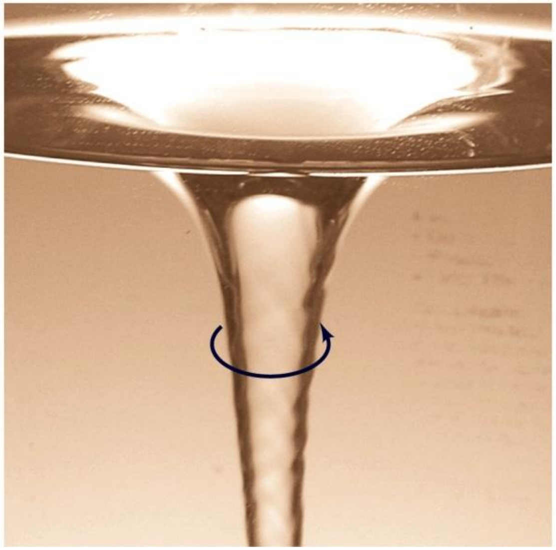
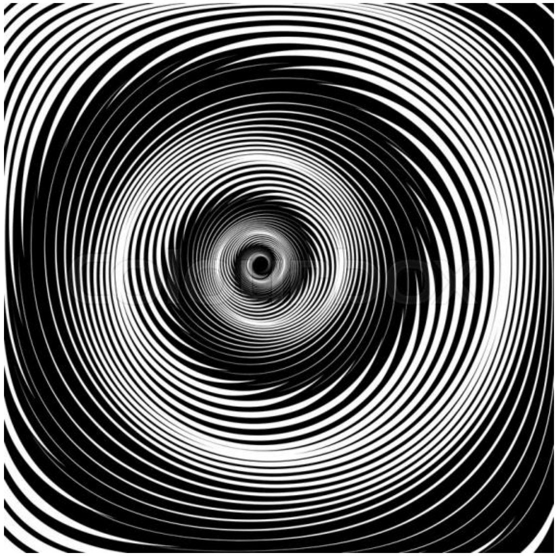
Lecture 15

Equations of Continuity and Motion (6/6)





Lecture 15 Equations of Continuity and Motion (6)

Contents

Objectives

- Derive 3D equations of continuity and motion
- Derive Navier-Stokes equation for Newtonian fluid
- Study solutions for simplified cases of laminar flow
- Derive Bernoulli equation for irrotational motion and frictionless flow
- Study solutions for vortex motions

15.1 Vortex Motion

- Vortex

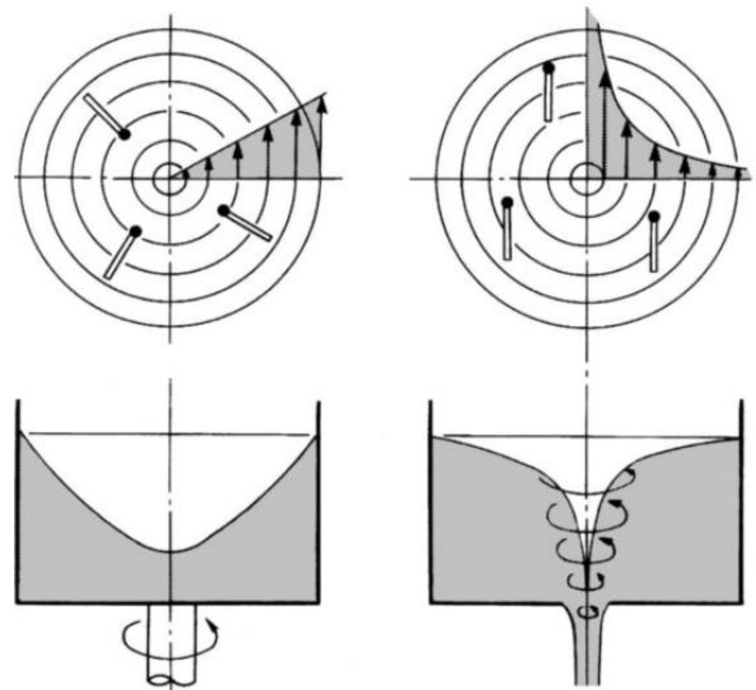
- fluid motion in which streamlines are concentric circles

