d\vec{A}\right) = \int q_n \rho q_n dA_2 - \int q_n \rho q_n dA_1 = V_2 \rho V_2 A_2$$

$$\therefore F_{px} = V_2 \rho V_2 A_2 = V_2 \rho Q_2$$

$$F_{px} = (24.05)(1.94)(2.405) = 112 \text{ lb}$$

7.2 The Moment of Momentum Equation for Finite Control Volumes

7.2.1 The Moment of momentum principle for inertial reference systems

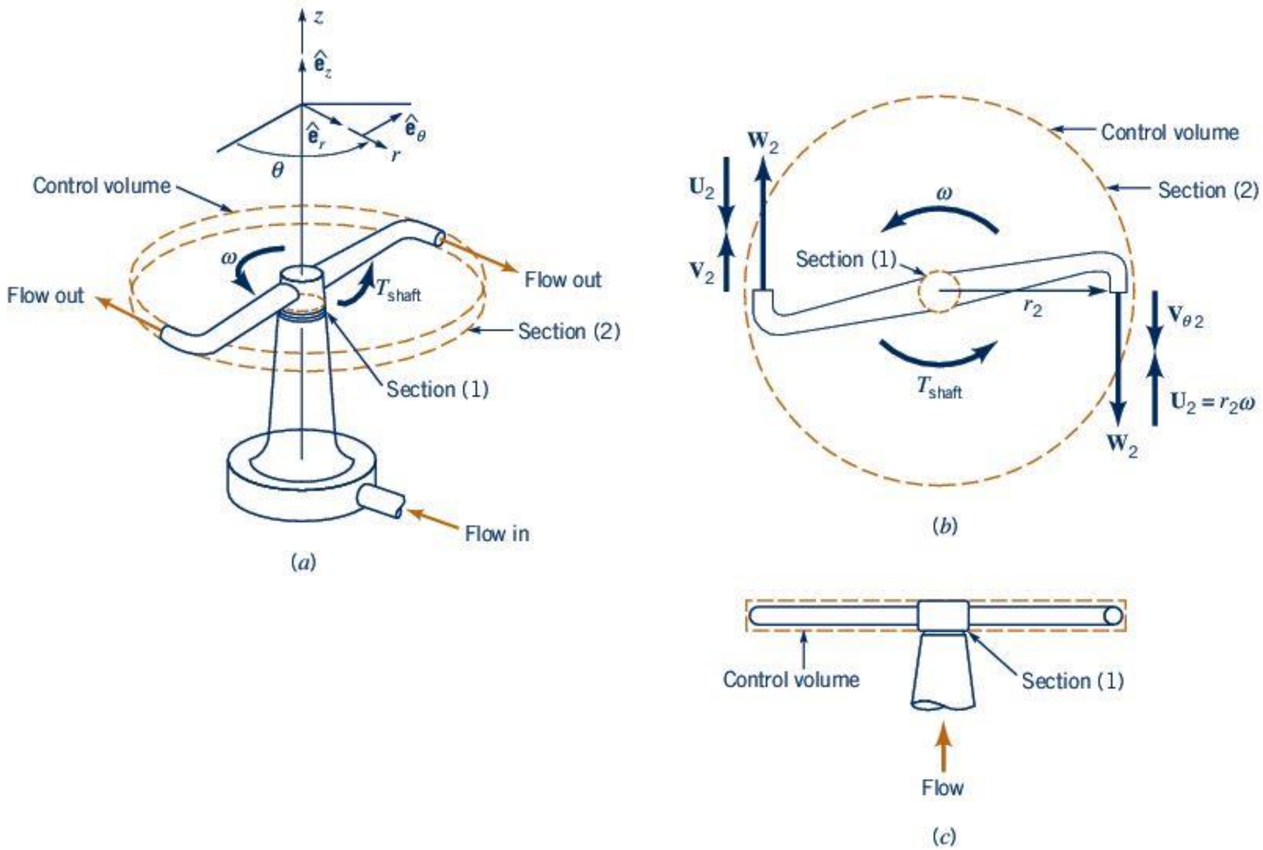


Apply Newton's 2nd law to rotating fluid masses

→ The vector sum of all the external moments acting on a fluid mass $\left(\vec{r}\times\vec{F}\right)$ equals the time rate of change of the moment of momentum (angular momentum) vector $\left(\vec{r}\times\vec{M}\right)$ of the fluid mass.

