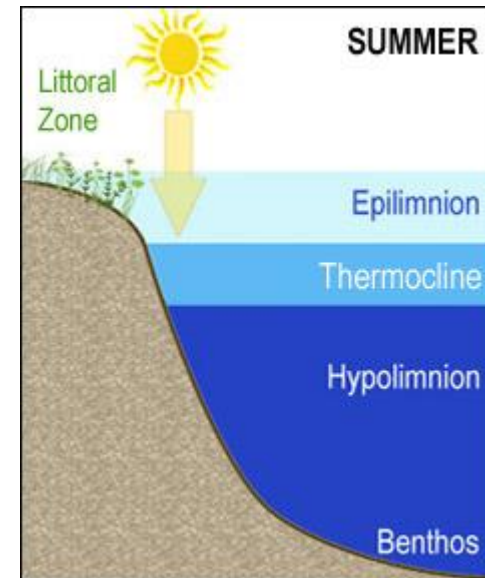


# Lecture 14

## Equations of Continuity and Motion (5/6)



# Lecture 14 Equations of Continuity and Motion (5)

## Contents

### 14.1 Frictionless Flow

## Objectives

- Derive 3D equations of continuity and motion
- Derive Navier-Stokes equation for Newtonian fluid
- Study solutions for simplified cases of laminar flow
- Derive Bernoulli equation for irrotational motion and frictionless flow
- Study solutions for vortex motions

# 14.1 Frictionless Flow

## *Bernoulli equations*

- Special case of the Navier-Stokes Equation
- 1) Irrotational motion of incompressible fluid in a real fluid
  - Bernoulli eq. can be obtained by integrating Navier-Stokes equation for entire flow field.
  - Viscous force term is dropped due to irrotational and incompressible fluid.
- 2) Frictionless flow of compressible fluid
  - Bernoulli eq. can be obtained by integrating Navier-Stokes equation along a streamline.
  - N-S eq. for ideal compressible fluid ( $\mu = 0$ ) is used without assuming irrotational motion.

# 14.1 Frictionless Flow

## 14.1.1 The Bernoulli equation for flow along a streamline

For inviscid flow

→ Assume no frictional (viscous) effects but compressible fluid flows

→ Bernoulli eq. can be obtained by integrating Navier-Stokes equation along a streamline.

Eq. (11.5): N-S eq. for ideal compressible fluid ( $\mu = 0$ )

$$\rho \vec{g} - \nabla p + \cancel{\mu \nabla^2 \vec{q}} + \cancel{\frac{\mu}{3} \nabla (\nabla \cdot \vec{q})} = \rho \frac{\partial \vec{q}}{\partial t} + \rho (\vec{q} \cdot \nabla) \vec{q}$$

$$\vec{g} - \frac{\nabla p}{\rho} = \frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla) \vec{q} \quad (14.1)$$

# 14.1 Frictionless Flow

→ **Euler's equation of motion** for inviscid (ideal) fluid flow

Substituting (11.7) into (14.1) leads to

$$-g\nabla h - \frac{\nabla p}{\rho} = \frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla) \vec{q} \quad (14.2)$$

$\vec{g} = -g\nabla h$

Multiply  $d\vec{r}$  (element of streamline length) and integrate along the streamline

$$-g \int \nabla h \cdot d\vec{r} - \int \frac{1}{\rho} \nabla p \cdot d\vec{r} = \int \left( \frac{\partial \vec{q}}{\partial t} \right) \cdot d\vec{r} + \int [(\vec{q} \cdot \nabla) \vec{q}] \cdot d\vec{r} + C(t) \quad (14.3)$$

where  $d\vec{r} = \vec{i}dx + \vec{j}dy + \vec{k}dz$

For a scalar function of space,  $\nabla \eta \cdot d\vec{r} = d\eta$

# 14.1 Frictionless Flow

$$-gh - \int \frac{dp}{\rho} = \int \left( \frac{\partial \vec{q}}{\partial t} \right) \cdot d\vec{r} + \int \underbrace{\left[ (\vec{q} \cdot \nabla) \vec{q} \right]}_{\text{I}} \cdot d\vec{r} + C(t) \quad (14.4)$$

$$I = \left[ (\vec{q} \cdot \nabla) \vec{q} \right] \cdot d\vec{r} = d\vec{r} \cdot \left[ (\vec{q} \cdot \nabla) \vec{q} \right] = \vec{q} \cdot \underbrace{\left[ (d\vec{r} \cdot \nabla) \vec{q} \right]}_{\text{II}}$$