• Forced vortex

- Rotational cylinder

• Free vortex

- Typhoon, drain hole vortex

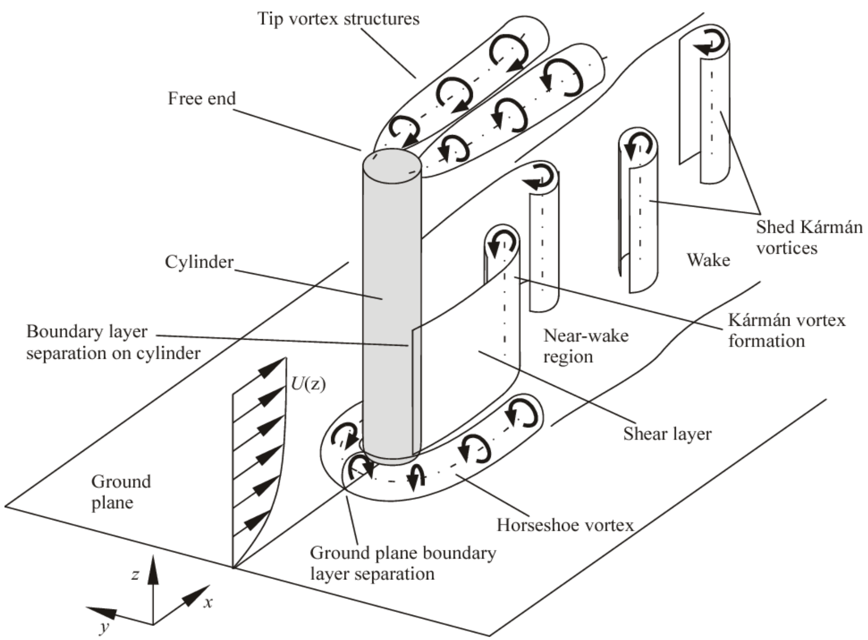


(a) Forced vortex flow

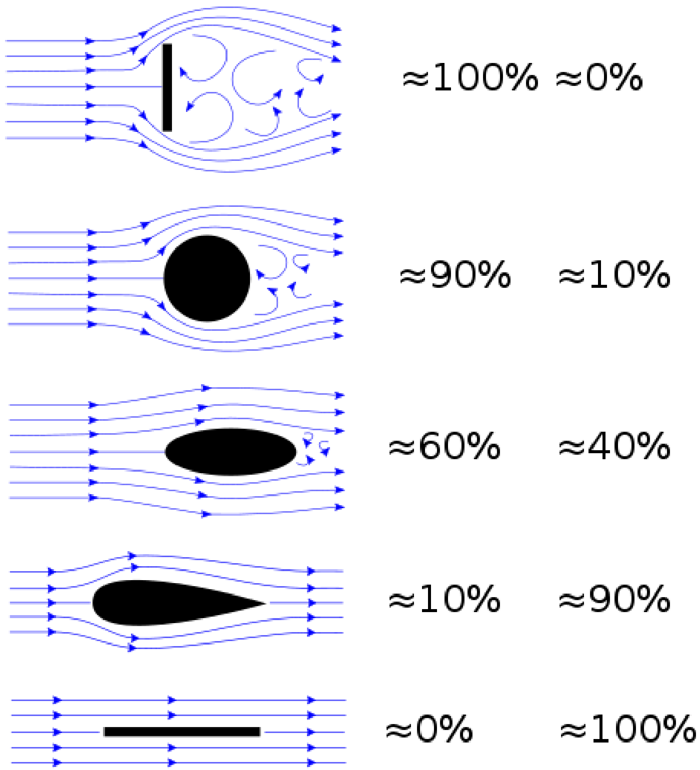
(b) Free vortex flow

15.1 Vortex Motion

- Vortex around the cylinder



- Form drag vs skin drag



15.1 Vortex Motion

15.1.1 Vortex

For steady flow of an incompressible fluid, apply Navier-Stokes equations in cylindrical coordinates

Assumptions:

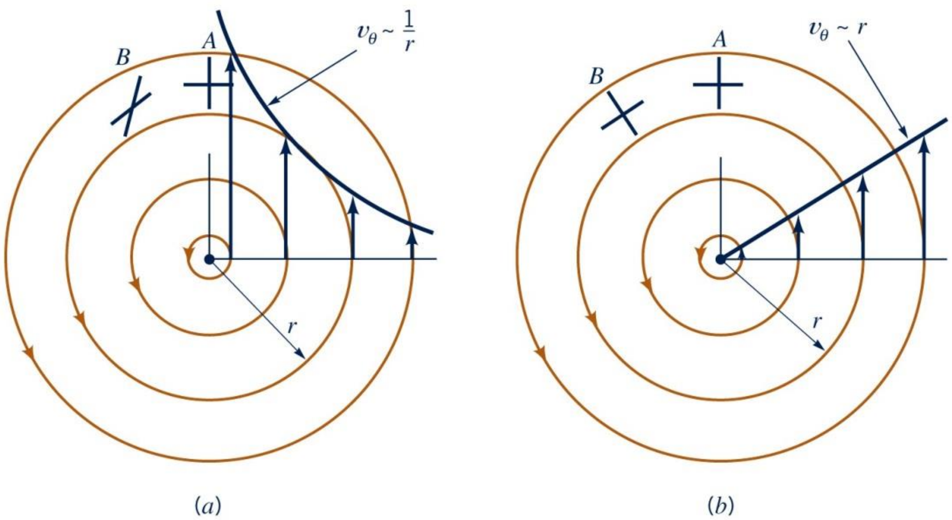
$$\frac{\partial(\ )}{\partial t} = 0$$

$v_\theta \neq 0$

$$v_r = 0; \quad v_z = 0; \quad \frac{\partial v_\theta}{\partial z} = 0$$

$$\frac{\partial p}{\partial \theta} = 0$$

$$\frac{\partial p}{\partial z} = \frac{\partial p}{\partial h}$$ (h =vertical direction)



15.1 Vortex Motion

Continuity Eq.: Eq. (11.10)

$$\frac{1}{r}\frac{\partial}{\partial r}\cancel{\left(rv_r\right)}+\frac{1}{r}\frac{\partial}{\partial\theta}\left(v_\theta\right)+\frac{\partial}{\partial z}\cancel{\left(v_z\right)}=0$$

$$\frac{1}{r}\frac{\partial}{\partial\theta}\left(v_\theta\right)=0\rightarrow\boxed{\frac{\partial v_\theta}{\partial\theta}=0}\qquad(15.1)$$

