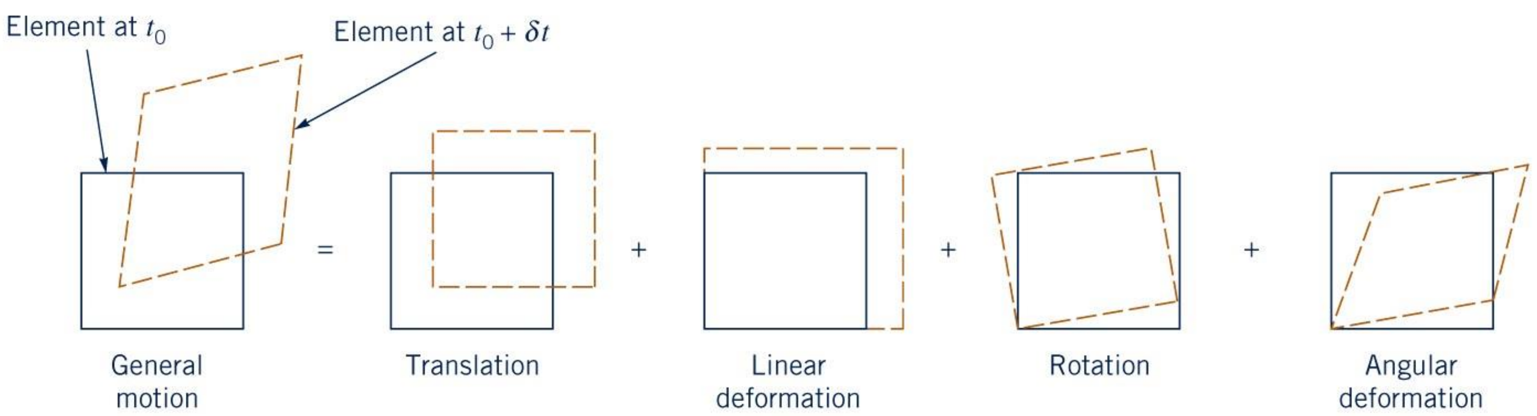
Lecture 9

Stress-Strain Relation (2)

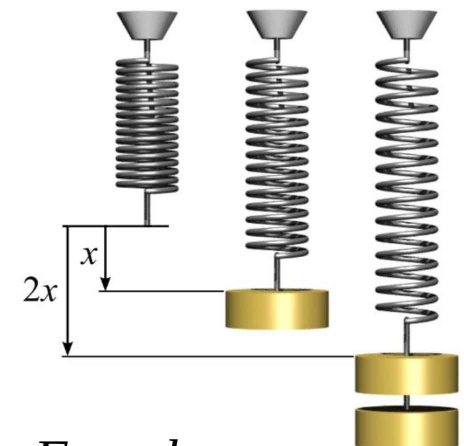


Lecture 9 Stress-Strain Relation (2)

Contents

Objectives

- Study Hooke's law of stress-strain relations for solid
- Study solid mechanics to deduce stress-rate of strain relations for fluid



$$F_x = k\,x$$

9.1 Relations between Stress and Strain for Elastic Solids

9.1.1 Normal Stresses

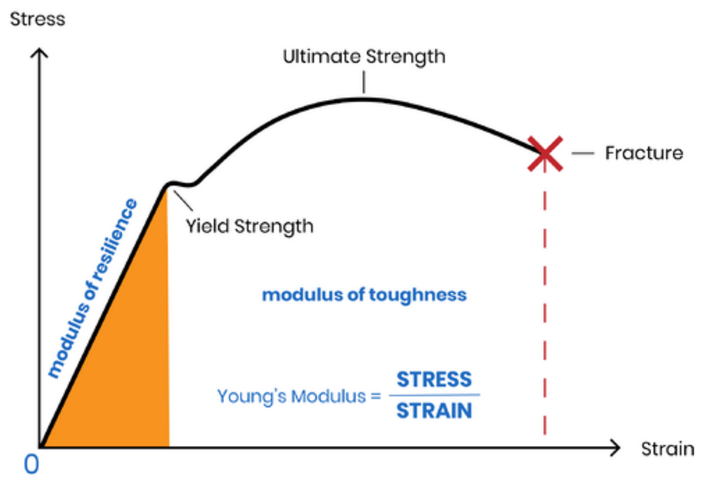
Hooke's law: Stress is linear with strain (linear elastic).