Example: rotary lawn sprinklers, ceiling fans, wind turbines

7.2 The Moment of Momentum Equation for Finite Control Volumes



7.2 The Moment of Momentum Equation for Finite Control Volumes

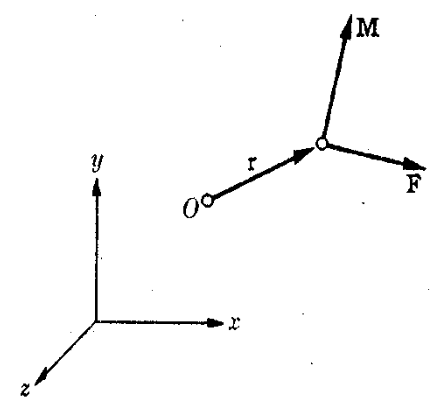
$$T = \vec{r} \times \vec{F} = \frac{d}{dt}\left(\vec{r} \times \vec{M}\right) \qquad (7.10)$$

where

T = torque

$\vec{r}$ = position vector of a mass in an arbitrary curvilinear motion

$\vec{M}$ = linear momentum



7.2 The Moment of Momentum Equation for Finite Control Volumes

[Re] Derivation of (7.10)

Eq. (7.1): $$\vec{F} = \frac{d\vec{M}}{dt}$$

Take the vector cross product of $\vec{r}$

$$\vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{M}}{dt}$$

By the way,

$$\frac{d}{dt}\left(\vec{r} \times \vec{M}\right) = \underbrace{\frac{d\vec{r}}{dt} \times \vec{M}}_{\text{I}} + \vec{r} \times \frac{d\vec{M}}{dt}$$

7.2 The Moment of Momentum Equation for Finite Control Volumes

$$I = \frac{d\vec{r}}{dt} \times \vec{M} = \vec{q} \times m\,\vec{q} = 0 \quad \left(\because \frac{d\vec{r}}{dt} = \vec{q} \right)$$

$$\left(\therefore\ \vec{q} \times \vec{q} = |\vec{q}|\,|\vec{q}| \sin 0^{\circ} = 0 \right)$$

$$\therefore \quad \left(\vec{r} \times \frac{d\vec{M}}{dt} \right) = \frac{d}{dt}\left(\vec{r} \times \vec{M} \right)$$

$$\therefore \therefore \quad \vec{r} \times \vec{F} = \frac{d}{dt}\left(\vec{r} \times \vec{M} \right)$$

where $\vec{r} \times \vec{M}$ = angular momentum (moment of momentum)

7.2 The Moment of Momentum Equation for Finite Control Volumes

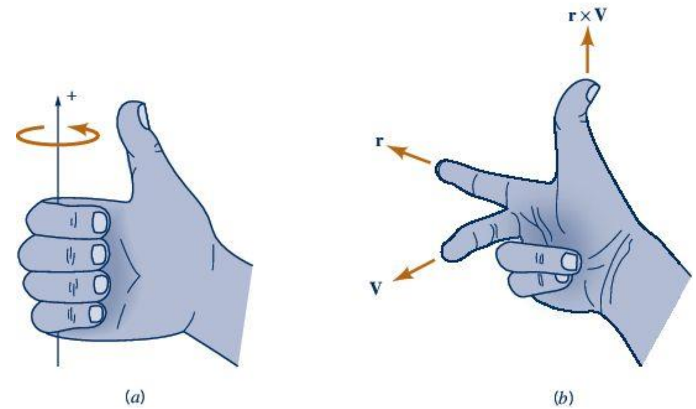
[Re] Torque $\vec{T} = \vec{r} \times \vec{F}$

- translational motion $\rightarrow$

Force - linear acceleration

- rotational motion $\rightarrow$

Torque - angular acceleration



[Re] Vector Product

$$\vec{V} = \vec{a} \times \vec{b}$$

Magnitude = $\left|\vec{V}\right| = \left|\vec{a}\right| \times \left|\vec{b}\right| \sin\gamma$ = area of parallelogram

direction = perpendicular to plane of $\vec{a}$ and $\vec{b}$ $\rightarrow$ right-handed triple

7.2 The Moment of Momentum Equation for Finite Control Volumes

$$\vec{b} \times \vec{a} = -\left(\vec{a} \times \vec{b}\right)$$

$$\left(k\vec{a}\right) \times \vec{b} = k\left(\vec{a} \times \vec{b}\right)$$

$$\vec{a} \times \left(\vec{b} + \vec{c}\right) = \left(\vec{a} \times \vec{b}\right) + \left(\vec{a} \times \vec{b}\right)$$