Navier-Stokes Eq.: Eq. (11.9)

1) r-comp.

$$\rho\left(\cancel{\frac{\partial v_r}{\partial t}}+v_r\cancel{\frac{\partial v_r}{\partial r}}+\frac{v_\theta}{r}\cancel{\frac{\partial v_r}{\partial\theta}}-\frac{v_\theta{}^2}{r}+v_z\cancel{\frac{\partial v_r}{\partial z}}\right)$$

15.1 Vortex Motion

$$= -\frac{\partial p}{\partial r} + \mu\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left[r\cancel{v_r}\right]\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_r}{\partial \theta^2}} - \frac{2}{r^2}\cancel{\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{\partial^2 v_r}{\partial z^2}}\right\} + \cancel{\rho g_r}$$

$$\boxed{\frac{{v_\theta}^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r}} \qquad (15.2a)$$

2) θ -comp.

$$\rho\left(\cancel{\frac{\partial v_\theta}{\partial t}} + \cancel{v_r}\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\cancel{\frac{\partial v_\theta}{\partial \theta}} - \cancel{\frac{v_r v_\theta}{r}} + \cancel{v_z}\frac{\partial v_\theta}{\partial z}\right)$$

$$= -\frac{1}{r}\cancel{\frac{\partial p}{\partial \theta}} + \mu\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left[r v_\theta\right]\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_\theta}{\partial \theta^2}} - \frac{2}{r^2}\cancel{\frac{\partial v_r}{\partial \theta}} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}}\right\} + \cancel{\rho g_\theta}$$

15.1 Vortex Motion

$$\therefore \boxed{0 = \frac{\mu}{\rho}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r v_\theta\right)\right]} \qquad (15.2b)$$

3) z-comp.

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + \cancel{v_r}\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\cancel{\frac{\partial v_z}{\partial \theta}} - \cancel{\frac{v_r v_\theta}{r}} + v_z\cancel{\frac{\partial v_z}{\partial z}}\right)$$

$$= -\frac{\partial p}{\partial z} + \mu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\cancel{\frac{\partial v_z}{\partial r}}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right\} + \rho g_z$$

$$\boxed{0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g_z = -\frac{1}{\rho}\frac{\partial p}{\partial h} - g} \qquad (15.2c)$$

15.1 Vortex Motion

Integrate θ -Eq. 15.2 b) w.r.t. r

$$0 = \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right]$$

$$C_1 = \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)$$

$$rC_1 = \frac{\partial}{\partial r}(rv_\theta)$$

Integrate again

$$\frac{r^2}{2}C_1 + C_2 = rv_\theta \qquad (A)$$

$$\boxed{v_\theta = \frac{C_1}{2}r + \frac{C_2}{r}} \qquad (B)$$

need 2 BCs (15.3)

15.1 Vortex Motion

z-Eq.

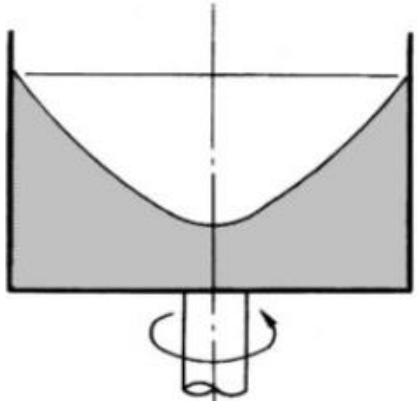
$$\frac{\partial p}{\partial h} = -\rho g = -\gamma \qquad (15.4)$$

$$p = -\gamma h \qquad (15.5)$$

→ hydrostatic pressure distribution

15.1 Vortex Motion

15.1.2 Forced Vortex - rotational flow



Consider cylindrical container of radius R is rotated at a constant angular velocity Ω about a vertical axis

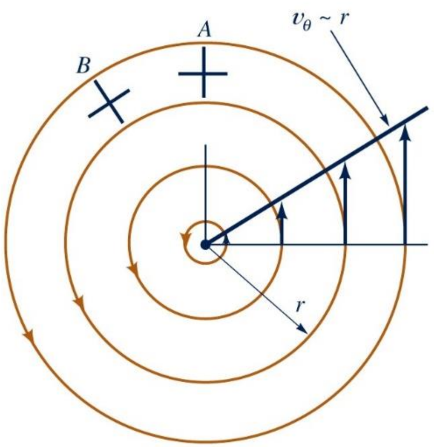
Substitute BCs into Eq. (15.3)

i) $r = 0, \quad v_\theta = 0$

$$\rightarrow (A):\ 0 + C_2 = 0 \qquad \therefore\ C_2 = 0$$

ii) $r = R,\ v_\theta = R\Omega$

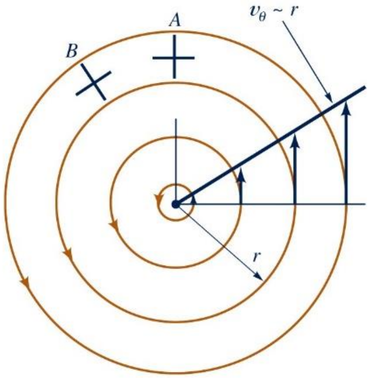
$$\rightarrow (B): R\Omega = \frac{C_1}{2} R \qquad \therefore\ C_1 = 2\Omega$$