$$\sigma_x \propto \varepsilon_x^{\circ}$$

$$\sigma_x = E\,\varepsilon_x^{\circ}$$

$$\varepsilon_x^{\circ} = \frac{1}{E}\,\sigma_x \qquad (9.1a)$$

$$\varepsilon_x = \frac{\partial \xi}{\partial x} \rightarrow \text{non-dimensional}$$



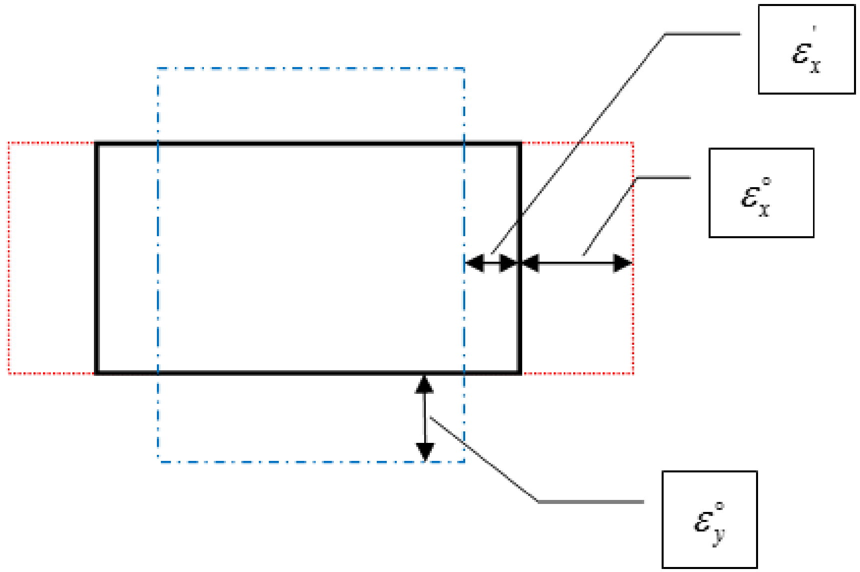
in which E = Young's modulus of elasticity (N/m^2) 탄성계수

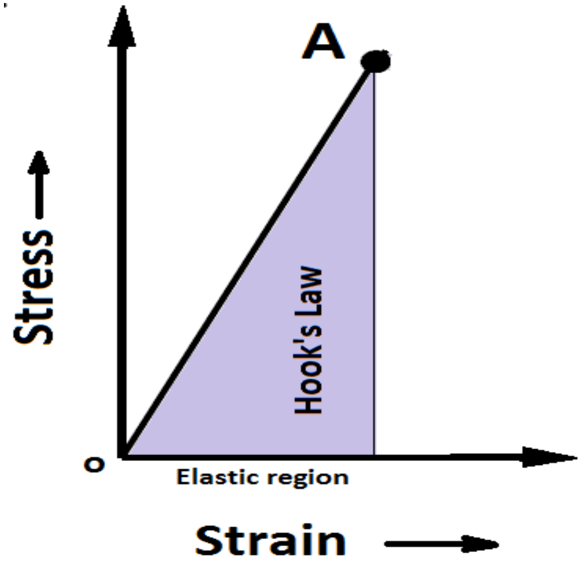
ε_x° = elongation in the *x- dir* due to normal stress, σ_x

9.1 Relations between Stress and Strain for Elastic Solids

$$y-dir.\ :\ \varepsilon_y^{\circ}=\frac{\sigma_y}{E} \qquad (9.1b)$$

$$z-dir.\ :\ \varepsilon_z^{\circ}=\frac{\sigma_z}{E} \qquad (9.1c)$$



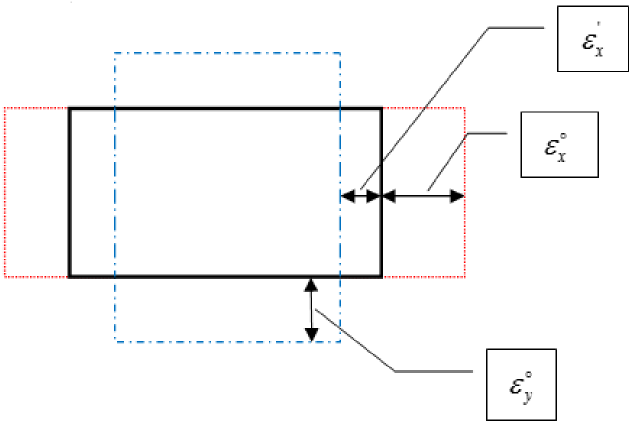


9.1 Relations between Stress and Strain for Elastic Solids

Now, we have to consider other elongations because of lateral contraction of matter under tension for isotropic material.