- External moments arise from external forces

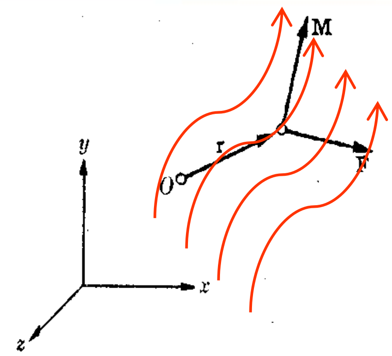
$$\underbrace{\left(\vec{r} \times \vec{F}_p\right)}_{\vec{T}_p} + \underbrace{\left(\vec{r} \times \vec{F}_s\right)}_{\vec{T}_s} + \underbrace{\left(\vec{r} \times \vec{F}_b\right)}_{\vec{T}_b} = \frac{d}{dt}\left(\vec{r} \times \vec{M}\right)$$

$$\vec{T}_p + \vec{T}_s + \vec{T}_b = \frac{d}{dt}\left(\vec{r} \times \vec{M}\right) \qquad (7.11)$$

where $\vec{T}_p$, $\vec{T}_s$, $\vec{T}_b$ = external torque

7.2 The Moment of Momentum Equation for Finite Control Volumes

7.2.2 The general moment of momentum equation



(7.3): $$\frac{d\vec{M}}{dt} = \oint_{CS} \vec{q}\rho\left(\vec{q}\cdot d\vec{A}\right) + \frac{\partial}{\partial t}\int_{CV} \vec{q}\rho\, dV$$

$$\therefore \quad \frac{d}{dt}\left(\vec{r}\times\vec{M}\right) = \oint_{CS} \left(\vec{r}\times\vec{q}\right)\rho\left(\vec{q}\cdot d\vec{A}\right) + \frac{\partial}{\partial t}\int_{CV} \left(\vec{r}\times\vec{q}\right)\rho\, dV$$

$$\vec{T}_p + \vec{T}_s + \vec{T}_b = \oint_{CS} \left(\vec{r}\times\vec{q}\right)\rho\left(\vec{q}\cdot d\vec{A}\right) + \frac{\partial}{\partial t}\int_{CV} \left(\vec{r}\times\vec{q}\right)\rho\, dV \qquad (7.12)$$

$$x-dir.: \quad \left|\left(\vec{r}\times\vec{q}\right)_{yz}\right| = r_{yz}\, q_{yz} \sin\left(\frac{\pi}{2} - \alpha_{yz}\right) = \left(rq\cos\alpha\right)$$

angle between q_{yz} and r_{yz}

7.2 The Moment of Momentum Equation for Finite Control Volumes

$$x-dir.:\quad \vec{T}_{px}+\vec{T}_{sx}+\vec{T}_{bx}=\oint_{CS}(rq\cos\alpha)_{yz}\,\rho(\vec{q}\cdot d\vec{A})+\frac{\partial}{\partial t}\int_{CV}(rq\cos\alpha)_{yz}\,\rho\, dV$$

$$y-dir.:\quad \vec{T}_{py}+\vec{T}_{sy}+\vec{T}_{by}=\oint_{CS}(rq\cos\alpha)_{zx}\,\rho(\vec{q}\cdot d\vec{A})+\frac{\partial}{\partial t}\int_{CV}(rq\cos\alpha)_{zx}\,\rho\, dV$$

$$z-dir.:\quad \vec{T}_{pz}+\vec{T}_{sz}+\vec{T}_{bz}=\oint_{CS}(rq\cos\alpha)_{xy}\,\rho(\vec{q}\cdot d\vec{A})+\frac{\partial}{\partial t}\int_{CV}(rq\cos\alpha)_{xy}\,\rho\, dV$$

(7.13)