By the way,

$$II = d\vec{r} \cdot \nabla = \frac{\partial(\quad)}{\partial x} dx + \frac{\partial(\quad)}{\partial y} dy + \frac{\partial(\quad)}{\partial z} dz$$

$$\therefore (d\vec{r} \cdot \nabla) \vec{q} = \frac{\partial \vec{q}}{\partial x} dx + \frac{\partial \vec{q}}{\partial y} dy + \frac{\partial \vec{q}}{\partial z} dz = d\vec{q}$$

# 14.1 Frictionless Flow

$$I = \vec{q} \cdot d\vec{q} = d\left(\frac{q^2}{2}\right)$$

$$\therefore \int [(\vec{q} \cdot \nabla) \vec{q}] \cdot d\vec{r} = \int d\left(\frac{q^2}{2}\right) = \frac{q^2}{2}$$

Thus, Eq. (14.4) becomes

$$\int \frac{dp}{\rho} + gh + \frac{q^2}{2} + \int \left(\frac{\partial q}{\partial t}\right) \cdot d\vec{r} = -C(t) \quad (14.5)$$

For steady motion,  $\frac{\partial \vec{q}}{\partial t} = 0$ ;  $C(t) \rightarrow C$

# 14.1 Frictionless Flow

$$\int \frac{dp}{\rho} + gh + \frac{q^2}{2} = \text{const.} \quad \text{along a streamline} \quad (14.6)$$

For incompressible fluids,  $\rho = \text{const.}$

$$\frac{p}{\rho} + gh + \frac{q^2}{2} = \text{const.}$$

Divide by  $g$

$$\frac{p}{\gamma} + h + \frac{q^2}{2g} = C \quad \text{along a streamline} \quad (14.7)$$



# 14.1 Frictionless Flow

- Bernoulli equation for steady, frictionless, incompressible fluid flow
- Eq. (14.7) is identical to Eq. (13.17).
- However, constant  $C$  is varying from one streamline to another in a rotational flow, Eq. (14.7); it is invariant throughout the fluid for irrotational flow, Eq. (13.17).
- **Summary of Bernoulli equation forms**
- Bernoulli equations for steady, incompressible flow
  - 1) For irrotational flow

$$H = \frac{p}{\gamma} + h + \frac{q^2}{2g} = \text{constant } \underline{\text{throughout the flow field}} \quad (14.8)$$

# 14.1 Frictionless Flow

2) For frictionless flow (rotational)

$$H = \frac{p}{\gamma} + h + \frac{q^2}{2g} = \text{constant along a streamline} \quad (14.9)$$

3) For 1-D frictionless flow (rotational)

$$H = \frac{p}{\gamma} + h + Ke \frac{V^2}{2g} = \text{constant along finite pipe} \quad (14.10)$$

4) For steady flow with friction → include head loss  $h_L$

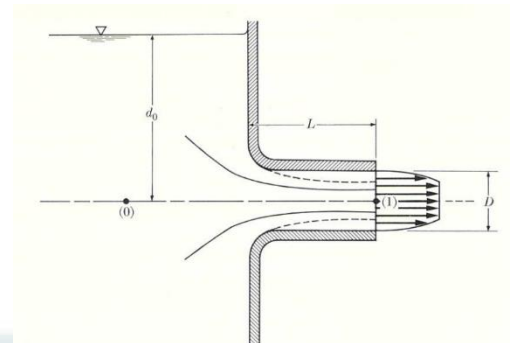
$$\frac{p_1}{\gamma} + h_1 + \frac{q_1^2}{2g} = \frac{p_2}{\gamma} + h_2 + \frac{q_2^2}{2g} + h_L \quad (14.11)$$

