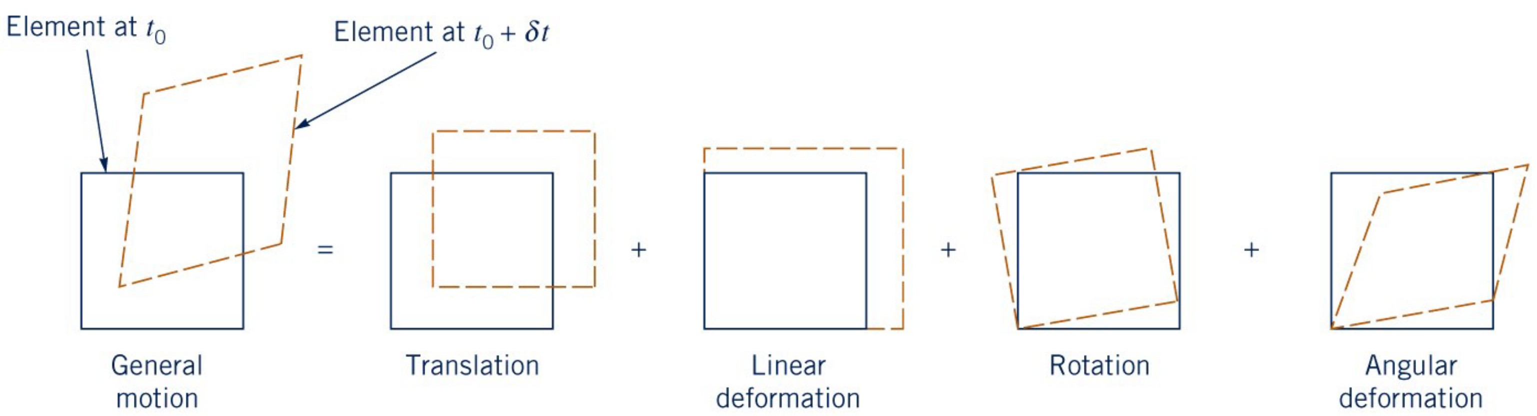
Chapter 5

Stress–Strain Relation



Chapter 5 Stress–Strain Relation

Contents

Objectives

- Understand tensor systems of stress and strain
- Study difference between displacement and deformation
- Study solid mechanics to deduce stress-rate of strain relations for fluid

5.2 Relations between Stress and Strain for Elastic Solids

5.2.1 Normal Stresses

Hooke's law: Stress is linear with strain. $\sigma_x \propto \varepsilon_x^\circ$

$$\sigma_x = E\,\varepsilon_x^\circ$$

$$\varepsilon_x^\circ = \frac{1}{E}\,\sigma_x$$

$$\varepsilon_x = \frac{\partial \xi}{\partial x} \rightarrow \text{non-dimensional}$$

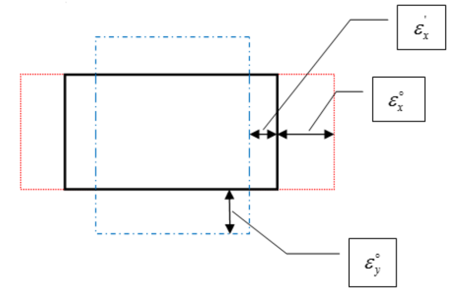
(5.8)

in which E = Young's modulus of elasticity (N/m^2)

ε_x° = elongation in the *x- dir* due to normal stress, σ_x

$$y-dir. \ : \ \varepsilon_y^\circ = \frac{\sigma_y}{E}$$

$$z-dir. \ : \ \varepsilon_z^\circ = \frac{\sigma_z}{E}$$

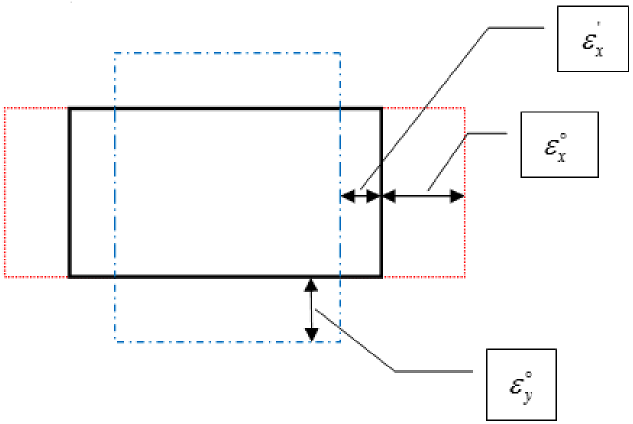


5.2 Relations between Stress and Strain for Elastic Solids

Now, we have to consider other elongations because of lateral contraction of matter under tension.

