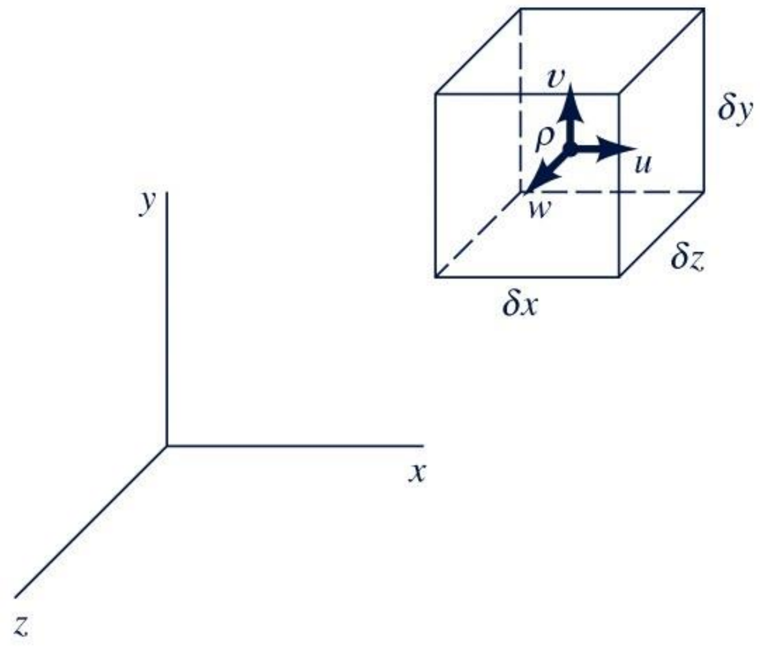
Lecture 11

Equations of Continuity and Motion (2/6)



Lecture 11 Equations of Continuity and Motion (2)

Contents

Objectives

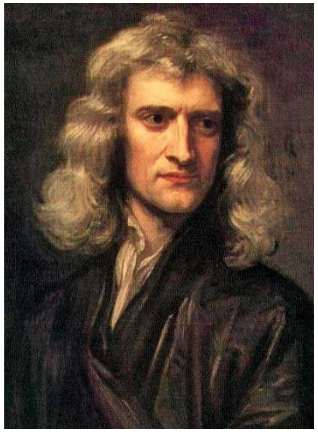
- Derive 3D equations of motion
- Derive Navier-Stokes equation for Newtonian fluid

11.1 Equations of Motion

- Apply Newton's 2nd law of motion

$$\sum \vec{F} = m\vec{a}$$

$$\Delta F_x = \Delta m a_x \qquad \text{(A)}$$



I. Newton (1643~1727)

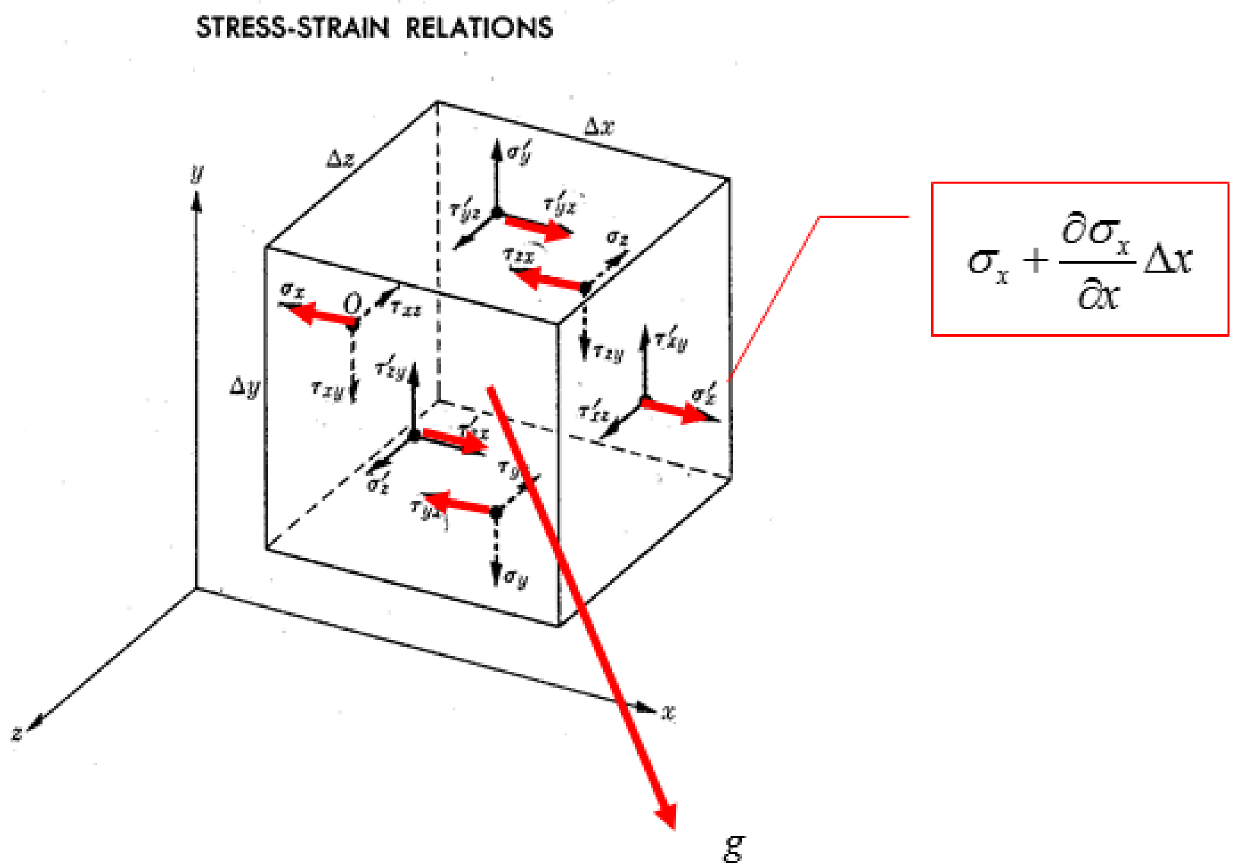
- External forces = surface force + body force
  - Surface force:

  ~ normal force + tangential force
  - Body forces:

  ~ due to gravitational or electromagnetic fields, <u>no contact</u>

  ~ act at the <u>centroid of the element</u> → <u>centroidal force</u>

11.1 Equations of Motion



11.1 Equations of Motion

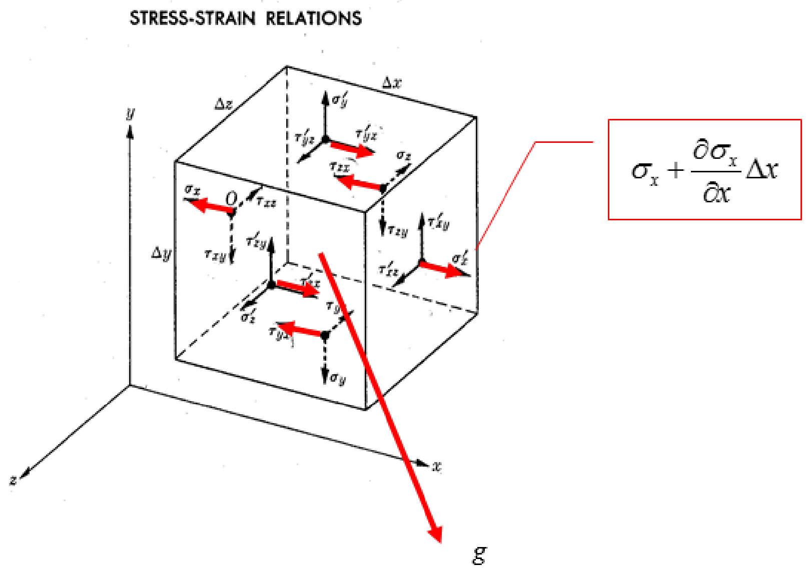
Consider only gravitational force

$$\vec{g} = \vec{i}g_x + \vec{j}g_y + \vec{k}g_z$$

LHS of (A):

$$\Delta F_x = \left(\rho \Delta x \Delta y \Delta z\right) g_x$$
$$-\sigma_x \Delta y \Delta z + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x}\Delta x\right)\Delta y \Delta z$$
$$-\tau_{yx} \Delta x \Delta z + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\Delta y\right)\Delta x \Delta z$$
$$-\tau_{zx} \Delta x \Delta y + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\Delta z\right)\Delta x \Delta y$$

(B)



11.1 Equations of Motion

Divide (B) by volume of element

$$\frac{\Delta F_x}{\Delta x \Delta y \Delta z} = \rho g_x + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \qquad \text{(C)}$$

RHS of (A):

$$\frac{\Delta m a_x}{\Delta x \Delta y \Delta z} = \rho a_x \qquad \text{(D)}$$

11.1 Equations of Motion

Combine (C) and (D)

$$\rho g_x + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho a_x$$

$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho a_y$$

$$\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = \rho a_z \qquad (11.1)$$

→ A general equation of motion containing 9 unknowns (6+3)

11.2 Navier-Stokes Equations

11.2.1 3D Navier-Stokes equations

- Eq (11.1) ~ general equation of motion containing 9 unknowns
- For Newtonian fluids (with single viscosity coeff.), use stress-strain relation given in (9.21) and (9.22) to reduce the number of unknowns