# 14.1 Frictionless Flow

## 14.1.2 Applications of Bernoulli's equation to flows of real fluids

### (1) Efflux from a short tube

- Zone of viscous action (boundary layer): frictional effects cannot be neglected.
- Flow in the reservoir and central core of the tube: primary forces are pressure and gravity forces. → irrotational flow
- Apply Bernoulli eq. along the centerline streamline between (0) and (1)



# 14.1 Frictionless Flow

$$\frac{p_0}{\gamma} + z_0 + \frac{q_0^2}{2g} = \frac{p_1}{\gamma} + z_1 + \frac{q_1^2}{2g}$$

$$p_0 = \text{hydrostatic pressure} = \gamma d_0, \quad p_1 = p_{\text{atm}} \rightarrow p_{1_{\text{gage}}} = 0$$

$$z_0 = z_1$$

$q_0 = 0$  (neglect velocity at the large reservoir)

$$\therefore \frac{q_1^2}{2g} = d_0 \quad q_1 = \sqrt{2gd_0} \quad \rightarrow \text{Torricelli's result} \quad (14.12)$$

If we neglect thickness of the zone of viscous influence

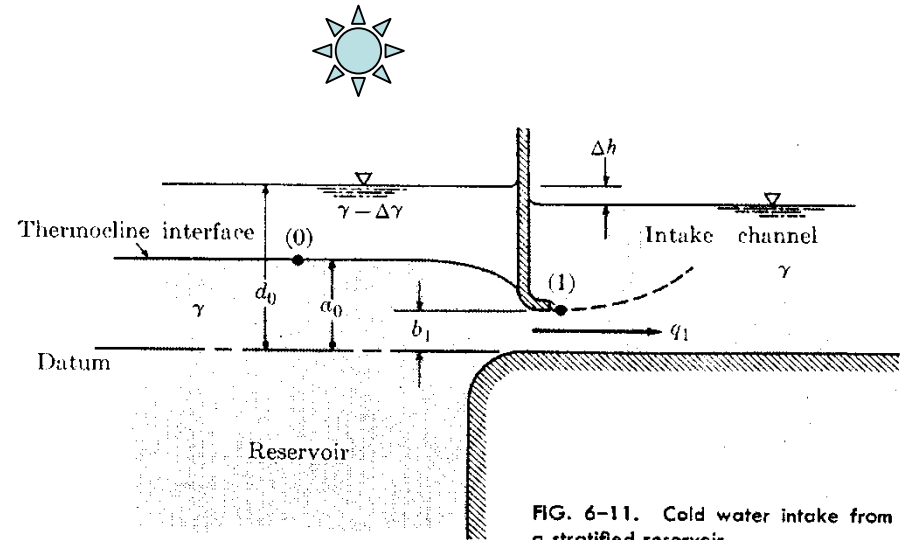
$$Q = \frac{\pi D^2}{4} q_1$$

# 14.1 Frictionless Flow

## (2) Stratified flow

During summer months, large reservoirs and lakes become thermally stratified.

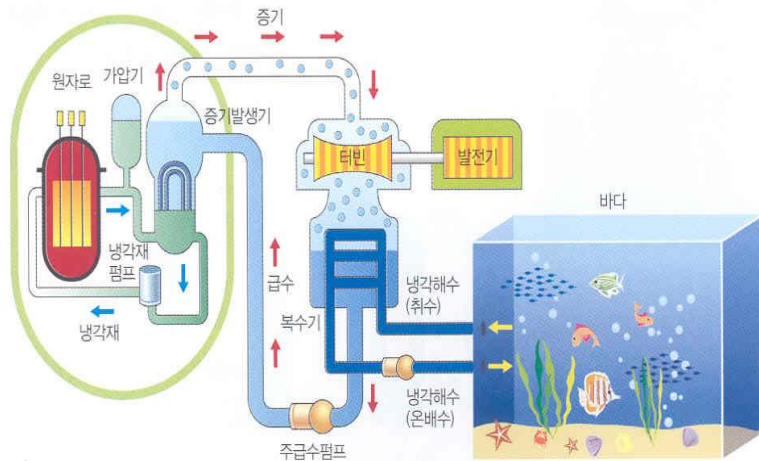
→ At thermocline, temperature changes rapidly with depth.