15.1 Vortex Motion

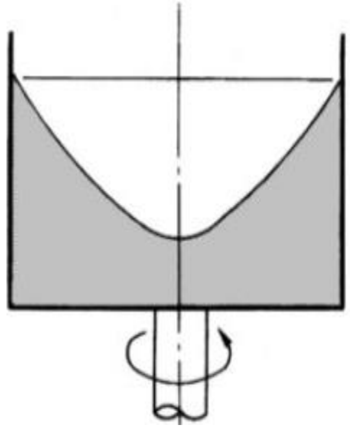
Eq. (B) becomes

$$\boxed{v_\theta = \frac{2\Omega}{2} r = \Omega r} \rightarrow \text{solid-body rotation} \quad (15.6)$$



$$r - Eq.: \ \frac{\Omega^2 r^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\rightarrow \quad \boxed{\frac{\partial p}{\partial r} = \rho \Omega^2 r} \qquad (15.7)$$



$$z - Eq.: \ \frac{\partial p}{\partial h} = -\gamma \qquad (15.4)$$

15.1 Vortex Motion

Consider total derivative *dp*

$$dp = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial h} dh = \rho \Omega^2 r dr - \gamma dh$$

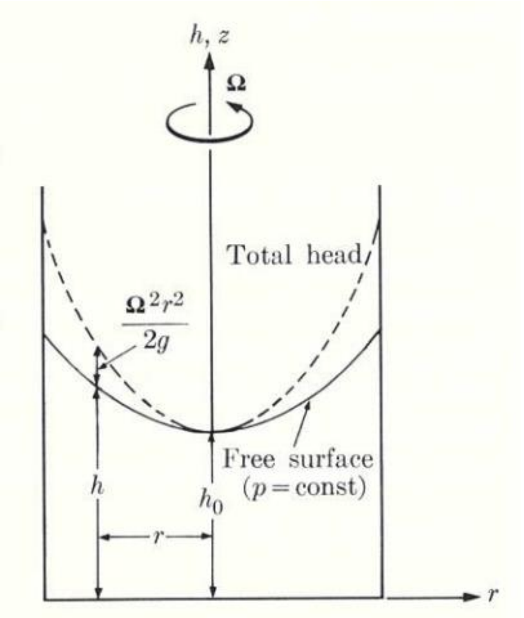
Integrate once

$$p = \rho \Omega^2 \frac{r^2}{2} - \gamma h + C_3$$

Incorporate B.C. to decide C_3

$$r = 0; \quad h = h_0 \quad \text{and} \quad p = p_0$$

$$p_0 = 0 - \gamma h_0 + C_3 \qquad \therefore \quad C_3 = p_0 + \gamma h_0$$



15.1 Vortex Motion

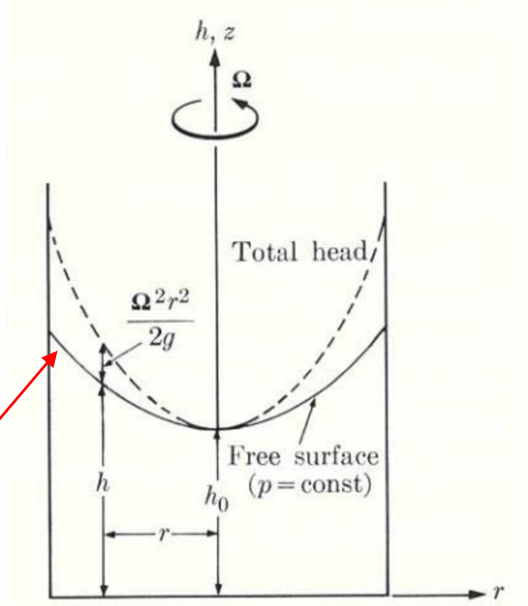
$$\boxed{p - p_0 = \rho \frac{\Omega^2 r^2}{2} - \gamma \left(h - h_0 \right)} \qquad (15.8)$$

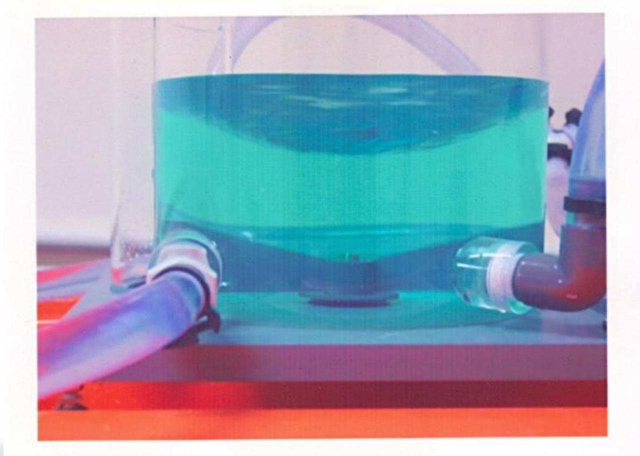
At free surface

$$p = p_0$$

$$\boxed{h = h_0 + \frac{\Omega^2}{2g} r^2} \qquad (15.9)$$

→ paraboloid of revolution





15.1 Vortex Motion

- Rotation components in cylindrical coordinates

Eq. (10.15):