→ Navier-Stokes equations

Eq. (9.21a):

$$\sigma_x = \underline{-p} + \underline{2\mu \frac{\partial u}{\partial x} - \left(\frac{2}{3}\right)\mu(\nabla \cdot \vec{q})}$$

pressure normal stress due to fluid deformation and viscosity

11.2 Navier-Stokes Equations

$$\sigma_y = -p + 2\mu \frac{\partial v}{\partial y} - \left(\frac{2}{3}\right)\mu(\nabla \cdot \vec{q}) \qquad (11.2)$$

$$\sigma_z = -p + 2\mu \frac{\partial w}{\partial z} - \left(\frac{2}{3}\right)\mu(\nabla \cdot \vec{q})$$

From Eq. (9.22):

$$\tau_{yx} = \tau_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \qquad (11.3)$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

11.2 Navier-Stokes Equations

Substitute Eqs. (11.2) & (11.3) into (11.1)

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu(\nabla\cdot\vec{q})\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] = \rho a_x$$

Assume constant viscosity (neglect effect of pressure and temperature on viscosity variation)

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\frac{\partial}{\partial x}\left[2\frac{\partial u}{\partial x} - \frac{2}{3}(\nabla\cdot\vec{q})\right] + \mu\frac{\partial}{\partial y}\left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \mu\frac{\partial}{\partial z}\left[\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] = \rho a_x$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

11.2 Navier-Stokes Equations

Expand and simplify

normal stress   shear stress

$$L.H.S = \rho g_x - \frac{\partial p}{\partial x} + 2\mu\frac{\partial^2 u}{\partial x^2} - \frac{2}{3}\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z}\right) + \mu\left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z}\right)$$

$$= \rho g_x - \frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + \frac{1}{3}\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$= \rho g_x - \frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + \frac{1}{3}\mu\frac{\partial}{\partial x}(\nabla \cdot \vec{q})$$

normal stress + shear stress

11.2 Navier-Stokes Equations

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + \frac{1}{3}\mu\frac{\partial}{\partial x}\left(\nabla\cdot\vec{q}\right) = \rho a_x$$

$$\rho g_y - \frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] + \frac{1}{3}\mu\frac{\partial}{\partial y}\left(\nabla\cdot\vec{q}\right) = \rho a_y$$

$$\rho g_z - \frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right] + \frac{1}{3}\mu\frac{\partial}{\partial z}\left(\nabla\cdot\vec{q}\right) = \rho a_z \qquad (11.4)$$

→ Navier-Stokes equation for compressible fluids with constant viscosity

11.2 Navier-Stokes Equations

♦ Vector form

$$\rho\vec{g} - \nabla p + \mu\nabla^2\vec{q} + \frac{\mu}{3}\nabla\left(\nabla\cdot\vec{q}\right) = \rho\frac{\partial\vec{q}}{\partial t} + \rho\left(\vec{q}\cdot\nabla\right)\vec{q} \qquad (11.5)$$

where $\vec{a} = \frac{d\vec{q}}{dt} = \frac{\partial\vec{q}}{\partial t} + \left(\vec{q}\cdot\nabla\right)\vec{q}$

1) For inviscid (ideal) fluid flow, $(\mu = 0)$ → viscous forces are neglected.

$$\rho\vec{g} - \nabla p = \rho\frac{\partial\vec{q}}{\partial t} + \rho\left(\vec{q}\cdot\nabla\right)\vec{q}$$

→ Euler equations for ideal fluid

11.2 Navier-Stokes Equations

2) For incompressible fluids, $\nabla \cdot \vec{q} = 0$ (Continuity Eq.)

$$\rho \vec{g} - \nabla p + \mu \nabla^2 \vec{q} = \rho \frac{\partial \vec{q}}{\partial t} + \rho \left(\vec{q} \cdot \nabla \right) \vec{q} \qquad (11.6)$$

Define acceleration due to gravity as

$$\left. \begin{aligned} g_x &= -g \frac{\partial h}{\partial x} \\ g_y &= -g \frac{\partial h}{\partial y} \\ g_z &= -g \frac{\partial h}{\partial z} \end{aligned} \right\} \quad \vec{g} = -g \nabla h$$

11.2 Navier-Stokes Equations

where h = vertical direction measured positive upward

For Cartesian axes oriented so that h and z coincide

$$g_x = g_y = 0 \quad , \quad \frac{\partial h}{\partial z} = 1$$

$$g_z = -g \qquad (11.7)$$

→ minus sign indicates that acceleration due to gravity is in the negative h direction

Then, N-S equation for incompressible fluids and isothermal flows are

11.2 Navier-Stokes Equations

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-g\frac{\partial h}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right]$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-g\frac{\partial h}{\partial y}-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right]$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-g\frac{\partial h}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\mu}{\rho}\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right] \quad (11.8)$$

Local acceleration

Convective acceleration

Body force per mass

Pressure force per mass

Viscosity force per mass

11.2 Navier-Stokes Equations

Eq. (11.8): 4 unknowns - *u, v, w, p* 4원 2차 비선형 편미분 연립 방정식

→ We need one more equation to obtain a solution when the boundary conditions are specified.