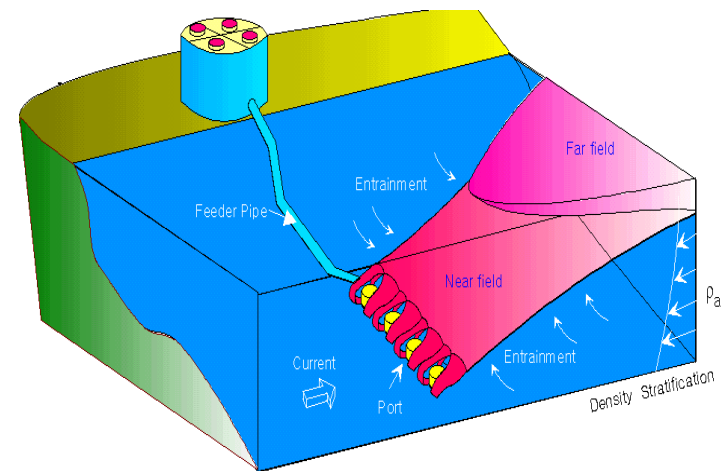
- **Selective withdrawal**: Colder water is withdrawn into the intake channel with a velocity  $q_1$  (uniform over the height  $b_1$ ) in order to provide cool condenser water for thermal (nuclear) power plant.

# 14.1 Frictionless Flow

- Cooling water for thermal (nuclear) power plant



- Heated water discharge from power plant



# 14.1 Frictionless Flow

Apply Bernoulli eq. between points (0) and (1) on the interfacial streamline

$$\frac{p_0}{\gamma} + a_0 + \frac{q_0^2}{2g} = \frac{p_1}{\gamma} + b_1 + \frac{q_1^2}{2g} \quad (14.13)$$

$$q_0 \cong 0$$

$$p_0 = \text{hydrostatic pressure} = (\gamma - \Delta\gamma)(d_0 - a_0)$$

$$p_1 = \gamma(d_0 - \Delta h - b_1) \quad (14.14)$$

$$\therefore \frac{q_1^2}{2g} = \Delta h - \frac{\Delta\gamma}{\gamma}(d_0 - a_0)$$

$$q_1 = \left[ 2g \left\{ \Delta h - \frac{\Delta\gamma}{\gamma}(d_0 - a_0) \right\} \right]^{\frac{1}{2}} \quad (14.15)$$