$$\omega_z = \frac{1}{2}\left(\cancel{-\frac{1}{r}\frac{\partial v_r}{\partial \theta}} + \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} \right)$$

$$= \frac{1}{2}\left(\frac{r\Omega}{r} + \frac{\partial}{\partial r}(r\Omega) \right) = \frac{1}{2}(\Omega + \Omega) = \Omega$$

vorticity $= 2\omega_z = 2\Omega \neq 0$

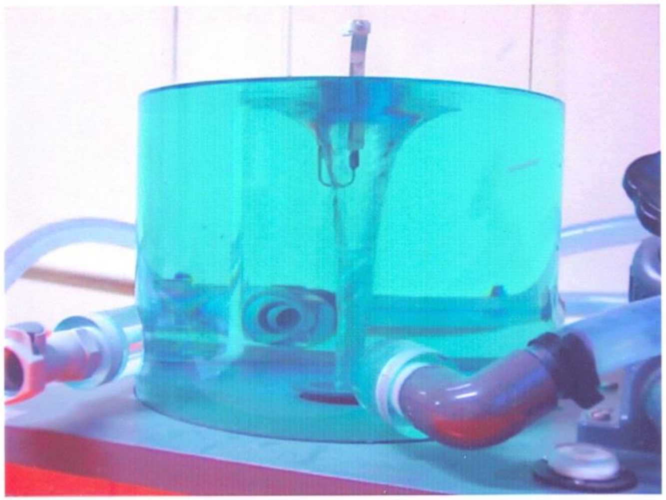
→ rotational flow

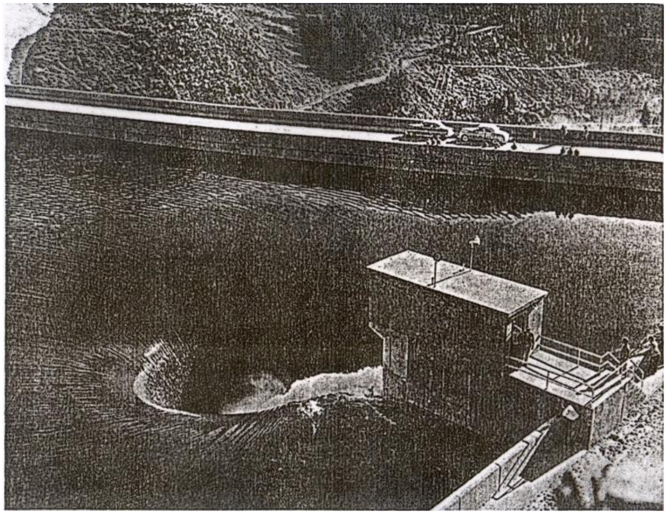
→ Forced vortex is generated by the transmission of tangential shear stresses.

15.1 Vortex Motion

15.1.3 Irrotational or free vortex

Free vortex: drain hole vortex, tornado, hurricane, morning glory spillway





15.1 Vortex Motion

For irrotational flow,

$$\frac{p}{\gamma}+h+\frac{v_\theta^{\ 2}}{2g}= \text{Constant throughout the fluid field}$$

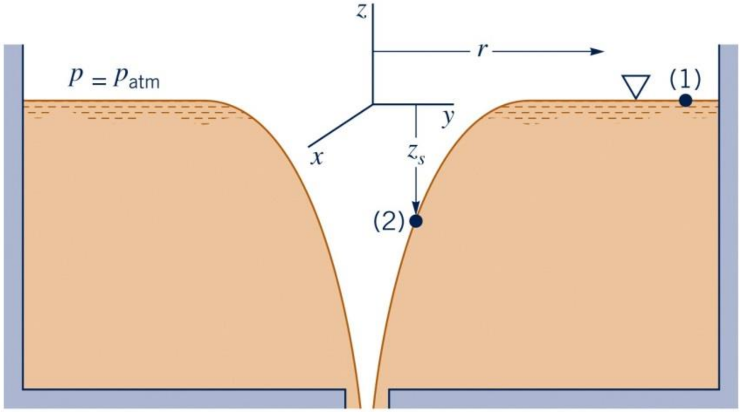
z coincides with h

$$\left(\frac{\partial h}{\partial r}=\frac{\partial h}{\partial \theta}=0,\ \frac{\partial h}{\partial z}=1\right)$$

Differentiate w.r.t r

$$\frac{1}{\gamma}\frac{\partial p}{\partial r}+\frac{\partial h}{\partial r}+\frac{1}{g}v_\theta\frac{\partial v_\theta}{\partial r}=0$$

$$\therefore\ \frac{\partial p}{\partial r}=-\rho v_\theta\frac{\partial v_\theta}{\partial r} \qquad \text{(A)}$$



15.1 Vortex Motion

Eq (15.2a): r-Eq. of N-S Eq.

$$\frac{\partial p}{\partial r} = \rho \frac{{v_\theta}^2}{r} \qquad \text{(B)}$$

Equate (A) and (B)

$$-\rho v_\theta \frac{\partial v_\theta}{\partial r} = \rho \frac{{v_\theta}^2}{r} \quad \rightarrow \quad -\frac{\partial v_\theta}{\partial r} r = v_\theta \qquad \text{(C)}$$