→ Eq. of continuity for incompressible fluid

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

♦ Boundary conditions

1) kinematic BC: velocity normal to any rigid boundary (wall) equal the boundary velocity (velocity = 0 for stationary boundary)

2) physical BC: no slip condition (continuum stick to a rigid boundary)

→ tangential velocity relative to the wall vanish at the wall surface

11.2 Navier-Stokes Equations

- General solutions for Navier-Stocks equations are not available because of the nonlinear, 2nd-order nature of the partial differential equations.

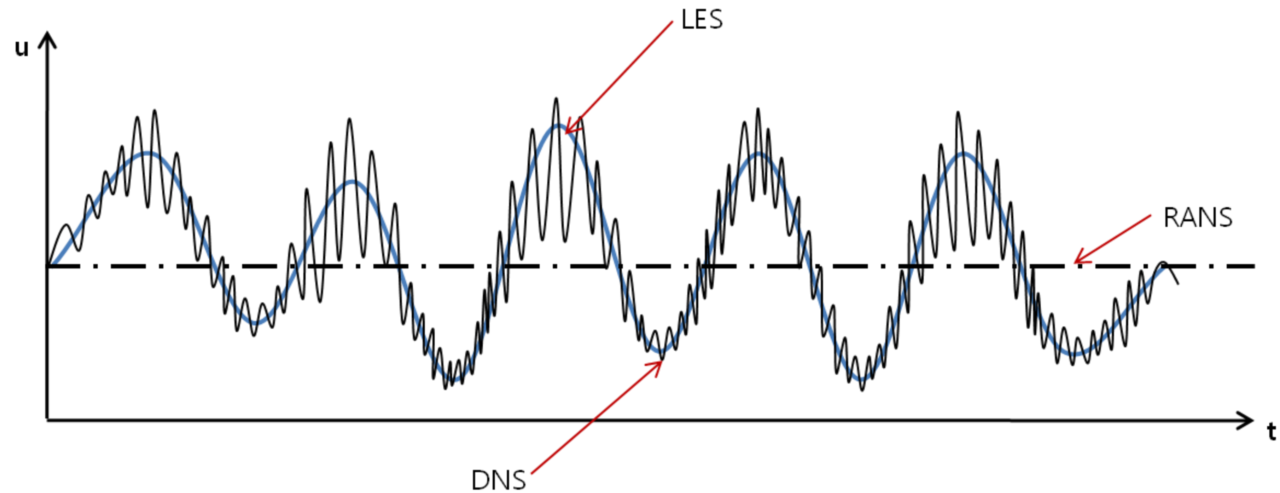
→ Only particular solutions may be obtained by simplifications.

→ Numerical solutions are usually sought.

① DNS

② RANS

③ LES



11.2 Navier-Stokes Equations

[Re] **Navier-Stokes existence and smoothness**

- Millennium Prize Problems by Clay Mathematics Institute (CMI)
- Prove or give a counter-example of the following statement: In three space dimensions and time, given an initial velocity field, there exists a vector velocity and a scalar pressure field, which are both smooth and globally defined, that solve the Navier-Stokes equations.

11.2 Navier-Stokes Equations

11.2.2 Navier-Stokes equations in cylindrical coordinates

- Navier-Stocks equations in cylindrical coordinates for constant density and viscosity

r - component:

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{{v_\theta}^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)$$

$$=\rho g_r-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r v_r\right)\right\}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right]$$

11.2 Navier-Stokes Equations

θ - component:

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}-\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)$$

$$=\rho g_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r v_\theta\right)\right\}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right]$$

z - component:

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)$$

$$=\rho g_z-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right] \qquad (11.9)$$

11.2 Navier-Stokes Equations

Continuity eq. for incompressible fluid

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta)+\frac{\partial}{\partial z}(v_z)=0 \qquad (11.10)$$

Normal & shear stresses for constant density and viscosity

$$\sigma_r=-p+2\mu\frac{\partial v_r}{\partial r}$$

$$\sigma_\theta=-p+2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}\right)$$

$$\sigma_z=-p+2\mu\frac{\partial v_z}{\partial z}$$

11.2 Navier-Stokes Equations

$$\tau_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau_{\theta z} = \mu\left[\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right]$$

$$\tau_{zr} = \mu\left[\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right]$$