$\varepsilon_x{}'$ = elongation in the $x-dir.$ due to σ_y

$\varepsilon_x{}''$ = elongation in the $x-dir.$ due to σ_z



Now, define

$$\varepsilon_x{}' = -n\varepsilon_y^\circ = -n\frac{\sigma_y}{E} \qquad (9.2)$$

$$\varepsilon_x{}'' = -n\varepsilon_z^\circ = -n\frac{\sigma_z}{E} \qquad (9.3)$$

where n = Poisson's ratio

9.1 Relations between Stress and Strain for Elastic Solids

Poisson's ratio: $n = 0\sim0.5$

rubber: ~ 0.5 (비압축성)

metal: ~ 0.3

concrete: 0.1~0.2

cork: ~ 0

S. D. Poisson (1781~1840)

9.1 Relations between Stress and Strain for Elastic Solids

[Practical Example]

- Cork in the wine bottle: The cork must be easily inserted and removed, yet it also must withstand the pressure from within the bottle.

- Corrugated rubber: It is used in shoe (sneaker) insoles and in vibration isolators for machinery. Foam ($n = 0.3$) is used in shoes and in wrestling mats.

9.1 Relations between Stress and Strain for Elastic Solids

- **Generalized Hooke's law**

The total strain ε_x is given as

$$\varepsilon_x = \varepsilon_x^{\circ} + \varepsilon_x ' + \varepsilon_x '' = \frac{\sigma_x}{E} - \frac{n}{E}\left(\sigma_y + \sigma_z\right) = \frac{1}{E}\left[\sigma_x - n\left(\sigma_y + \sigma_z\right)\right] \qquad (9.4a)$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - n\left(\sigma_z + \sigma_x\right)\right] \qquad (9.4b)$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - n\left(\sigma_x + \sigma_y\right)\right] \qquad (9.4c)$$

9.1 Relations between Stress and Strain for Elastic Solids

9.1.2 Shear Stress

~ Hooke's law $\tau_{xy} = G\,\gamma_{xy}$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y} \tag{9.5a}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} = \frac{\partial \zeta}{\partial y} + \frac{\partial \eta}{\partial z} \tag{9.5b}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G} = \frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial x} \tag{9.5c}$$

9.1 Relations between Stress and Strain for Elastic Solids

where G = shear modulus of elasticity (N/m^2) 전단탄성계수

$$G = \frac{E}{2(1+n)} \quad (9.6)$$

- Most materials resist a change in volume as determined by the Young's modulus E more than they resist a change in shape, as determined by the shear modulus G.

$$E = 2G \ \ for\, n = 0.0\,(\text{compressible material})$$
$$E = 3G \ \ for\, n = 0.5\,(\text{incompressible material})$$

9.1 Relations between Stress and Strain for Elastic Solids

■ Volume dilation (dilatation; 팽창)

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1}{E}\left[\sigma_x - n\left(\sigma_y + \sigma_z\right)\right]$$

$$+\frac{1}{E}\left[\sigma_y - n\left(\sigma_z + \sigma_x\right)\right]$$

$$+\frac{1}{E}\left[\sigma_z - n\left(\sigma_x + \sigma_y\right)\right]$$

$$e = \frac{1}{E}\left[(1-2n)\left(\sigma_x + \sigma_y + \sigma_z\right)\right] \quad (9.7)$$

$$e = \nabla \cdot \vec{\delta} = \nabla \cdot \left(\xi \vec{i} + \eta \vec{j} + \zeta \vec{k}\right) = \frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z}$$

9.1 Relations between Stress and Strain for Elastic Solids

- $\bar{\sigma}$ = arithmetic mean of 3 normal stresses

$$\bar{\sigma} = \frac{1}{3}\left(\sigma_x + \sigma_y + \sigma_z\right) \tag{9.8}$$