Integrate using separation of variables

$$\int \frac{1}{v_\theta} \partial v_\theta = \int -\frac{1}{r} \partial r$$

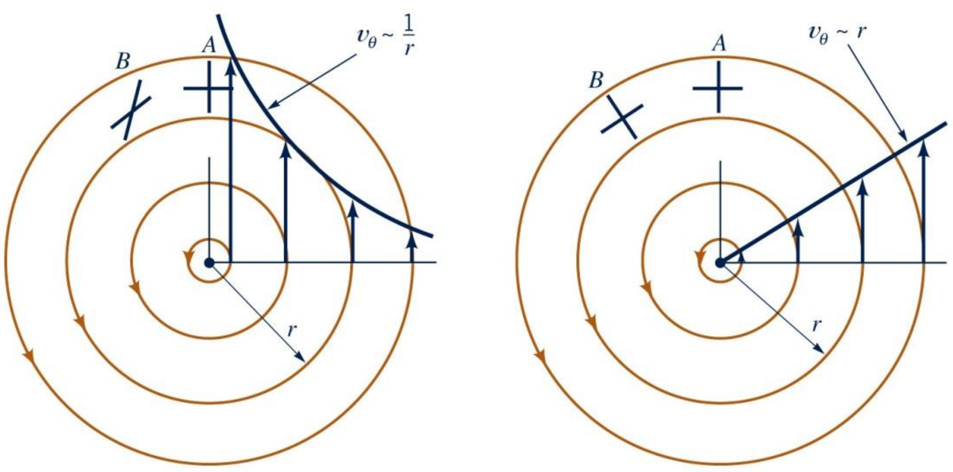
15.1 Vortex Motion

$$\ln v_\theta = -\ln r + C$$

$$\ln v_\theta + \ln r = \ln\left(v_\theta r\right) = C$$

$v_\theta r = C_4$ ~ constant angular momentum (D)

$$\boxed{v_\theta = \frac{C_4}{r}} \qquad (15.10)$$



(a) (b)

[Cf] Forced vortex

$$v_\theta = \Omega r \qquad (15.6)$$

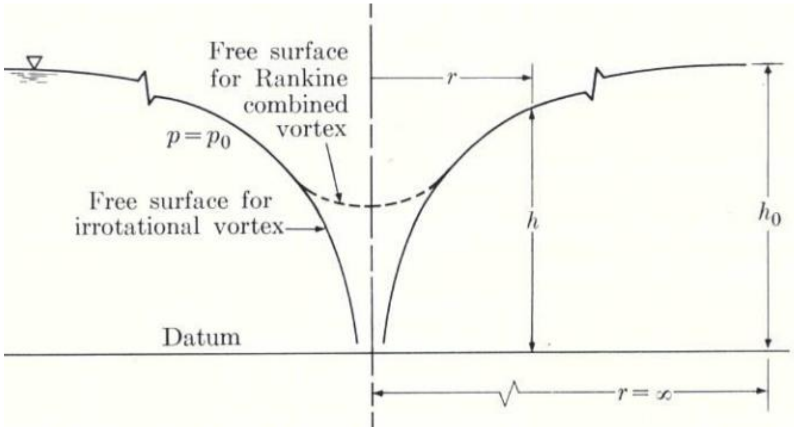
15.1 Vortex Motion

- Radial pressure gradient

(D)

(B):

$$\frac{\partial p}{\partial r} = \rho \frac{v_\theta^{\ 2}}{r} = \rho \frac{\left(v_\theta r\right)^2}{r^3} = \rho \frac{C_4^{\ 2}}{r^3}$$



- Total derivative

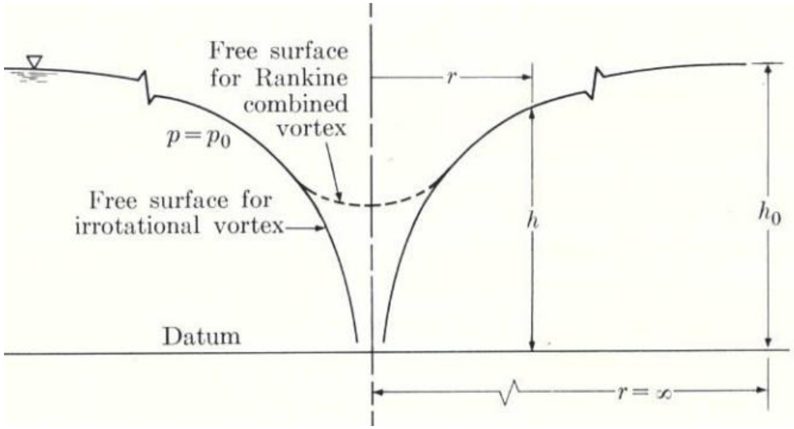
$$\boxed{\frac{\partial p}{\partial h} = -\gamma}$$

$$dp = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial h} dh = \rho \frac{C_4^{\ 2}}{r^3} dr - \gamma dh$$

15.1 Vortex Motion

Integrate once

$$p = -\rho \frac{C_4^{\ 2}}{2r^2} - \gamma h + C_5 \qquad \text{(E)}$$



B.C.: $r = \infty:\quad h = h_0$ and $p = p_0$

Substitute B.C. into Eq. (E)

$$p_0 = -\gamma h_0 + C_5 \qquad C_5 = p_0 + \gamma h_0$$

$$\boxed{p - p_0 = \gamma\left(h_0 - h\right) - \rho \frac{C_4^{\ 2}}{2r^2}} \qquad (15.11)$$