$\varepsilon_x{}'$ = elongation in the $x-dir.$ due to σ_y

$\varepsilon_x{}''$ = elongation in the $x-dir.$ due to σ_z



Now, define

$$\varepsilon_x{}' = -n\varepsilon_y^\circ = -n\frac{\sigma_y}{E} \qquad (5.9)$$

$$\varepsilon_x{}'' = -n\varepsilon_z^\circ = -n\frac{\sigma_z}{E} \qquad (5.10)$$

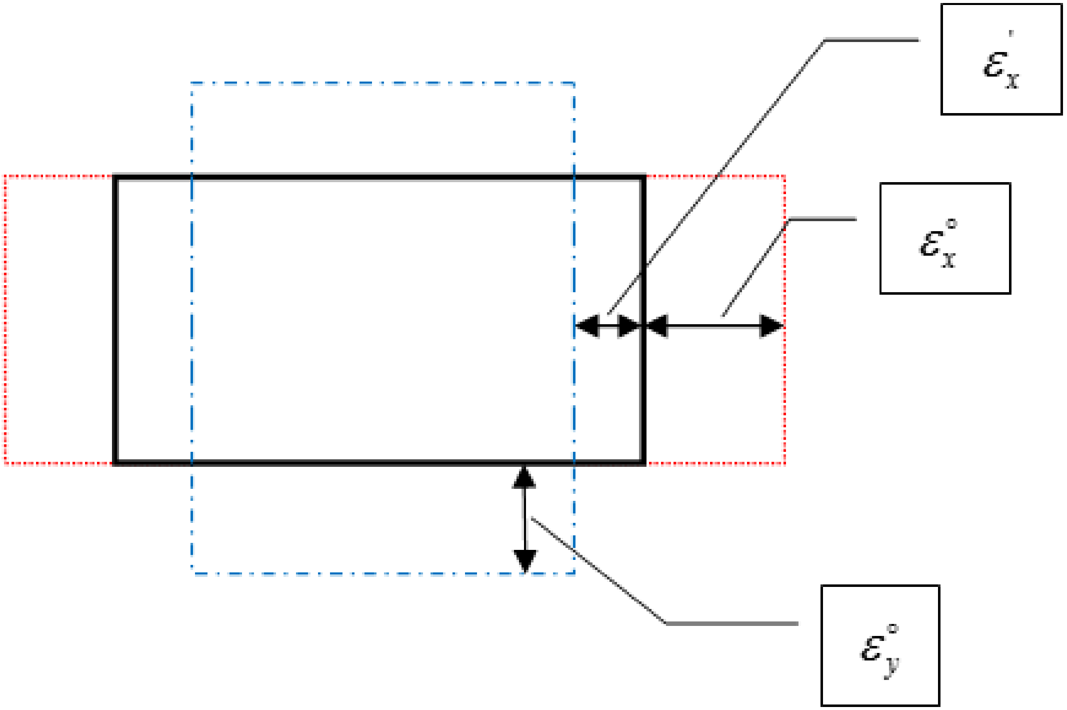
where n = Poisson's ratio

5.2 Relations between Stress and Strain for Elastic Solids

Poisson's ratio: n = 0~0.5

water~0.5; metal~0.3

cork: Poisson's ratio is high.