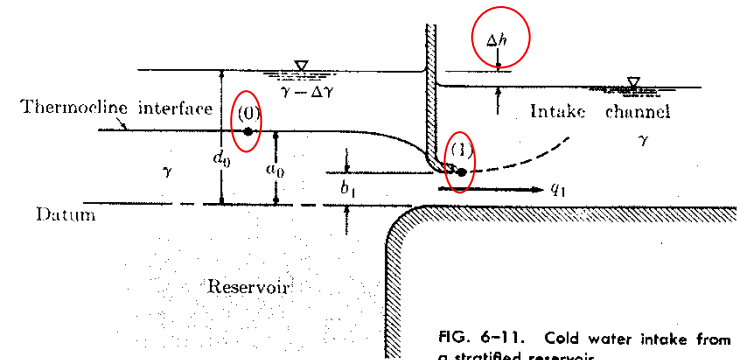


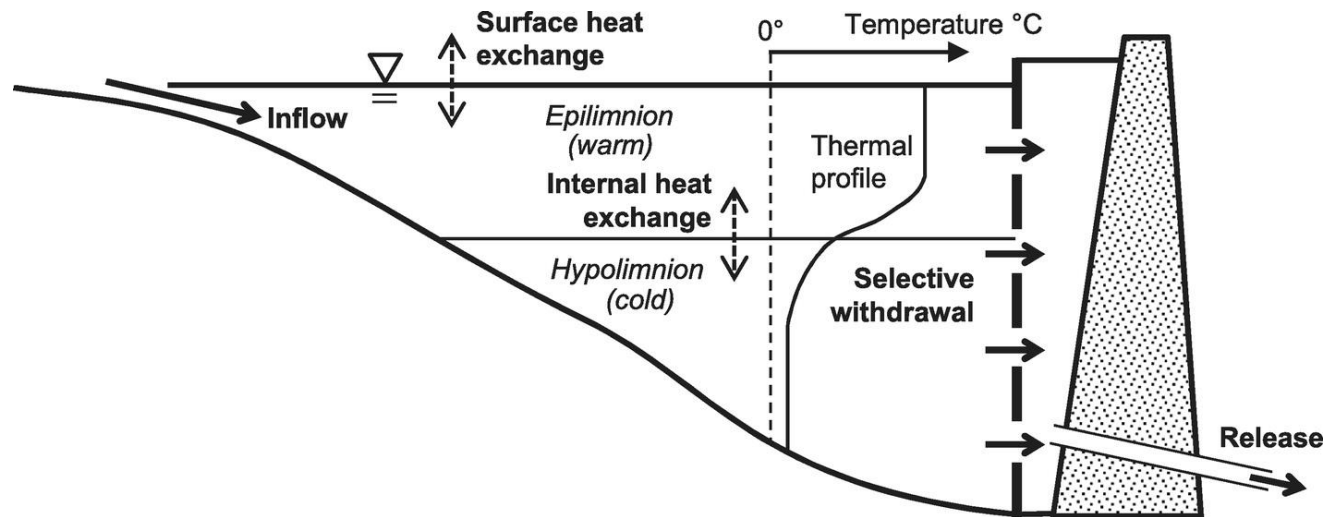
FIG. 6-11. Cold water intake from a stratified reservoir.

# 14.1 Frictionless Flow

For isothermal (unstratified) case,  $a_0 = d_0$

$$q_1 = \sqrt{2g\Delta h} \rightarrow \text{Torricelli's result} \quad (14.16)$$

- Selective withdrawal





# 14.1 Frictionless Flow

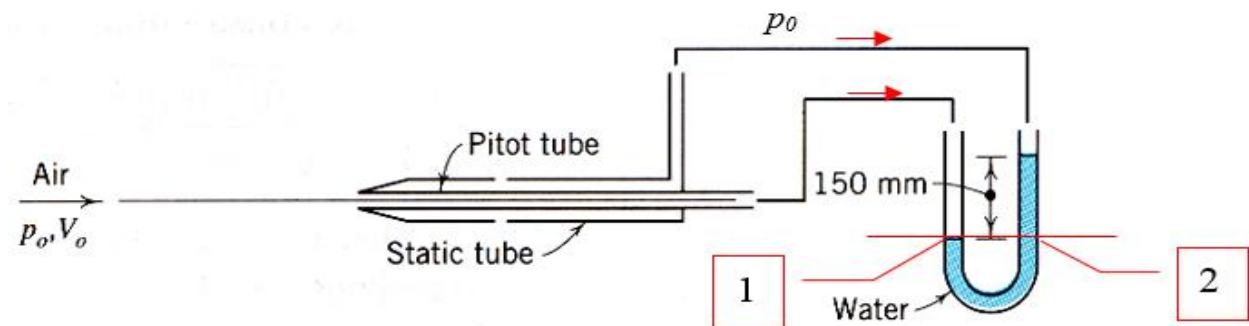
(3) Velocity measurements with the Pitot tube (Henri Pitot, 1732)

→ Measure velocity from stagnation or impact pressure

$$\frac{p_0}{\gamma} + h_0 + \frac{q_0^2}{2g} = \frac{p_s}{\gamma} + h_s + \frac{q_s^2}{2g}$$

$$h_0 = h_s, \quad q_s = 0 \quad (14.17)$$

$$\therefore \frac{q_0^2}{2g} = \frac{p_s - p_0}{\gamma} = \Delta h \quad (14.18)$$