Combine Eqs. (9.4), (9.6) and (9.7)

$$\sigma_x = 2G\left[\varepsilon_x + \frac{ne}{1-2n}\right] \tag{9.9}$$

9.1 Relations between Stress and Strain for Elastic Solids

Therefore

$$\sigma_x - \overline{\sigma} = 2G\left(\varepsilon_x - \frac{e}{3}\right)$$

$$\sigma_y - \overline{\sigma} = 2G\left(\varepsilon_y - \frac{e}{3}\right)$$

$$\sigma_z - \overline{\sigma} = 2G\left(\varepsilon_z - \frac{e}{3}\right) \qquad (9.10)$$

$$\tau_{xy} = \tau_{yx} = G\left(\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}\right)$$

9.1 Relations between Stress and Strain for Elastic Solids

$$\tau_{zy} = \tau_{yz} = G\left(\frac{\partial \zeta}{\partial y} + \frac{\partial \eta}{\partial z}\right)$$

$$\tau_{xz} = \tau_{zx} = G\left(\frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial x}\right) \qquad (9.11)$$

[Proof] Derivation of Eqs. (9.9) & (9.10)

(9.7) → $$e = \frac{1}{E}(1-2n)\left(\sigma_x + \sigma_y + \sigma_z\right) \qquad \text{(A)}$$

(9.4) → $$\varepsilon_x = \frac{1}{E}\left[\sigma_x - n\left(\sigma_y + \sigma_z\right)\right] \qquad \text{(B)}$$

(9.6) → $$G = \frac{E}{2(1+n)} \rightarrow E = 2G(1+n) \qquad \text{(C)}$$

9.1 Relations between Stress and Strain for Elastic Solids

i) Combine (A) and (B)

$$\frac{n}{(1-2n)}\times e = \frac{n}{(1-2n)}\frac{(1-2n)}{E}\left(\sigma_x+\sigma_y+\sigma_z\right) = \frac{n}{E}\left(\sigma_x+\sigma_y+\sigma_z\right)$$

$$+\quad \varepsilon_x = \frac{1}{E}\left[\sigma_x - n\left(\sigma_y+\sigma_z\right)\right]$$

$$\frac{n}{(1-2n)}e+\varepsilon_x = \frac{1+n}{E}\sigma_x$$

$$G = \frac{E}{2(1+n)}$$

$$\therefore \sigma_x = \frac{E}{1+n}\left[\varepsilon_x + \frac{n}{(1-2n)}e\right] \qquad \text{(D)}$$

9.1 Relations between Stress and Strain for Elastic Solids

Substitute (C) into (D)

$$\therefore \sigma_x = 2G\left[\varepsilon_x + \frac{n}{(1-2n)}e\right] \quad \rightarrow \quad \text{Eq. (9.9)}$$

ii) Subtract (9.8) from (9.9)

$$\sigma_x - \bar{\sigma} = 2G\left[\varepsilon_x + \frac{n}{(1-2n)}e\right] - \frac{1}{3}\left(\sigma_x + \sigma_y + \sigma_z\right) \qquad \text{(E)}$$

Substitute (A) into (E); $(A): \ \sigma_x + \sigma_y + \sigma_z = \dfrac{E}{(1-2n)}e$

9.1 Relations between Stress and Strain for Elastic Solids

$$\therefore\ RHS\ of\ (E) = 2G\left[\varepsilon_x + \frac{n}{(1-2n)}e\right] - \frac{1}{3}\frac{E}{(1-2n)}e$$

$$= 2G\varepsilon_x + \left[\frac{2Gn}{(1-2n)} - \frac{1}{3}\frac{2G(1+n)}{(1-2n)}\right]e = 2G\left\{\varepsilon_x\left[\frac{n}{(1-2n)} - \frac{\frac{1+n}{3}}{(1+2n)}\right]e\right\}$$

$$= 2G\left\{\varepsilon_x + \frac{-\frac{1}{3}(1-2n)}{(1-2n)}e\right\} = 2G\left(\varepsilon_x - \frac{1}{3}e\right)$$

$\rightarrow$ Eq. (9.10)

9.2 Relations between Stress and Rate of Strain for Newtonian Fluids

Experimental evidence suggests that, in fluid, stress is linear with time rate of strain.

$$\rightarrow stress \propto \frac{\partial}{\partial t}(strain)$$

→ Newtonian fluid (**Newton's law of viscosity**)

[Cf] For solid,

$$stress \propto strain$$

9.2 Relations between Stress and Rate of Strain for Newtonian Fluids

9.2.1 Normal stress

For solid, Eq. (9.10) can be used as

Non-dimensional

Hookeian elastic solid: $$\sigma_x - \overline{\sigma} = 2\left(\frac{F}{L^2}\right)\left(\varepsilon_x - \frac{e}{3}\right)$$