5.2 Relations between Stress and Strain for Elastic Solids

Thus, total strain ε_x is

$$\varepsilon_x = \varepsilon_x^{\circ} + \varepsilon_x' + \varepsilon_x'' = \frac{\sigma_x}{E} - \frac{n}{E}\left(\sigma_y + \sigma_z\right) = \frac{1}{E}\left[\sigma_x - n\left(\sigma_y + \sigma_z\right)\right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - n\left(\sigma_z + \sigma_x\right)\right] \quad (5.11)$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - n\left(\sigma_x + \sigma_y\right)\right] \quad (5.12)$$

5.2 Relations between Stress and Strain for Elastic Solids

5.2.2 Shear Stress

~ Hooke's law $\tau_{xy} = G\,\gamma_{xy}$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} = \frac{\partial \zeta}{\partial y} + \frac{\partial \eta}{\partial z}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G} = \frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial x} \qquad (5.13)$$

5.2 Relations between Stress and Strain for Elastic Solids

where G = shear modulus of elasticity (N/m^2)

$$G = \frac{E}{2(1+n)} \tag{5.14}$$

- Volume dilation

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1}{E}\left[\sigma_x - n\left(\sigma_y + \sigma_z\right)\right]$$

$$+ \frac{1}{E}\left[\sigma_y - n\left(\sigma_z + \sigma_x\right)\right]$$

$$+ \frac{1}{E}\left[\sigma_z - n\left(\sigma_x + \sigma_y\right)\right]$$

$$= \frac{1}{E}\left[(1-2n)\left(\sigma_x + \sigma_y + \sigma_z\right)\right] \tag{5.15}$$

5.2 Relations between Stress and Strain for Elastic Solids

- $\bar{\sigma}$ = arithmetic mean of 3 normal stresses

$$\bar{\sigma} = \frac{1}{3}\left(\sigma_x + \sigma_y + \sigma_z\right) \tag{5.16}$$

Combine Eqs. (5.12), (5.14) and (5.15)

$$\sigma_x = 2G\left[\varepsilon_x + \frac{ne}{1-2n}\right] \tag{5.17}$$

5.2 Relations between Stress and Strain for Elastic Solids

Therefore

$$\sigma_x - \overline{\sigma} = 2G\left(\varepsilon_x - \frac{e}{3}\right)$$

$$\sigma_y - \overline{\sigma} = 2G\left(\varepsilon_y - \frac{e}{3}\right)$$

$$\sigma_z - \overline{\sigma} = 2G\left(\varepsilon_z - \frac{e}{3}\right) \qquad (5.18)$$

$$\tau_{xy} = \tau_{yx} = G\left(\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}\right)$$

5.2 Relations between Stress and Strain for Elastic Solids

$$\tau_{zy} = \tau_{yz} = G\left(\frac{\partial \zeta}{\partial y} + \frac{\partial \eta}{\partial z}\right)$$

$$\tau_{xz} = \tau_{zx} = G\left(\frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial x}\right) \qquad (5.19)$$

[Proof] Derivation of Eqs. (5.17) & (5.18)

(5.15) → $$e = \frac{1}{E}(1-2n)\left(\sigma_x + \sigma_y + \sigma_z\right) \qquad (A)$$

(5.12) → $$\varepsilon_x = \frac{1}{E}\left[\sigma_x - n\left(\sigma_y + \sigma_z\right)\right] \qquad (B)$$

(5.14) → $$G = \frac{E}{2(1+n)} \rightarrow E = 2G(1+n) \qquad (C)$$

5.2 Relations between Stress and Strain for Elastic Solids

i) Combine (A) and (B)

$$
\begin{aligned}
&\frac{n}{(1-2n)}\times e = \frac{n}{(1-2n)}\frac{(1-2n)}{E}\left(\sigma_x+\sigma_y+\sigma_z\right) = \frac{n}{E}\left(\sigma_x+\sigma_y+\sigma_z\right)\\
+\;&\varepsilon_x = \frac{1}{E}\left[\sigma_x - n\left(\sigma_y+\sigma_z\right)\right]
\end{aligned}
$$

$$\frac{n}{(1-2n)}e+\varepsilon_x = \frac{1+n}{E}\sigma_x \qquad G=\frac{E}{2(1+n)}$$

$$\therefore \sigma_x = \frac{E}{1+n}\left[\varepsilon_x + \frac{n}{(1-2n)}e\right] \qquad \text{(D)}$$

5.2 Relations between Stress and Strain for Elastic Solids

Substitute (C) into (D)

$$\therefore \sigma_x = 2G\left[\varepsilon_x + \frac{n}{(1-2n)}e\right] \quad \rightarrow \quad \text{Eq. (5.17)}$$

ii) Subtract (5.16) from (5.17)

$$\sigma_x - \overline{\sigma} = 2G\left[\varepsilon_x + \frac{n}{(1-2n)}e\right] - \frac{1}{3}\left(\sigma_x + \sigma_y + \sigma_z\right) \qquad \text{(E)}$$