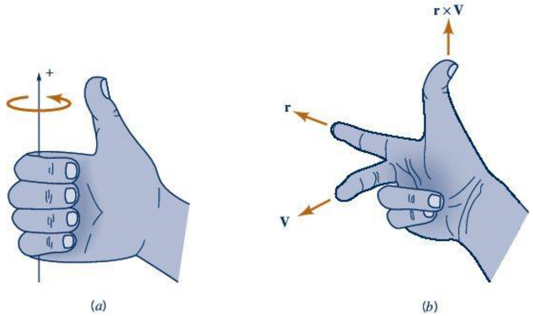


fig_05_05

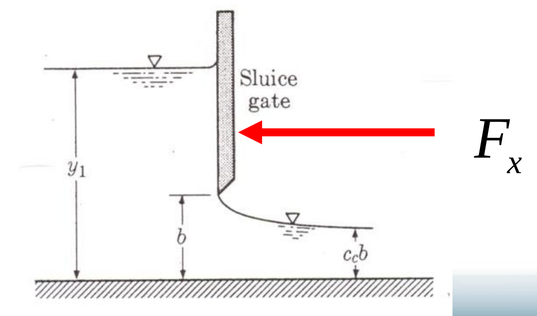
Problems

Homework Assignment # 3

Due: 1 week from today

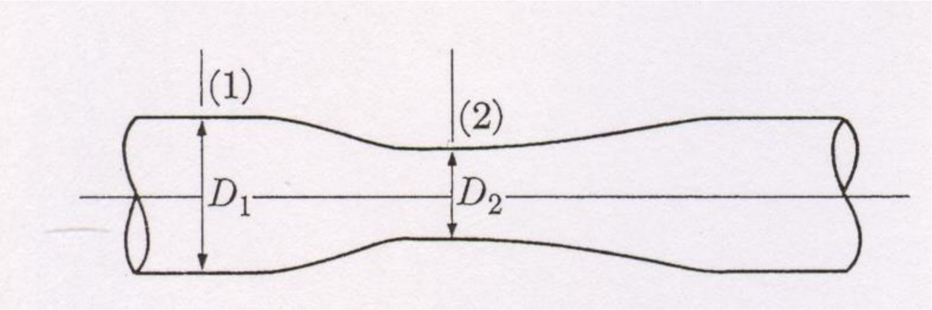
7-1. Derive the equation for the volume rate of flow per unit width for the sluice gate shown in Fig. 4-20 in terms of the geometric variable b, y_1, and C_C. Assume the pressure in hydrostatic at y_1 and $c_c b$ and the velocity is constant over the depth at each of these sections.

7-2. Derive the expression for the total force per unit width exerted by the sluice gate on the fluid in terms of vertical distances shown in Fig. 4-20.



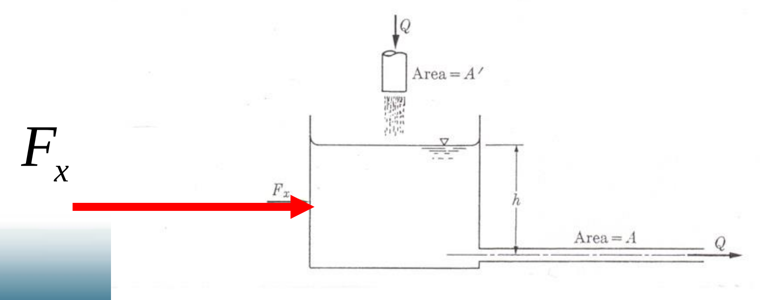
Problems

7-3. Consider the flow of an incompressible fluid through the Venturi meter shown in Fig. 4-22. Assuming uniform flow at sections (1) and (2) neglecting all losses, find the pressure difference between these sections as a function of the flow rate *Q*, the diameters of the sections, and the density of the fluid, ρ. Note that for a given configuration, *Q* is a function of only the pressure drop and fluid density.



Problems

7-4. Water flows into a tank from a supply line and out of the tank through a horizontal pipe as shown in Fig. 4-23. The rates of <u>inflow and outflow are the same</u>, and the water surface in the tank remains a distance *h* above the discharge pipe centerline. All velocities in the tank are negligible compared to those in the pipe. The head loss between the tank and the pipe exit is H_L (a) Find the <u>discharge</u> Q in terms of *h, A,* and H_L. (b) What is the <u>horizontal force</u>, F_X required to keep the tank from moving? (c) If the supply line has an area *A'*, what is the vertical force exerted on the water in the tank by the vertical jet?



Problems

7-5. Derive the one-dimensional continuity equation for the unsteady, non-uniform flow of an incompressible liquid in a horizontal open channel as shown in Fig. 4-29. The channel has a rectangular cross section of a constant width, b. Both the depth, y_0 and the mean velocity, V are functions of x and t.

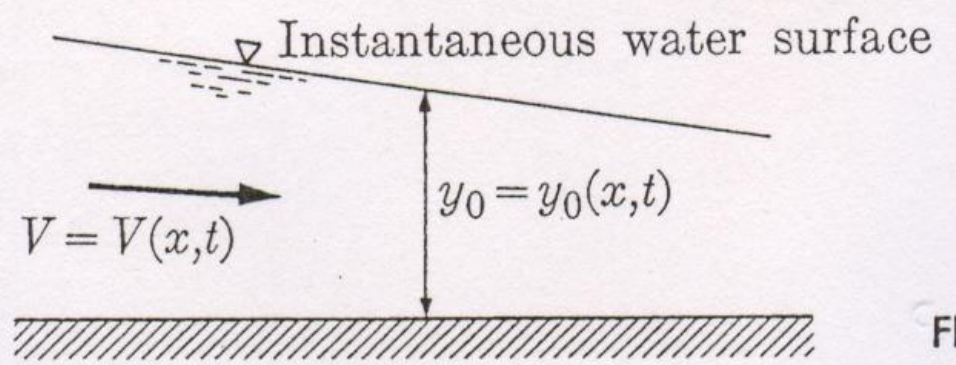


FIGURE 4–29