G

By analogy,

Newtonian fluid: $$\sigma_x - \overline{\sigma} = 2\left(\frac{Ft}{L^2}\right)\frac{\partial}{\partial t}\left(\varepsilon_x - \frac{e}{3}\right) \qquad (9.12)$$

Time rate of strain [1/t]

Now set $\mu \equiv \dfrac{Ft}{L^2}$ = dynamic viscosity ($N \cdot s/m^2$)

9.2 Relations between Stress and Rate of Strain for Newtonian Fluids

Then,

$$\sigma_x - \overline{\sigma} = 2\mu \frac{\partial \varepsilon_x}{\partial t} - \frac{2}{3}\mu \frac{\partial e}{\partial t} \qquad (9.13)$$

By the way,

$$\varepsilon_x = \frac{\partial \xi}{\partial x}, \quad e = \nabla \cdot \vec{\delta}$$

$$u = \frac{\partial \xi}{\partial t}, \quad v = \frac{\partial \eta}{\partial x}, \quad w = \frac{\partial \zeta}{\partial t} \quad (\xi, \eta, \zeta = \text{displacement})$$

Therefore,

$$\frac{\partial \varepsilon_x}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial \xi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial t}\right) = \frac{\partial u}{\partial x} \qquad (9.14)$$

9.2 Relations between Stress and Rate of Strain for Newtonian Fluids

$$\frac{\partial e}{\partial t} = \nabla \cdot \frac{\partial \vec{\delta}}{\partial t} = \nabla \cdot \vec{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \qquad (9.15)$$

$$\vec{\delta} = \xi \vec{i} + \eta \vec{j} + \zeta \vec{k}$$

$$\vec{q} = \frac{\partial \vec{\delta}}{\partial t} = u\vec{i} + v\vec{j} + w\vec{k}$$

$$\nabla \cdot \vec{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Eq. (9.13) becomes

$$\sigma_x = \overline{\sigma} + 2\mu \frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\nabla \cdot \vec{q}\right)$$

9.2 Relations between Stress and Rate of Strain for Newtonian Fluids

For compressible fluid,

$$\sigma_x = \bar{\sigma} + 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\nabla \cdot \vec{q}\right)$$

$$\sigma_y = \bar{\sigma} + 2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\left(\nabla \cdot \vec{q}\right) \qquad (9.16)$$

$$\sigma_z = \bar{\sigma} + 2\mu\frac{\partial w}{\partial z} - \frac{2}{3}\mu\left(\nabla \cdot \vec{q}\right)$$

For incompressible fluid,

$$\frac{de}{dt} = \nabla \cdot \vec{q} = 0$$ ← time rate of volume expansion=0

9.2 Relations between Stress and Rate of Strain for Newtonian Fluids

$$\rightarrow \nabla \cdot \vec{q} = 0 \quad \rightarrow \text{Continuity Eq.}$$

Therefore, Eq. (9.16) becomes

$$\sigma_x = \bar{\sigma} + 2\mu \frac{\partial u}{\partial x}$$

$$\sigma_y = \bar{\sigma} + 2\mu \frac{\partial v}{\partial y} \qquad (9.17)$$

$$\sigma_z = \bar{\sigma} + 2\mu \frac{\partial w}{\partial z}$$

9.2 Relations between Stress and Rate of Strain for Newtonian Fluids

▪ **Relation between thermodynamic pressure p and mean normal stress $\bar{\sigma}$**

1) Assume viscous effects are completely represented by the viscosity μ for incompressible fluid

$$\bar{\sigma} = -p = \frac{1}{3}\left(\sigma_x + \sigma_y + \sigma_z\right) \qquad (9.18)$$

~ minus sign accounts for pressure (compression)