Substitute (A) into (E); $(A):\ \sigma_x + \sigma_y + \sigma_z = \frac{E}{(1-2n)}e$

5.2 Relations between Stress and Strain for Elastic Solids

$$\therefore \; RHS \; of \; (E) = 2G\left[\varepsilon_x + \frac{n}{(1-2n)}e\right] - \frac{1}{3}\frac{E}{(1-2n)}e$$

$$= 2G\varepsilon_x + \left[\frac{2Gn}{(1-2n)} - \frac{1}{3}\frac{2G(1+n)}{(1-2n)}\right]e \; = \; 2G\left\{\varepsilon_x\left[\frac{n}{(1-2n)} - \frac{\frac{1+n}{3}}{(1+2n)}\right]e\right\}$$

$$= 2G\left\{\varepsilon_x + \frac{-\frac{1}{3}(1-2n)}{(1-2n)}e\right\} \; = 2G\left(\varepsilon_x - \frac{1}{3}e\right)$$

→ Eq. (5.18)

5.3 Relations between Stress and Rate of Strain for Newtonian Fluids

Experimental evidence suggests that, in fluid, stress is linear with time rate of strain.

$$\rightarrow \ stress \propto \frac{\partial}{\partial t}(strain)$$

→ Newtonian fluid (**Newton's law of viscosity**)

[Cf] For solid,

$$stress \propto strain$$

5.3 Relations between Stress and Rate of Strain for Newtonian Fluids

5.3.1 Normal stress

For solid, Eq. (5.18) can be used as

Hookeian elastic solid: $$\sigma_x - \overline{\sigma} = 2\left(\frac{F}{L^2}\right)\left(\varepsilon_x - \frac{e}{3}\right)$$

$\frac{F}{L^2}$: G

$\left(\varepsilon_x - \frac{e}{3}\right)$: Non-dimensional

By analogy,

Newtonian fluid: $$\sigma_x - \overline{\sigma} = 2\left(\frac{Ft}{L^2}\right)\frac{\partial}{\partial t}\left(\varepsilon_x - \frac{e}{3}\right) \qquad (5.20)$$

$\frac{\partial}{\partial t}\left(\varepsilon_x - \frac{e}{3}\right)$: Time rate of strain [1/t]

Now set $\mu \equiv \frac{Ft}{L^2}$ = dynamic viscosity (N·s/m^2)

5.3 Relations between Stress and Rate of Strain for Newtonian Fluids

Then,

$$\sigma_x - \overline{\sigma} = 2\mu \frac{\partial \varepsilon_x}{\partial t} - \frac{2}{3}\mu \frac{\partial e}{\partial t} \qquad (5.21)$$

By the way,

$$\varepsilon_x = \frac{\partial \xi}{\partial x}, \quad e = \nabla \cdot \vec{\delta}$$

$$u = \frac{\partial \xi}{\partial t}, \quad v = \frac{\partial \eta}{\partial x}, \quad w = \frac{\partial \zeta}{\partial t} \quad (\xi, \eta, \zeta = \text{displacement})$$

Therefore,

$$\frac{\partial \varepsilon_x}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial \xi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial t}\right) = \frac{\partial u}{\partial x} \qquad (5.22)$$

5.3 Relations between Stress and Rate of Strain for Newtonian Fluids

$$\frac{\partial e}{\partial t} = \nabla \cdot \frac{\partial \vec{\delta}}{\partial t} = \nabla \cdot \vec{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \qquad (5.23)$$

$$\vec{\delta} = \xi \vec{i} + \eta \vec{j} + \zeta \vec{k}$$

$$\vec{q} = \frac{\partial \vec{\delta}}{\partial t} = u\vec{i} + v\vec{j} + w\vec{k}$$

$$\nabla \cdot \vec{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Eq. (5.21) becomes

$$\sigma_x = \overline{\sigma} + 2\mu \frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\nabla \cdot \vec{q}\right)$$