[Cf] Forced vortex: $p - p_0 = \dfrac{\rho}{2}\Omega^2 r^2 + \gamma\left(h_0 - h\right)$ (15.8)

15.1 Vortex Motion

- Locus of free surface is given when $p = p_0$

$$h = h_0 - \frac{{C_4}^2}{2gr^2} \quad \rightarrow \text{hyperboloid of revolution} \qquad (15.12)$$

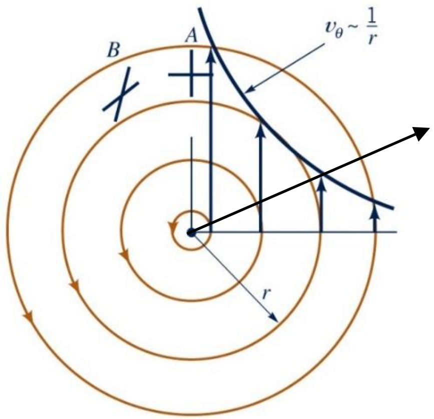
[Cf] Forced vortex: $h = h_0 + \dfrac{\Omega^2}{2g} r^2$ (15.9)

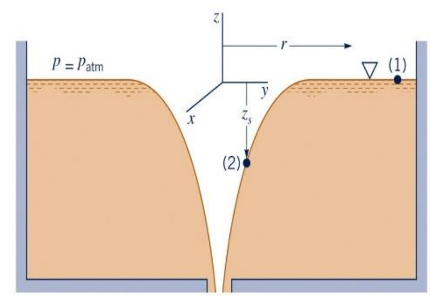
- Circulation

$$\boxed{ds = r d\theta}$$

$$\Gamma = \oint \vec{q} \cdot d\vec{s} = \int_0^{2\pi} v_\theta r d\theta = \left[C_4 \theta\right]_0^{2\pi} = 2\pi C_4 \neq 0$$

$$\boxed{v_\theta r = C_4}$$





15.1 Vortex Motion

→ Even though flow is irrotational, circulation for a contour enclosing the origin is not zero because of the singularity point.

- Vorticity component ω_z

$$\omega_z = -\frac{1}{r}\cancel{\frac{\partial v_r}{\partial \theta}} + \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r}$$

Substitute $v_\theta = \dfrac{C_4}{r}$

$$\omega_z = \frac{C_4}{r^2} + \frac{\partial}{\partial r}\left(\frac{C_4}{r}\right) = \frac{C_4}{r^2} - \frac{C_4}{r^2} = 0$$

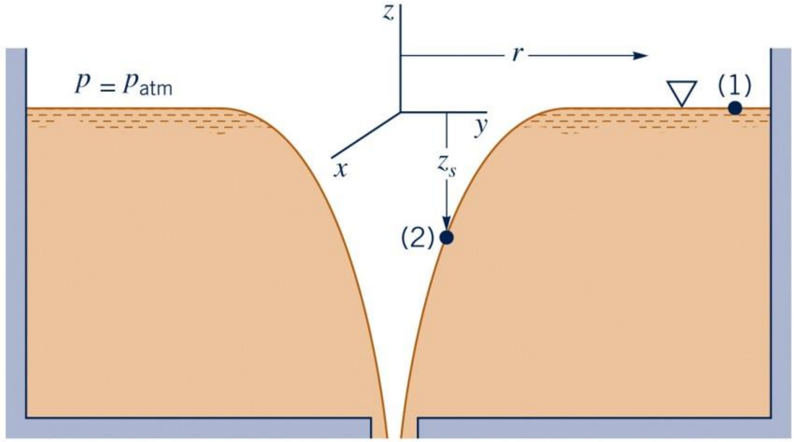
→ Irrotational motion

15.1 Vortex Motion

At $r = 0$ of drain hole vortex, either fluid does not occupy the space or fluid is rotational (forced vortex) when drain in the tank bottom is suddenly closed.

→ Rankine combined vortex

→ Fluid motion is ultimately dissipated through viscous action.



15.1 Vortex Motion

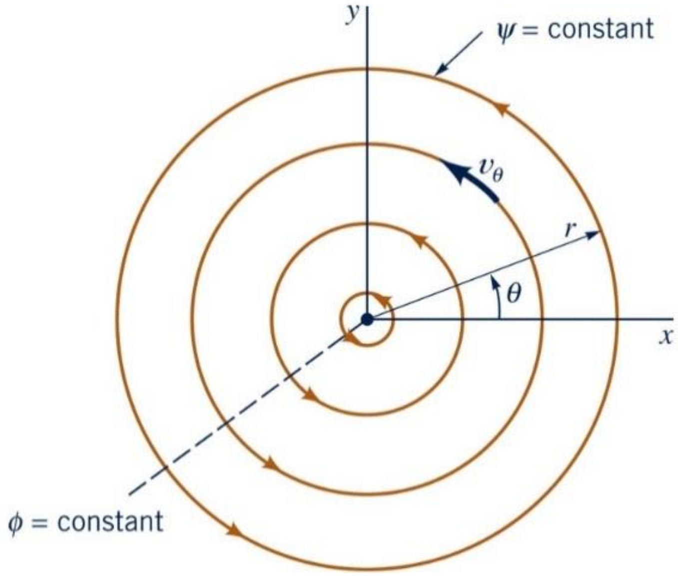
- Stream function, ψ

$$v_\theta = \frac{\partial \psi}{\partial r} = \frac{C_4}{r} = \frac{\Gamma}{2\pi r}$$