2) For compressible fluid

$$\bar{\sigma} = -p + \mu'\left(\nabla \cdot \vec{q}\right) \qquad (9.19)$$

in which μ ‘= 2nd coefficient of viscosity associated solely with dilation

= bulk viscosity

9.2 Relations between Stress and Rate of Strain for Newtonian Fluids

Since, dilation effect is small for most cases

$$\nabla \cdot \vec{q} \to 0 \qquad \therefore \bar{\sigma} = -p \tag{9.20}$$

For zero-dilation effects, (9.16) becomes

$$\sigma_x = -p + 2\mu \frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\nabla \cdot \vec{q}\right)$$

$$\sigma_y = -p + 2\mu \frac{\partial v}{\partial y} - \frac{2}{3}\mu\left(\nabla \cdot \vec{q}\right) \tag{9.21}$$

$$\sigma_z = -p + 2\mu \frac{\partial w}{\partial z} - \frac{2}{3}\mu\left(\nabla \cdot \vec{q}\right)$$

Normal stress

pressure

Viscous effects

9.2 Relations between Stress and Rate of Strain for Newtonian Fluids

9.2.2. Shear stress

By following the same analogy

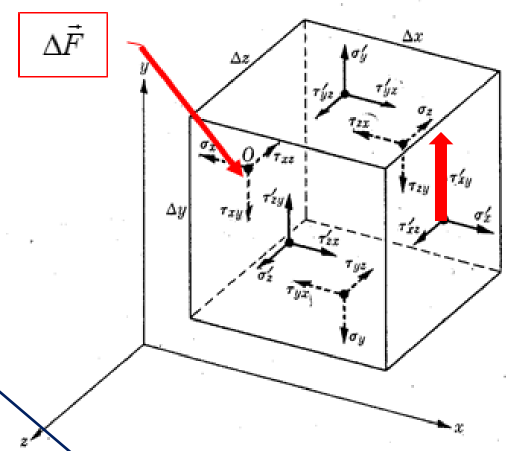
$$\mu$$

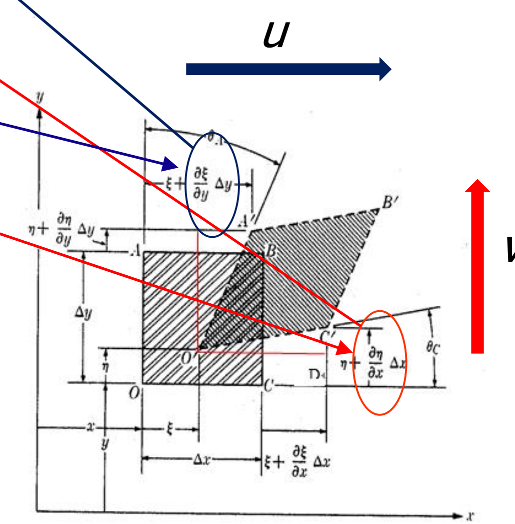
$$\tau_{xy} = G\left(\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}\right) = \left(\frac{Ft}{L^2}\right)\frac{\partial}{\partial t}\left(\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}\right)$$

$$= \mu \frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial t}\right) + \mu \frac{\partial}{\partial y}\left(\frac{\partial \xi}{\partial t}\right) = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\frac{\partial \eta}{\partial t} = v$$

$$\frac{\partial \xi}{\partial t} = u$$





9.2 Relations between Stress and Rate of Strain for Newtonian Fluids

■ Shear stresses in a real fluid

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \qquad (9.22)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

For zero viscous effects ($\mu = 0$) → ideal fluids in motion and for all fluids at rest

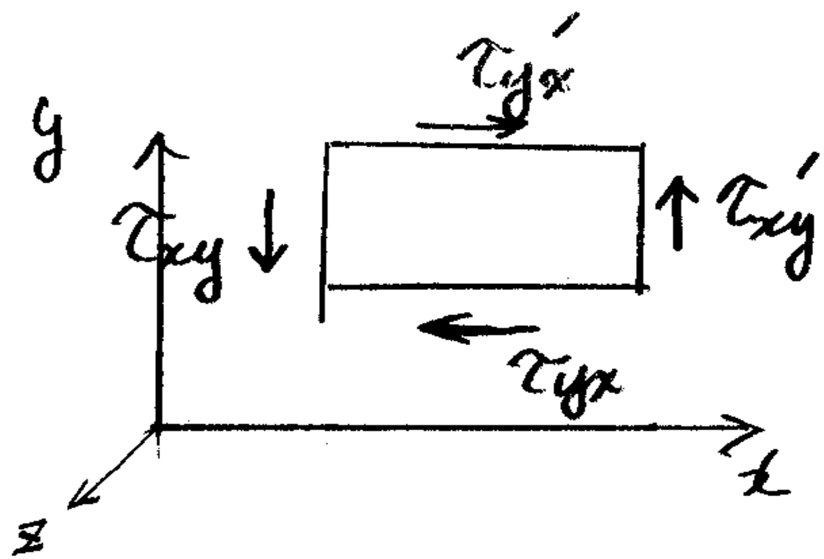
$$\sigma_x = \sigma_y = \sigma_z = \bar{\sigma} = -p$$

$$\tau_{xy} = \tau_{yz} = \tau_{zx} = 0$$

Appendix

[Appendix 1]