5.3 Relations between Stress and Rate of Strain for Newtonian Fluids

For compressible fluid,

$$\sigma_x = \overline{\sigma} + 2\mu \frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\nabla \cdot \vec{q}\right)$$

$$\sigma_y = \overline{\sigma} + 2\mu \frac{\partial v}{\partial y} - \frac{2}{3}\mu\left(\nabla \cdot \vec{q}\right)$$

$$\sigma_z = \overline{\sigma} + 2\mu \frac{\partial w}{\partial z} - \frac{2}{3}\mu\left(\nabla \cdot \vec{q}\right) \qquad (5.24)$$

For incompressible fluid,

$$\frac{de}{dt} = \nabla \cdot \vec{q} = 0$$ ← time rate of volume expansion=0

5.3 Relations between Stress and Rate of Strain for Newtonian Fluids

$$\rightarrow \nabla \cdot \vec{q} = 0 \quad \rightarrow \text{Continuity Eq.}$$

Therefore, Eq. (5.24) becomes

$$\sigma_x = \overline{\sigma} + 2\mu \frac{\partial u}{\partial x}$$

$$\sigma_y = \overline{\sigma} + 2\mu \frac{\partial v}{\partial y}$$

$$\sigma_z = \overline{\sigma} + 2\mu \frac{\partial w}{\partial z}$$

5.3 Relations between Stress and Rate of Strain for Newtonian Fluids

5.3.2. Shear stress

By following the same analogy

$$\tau_{xy} = G\left(\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}\right) = \left(\frac{Ft}{L^2}\right)\frac{\partial}{\partial t}\left(\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}\right)$$

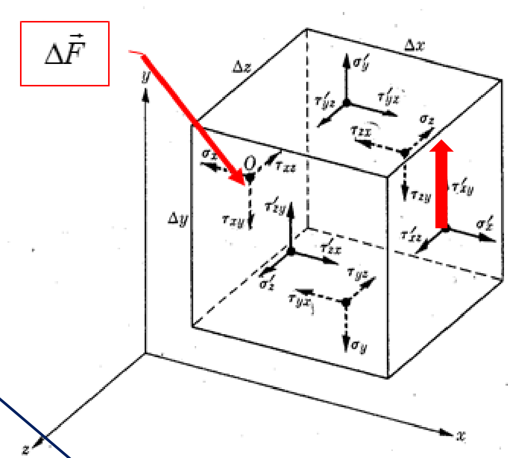
μ

$$= \mu\frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial t}\right) + \mu\frac{\partial}{\partial y}\left(\frac{\partial \xi}{\partial t}\right) = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\frac{\partial \eta}{\partial t} = v$$

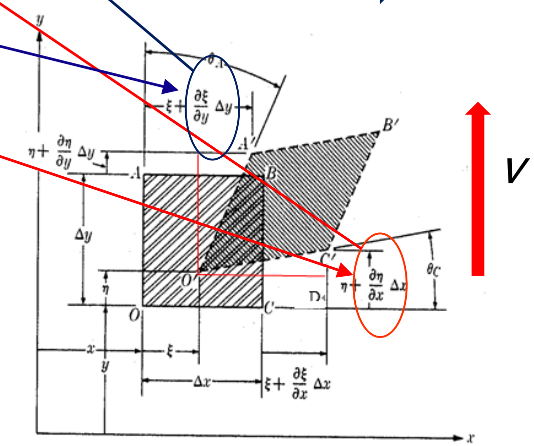
$$\frac{\partial \xi}{\partial t} = u$$

$\Delta\vec{F}$



u

v



5.3 Relations between Stress and Rate of Strain for Newtonian Fluids

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

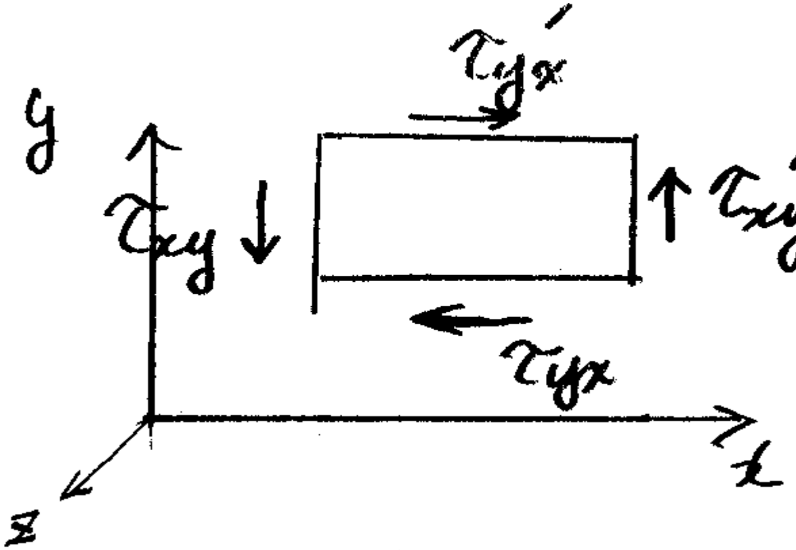
$$\tau_{zy} = \tau_{yz} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \qquad (5.25)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