# 14.1 Frictionless Flow

- Pitot-static tube

$$q_0 = \sqrt{\frac{2(p_s - p_0)}{\rho}} \quad (\text{A})$$

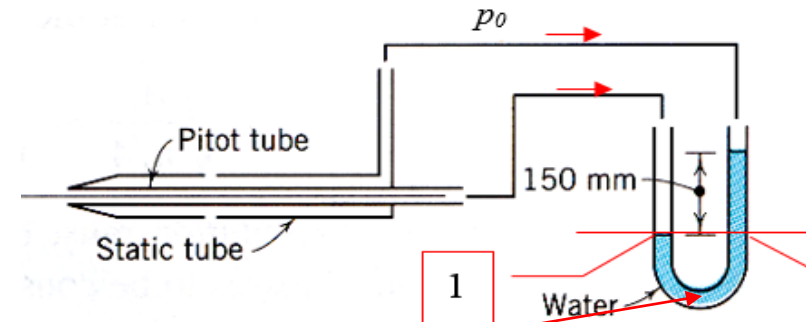
By the way,

$$p_1 = p_s + \gamma \Delta h = p_2 = p_0 + \gamma_m \Delta h$$

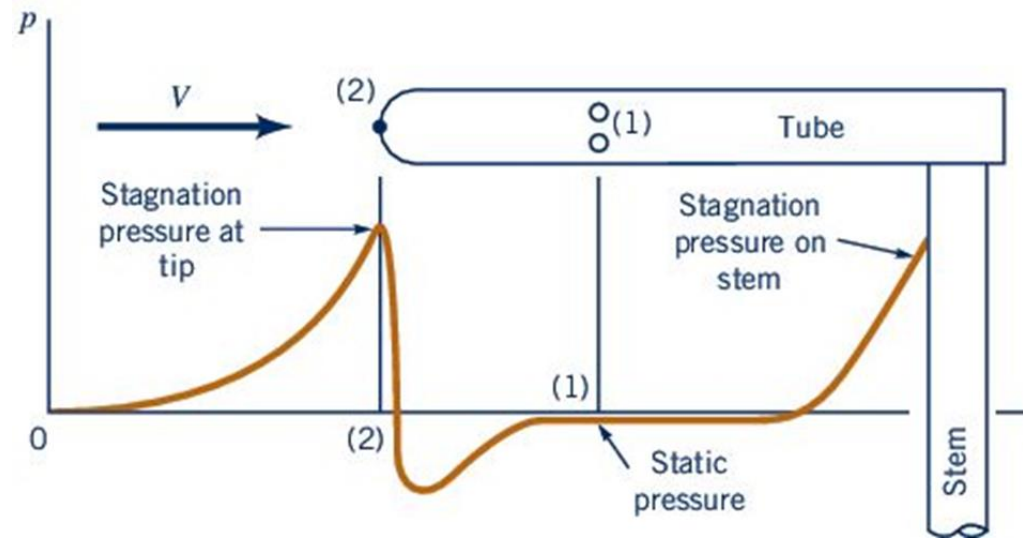
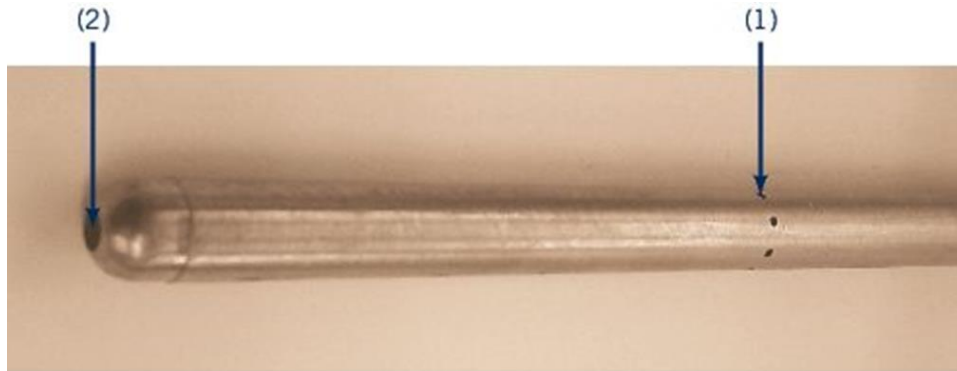
$$p_s - p_0 = \Delta h(\gamma_m - \gamma) \quad (\text{B})$$

Combine (A) and (B)

$$q_0 = \sqrt{\frac{2\Delta h(\gamma_m - \gamma)}{\rho}}$$



# 14.1 Frictionless Flow



# 14.1 Frictionless Flow