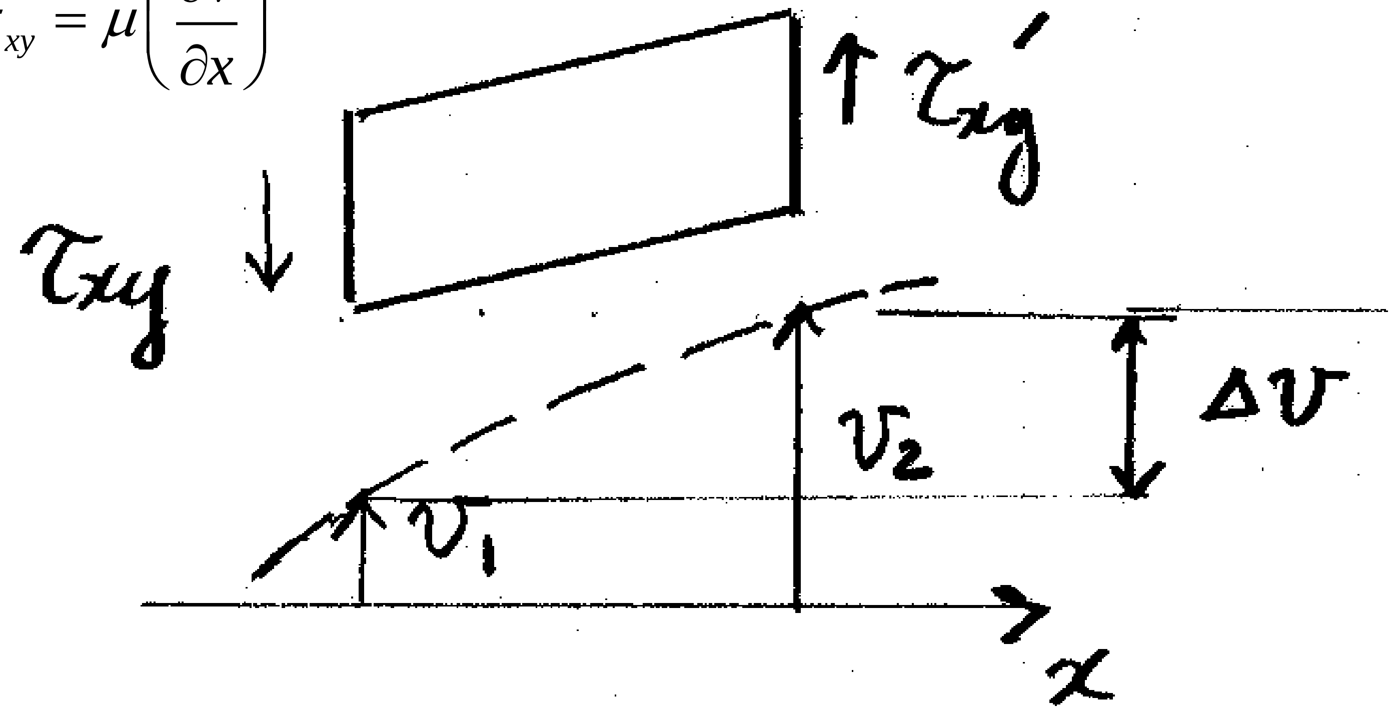
$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$



Appendix

i) τ_{xy} , τ_{xy} '