5.3 Relations between Stress and Rate of Strain for Newtonian Fluids

[Appendix 1]

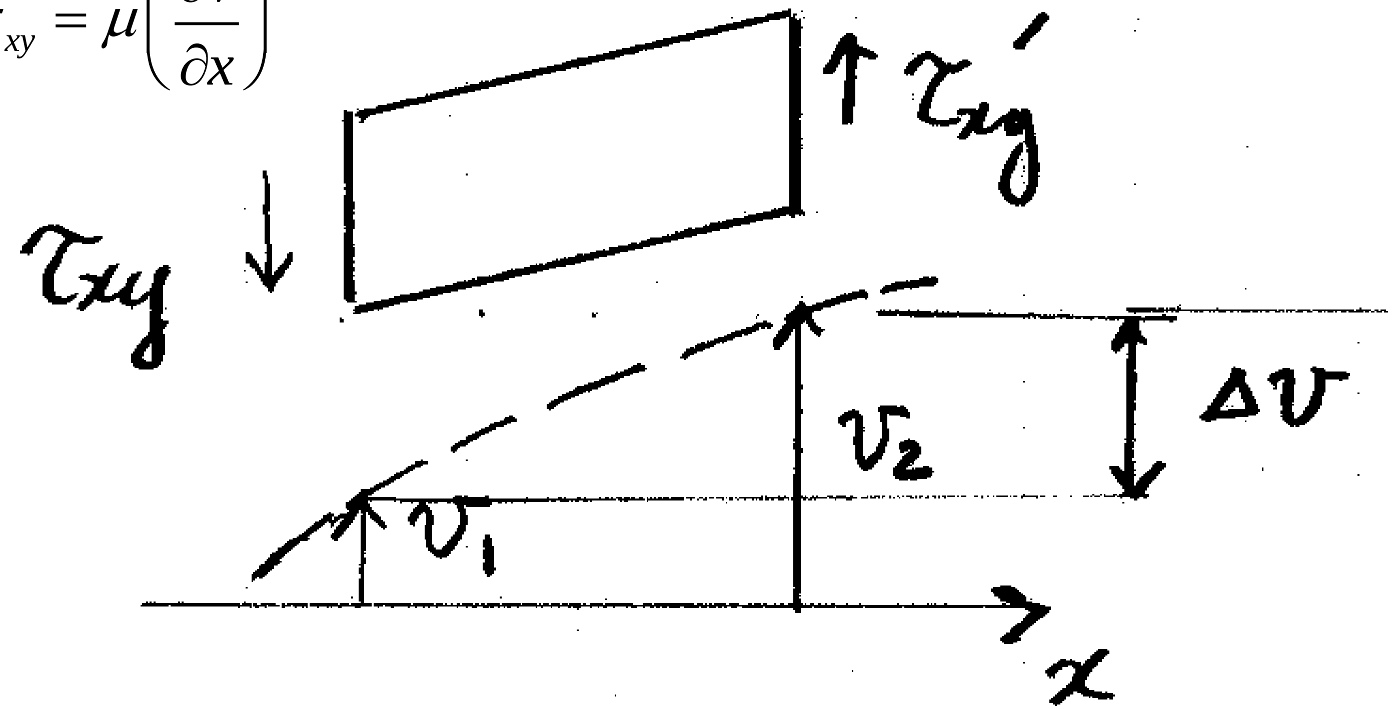
$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$



5.3 Relations between Stress and Rate of Strain for Newtonian Fluids

i) τ_{xy} , $\tau_{xy}{}'$

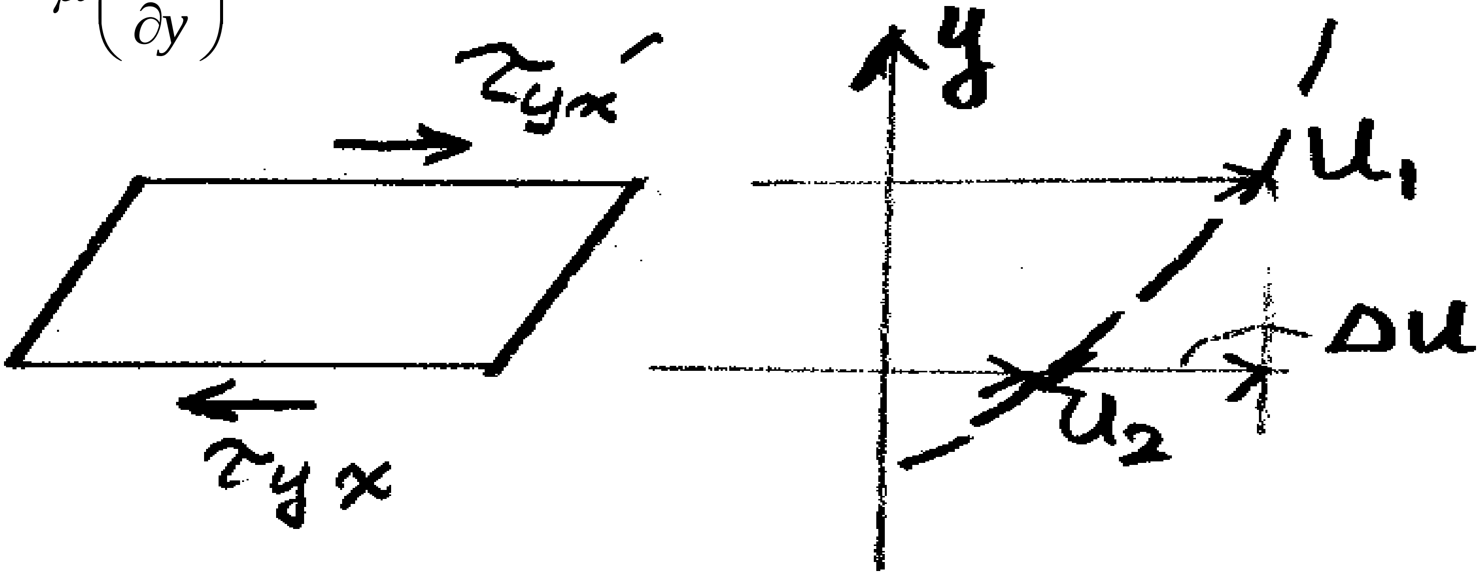
$$\tau_{xy} = \mu\left(\frac{\partial v}{\partial x}\right)$$