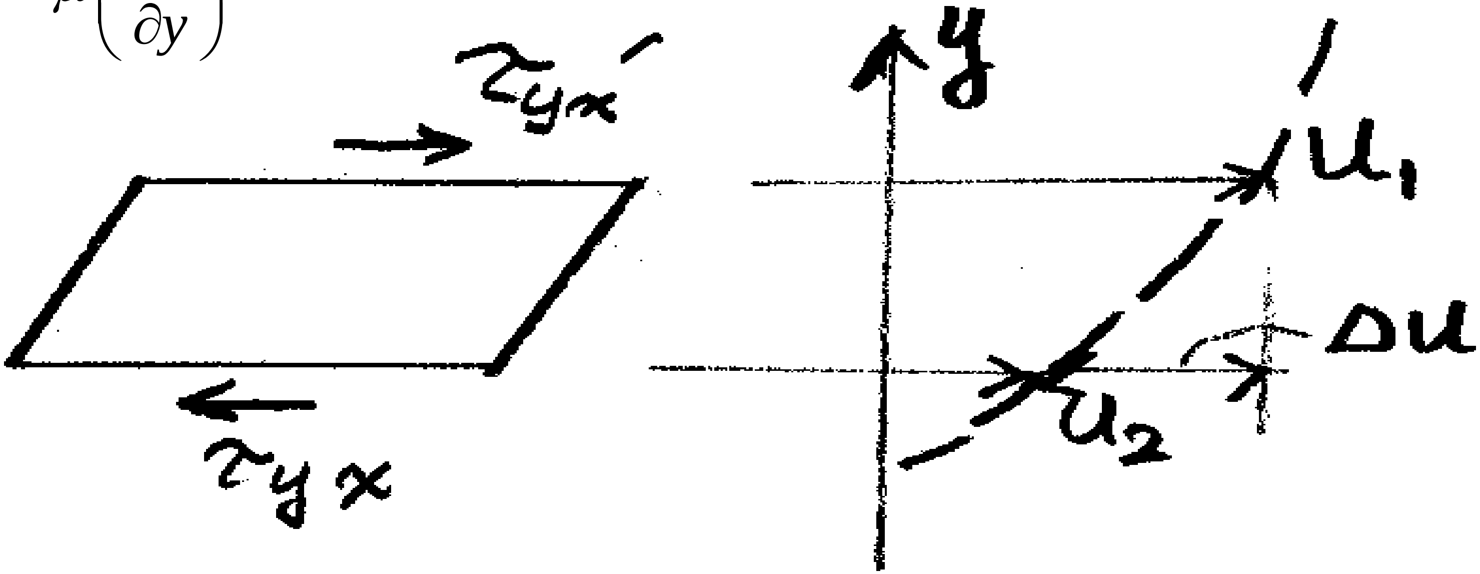
$$\tau_{xy} = \mu\left(\frac{\partial v}{\partial x}\right)$$



Appendix

ii) τ_{yx} , $\tau_{yx}{}'$

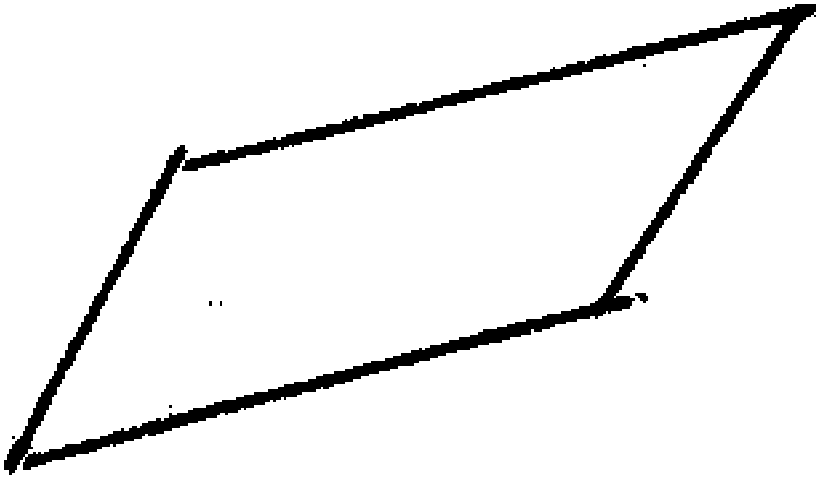
$$\tau_{yx} = \mu\left(\frac{\partial u}{\partial y}\right)$$



Appendix

iii) composition

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$



Problems

Homework Assignment # 4

Due: 1 week from today

1. Consider a fluid element under a general state of stress as illustrated in Fig. 5-1 (Daily & Harleman, 1966). Given that the element is in a gravity field, show that the equilibrium requirement between surface, body and inertial forces leads to the equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x = \rho a_x$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial x} + \rho g_y = \rho a_y$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \rho g_z = \rho a_z$$

Problems

2. Prove the equation given below:

$$G = \frac{E}{2(1+n)}$$