5.3 Relations between Stress and Rate of Strain for Newtonian Fluids

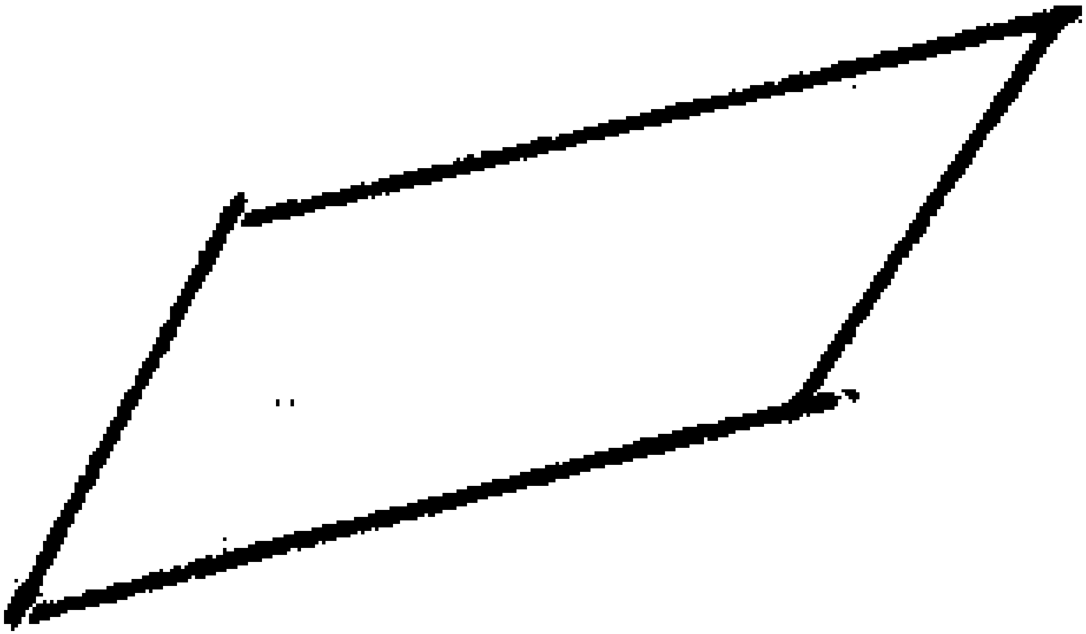
ii) τ_{yx} , $\tau_{yx}{}'$

$$\tau_{yx} = \mu \left(\frac{\partial u}{\partial y} \right)$$



5.3 Relations between Stress and Rate of Strain for Newtonian Fluids

iii) composition



5.3 Relations between Stress and Rate of Strain for Newtonian Fluids

- **Relation between thermodynamic pressure p and mean normal stress $\bar{\sigma}$**

1) Assume viscous effects are completely represented by the viscosity μ for incompressible fluid

$$\bar{\sigma} = -p = \frac{1}{3}\left(\sigma_x + \sigma_y + \sigma_z\right) \qquad (5.26)$$

~ minus sign accounts for pressure (compression)

2) For compressible fluid

$$\bar{\sigma} = -p + \mu'\left(\nabla \cdot \vec{q}\right)$$

in which μ '= 2nd coefficient of viscosity associated solely with dilation

= bulk viscosity

5.3 Relations between Stress and Rate of Strain for Newtonian Fluids

Since, dilation effect is small for most cases

$$\mu'(\nabla \cdot \vec{q}) \to 0 \qquad \therefore \bar{\sigma} = -p$$

For zero-dilation viscosity effects (μ ‘= 0), (5.24) becomes

$$\sigma_x = -p + 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu(\nabla \cdot \vec{q}) \qquad (5.27)$$

$$\sigma_y = -p + 2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu(\nabla \cdot \vec{q}) \qquad (5.28)$$

$$\sigma_z = -p + 2\mu\frac{\partial w}{\partial z} - \frac{2}{3}\mu(\nabla \cdot \vec{q}) \qquad (5.29)$$

Normal stress

pressure

Viscous effects

5.3 Relations between Stress and Rate of Strain for Newtonian Fluids

- Shear stresses in a real fluid

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \quad (5.30)$$

For zero viscous effects ($\mu = 0$) → ideal fluids in motion and for all fluids at rest

$$\sigma_x = \sigma_y = \sigma_z = \bar{\sigma} = -p$$

$$\tau_{xy} = \tau_{yz} = \tau_{zx} = 0$$

5.3 Relations between Stress and Rate of Strain for Newtonian Fluids

[Appendix 2] Normal stress

Normal stress = pressure + deviation from it

$$\sigma_x = -p + \sigma_x'$$

$$\sigma_y = -p + \sigma_y'$$

$$\sigma_z = -p + \sigma_z'$$

Thus, stress matrix becomes

$$\begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} + \begin{pmatrix} \sigma_x' & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y' & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z' \end{pmatrix}$$

5.3 Relations between Stress and Rate of Strain for Newtonian Fluids

Normal stresses are proportional to the volume change (compressibility) and corresponding components of linear deformation, a, b, c.

Thus,

$$\sigma_x = -p + \lambda(a + b + c) + 2\mu a$$

$$\sigma_y = -p + \lambda(a + b + c) + 2\mu b$$

$$\sigma_z = -p + \lambda(a + b + c) + 2\mu c$$

where λ = compressibility coefficient

Homework Assignment # 5

Due: 1 week from today

1. (5-3) Consider a fluid element under a general state of stress as illustrated in Fig. 5-1. Given that the element is in a gravity field, show that the equilibrium requirement between surface, body and inertial forces leads to the equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x = \rho a_x$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial x} + \rho g_y = \rho a_y$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \rho g_z = \rho a_z$$