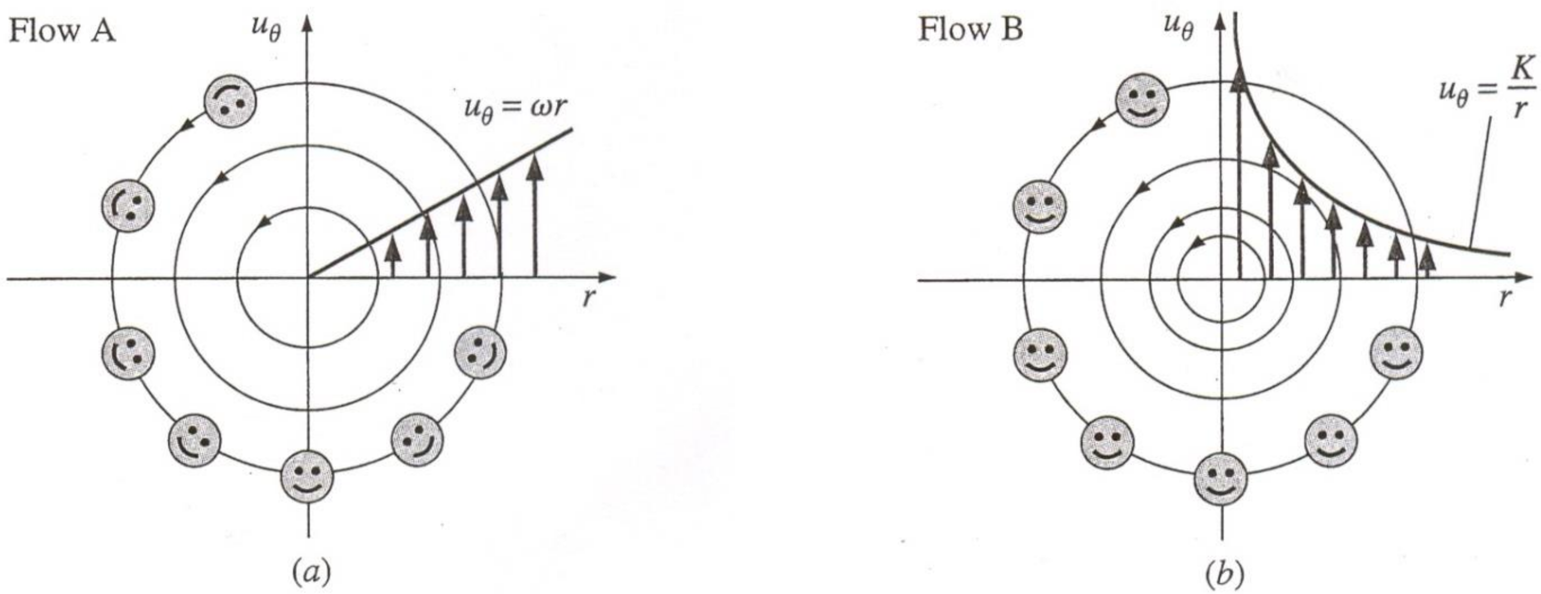
$$C_4 = \frac{\Gamma}{2\pi}$$

$$\boxed{\psi = \frac{\Gamma}{2\pi}\int \frac{dr}{r} = \frac{\Gamma}{2\pi}\ln r = K \ln r}$$

where Γ = vortex strength



15.1 Vortex Motion



15.1 Vortex Motion

(a) (b)

Problems

Homework Assignment # 5

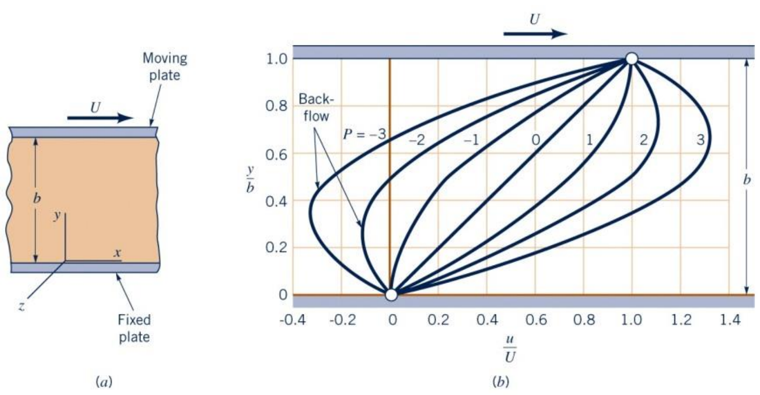
Due: 2 weeks from today

1. (6-4) Consider an incompressible two-dimensional flow of a viscous fluid in the xy-plane in which the body force is due to gravity. (a) Prove that the divergence of the vorticity vector is zero. (This expresses the conservation of vorticity, $\nabla \cdot \vec{\zeta} = 0$.) (b) Show that the Navier- Stokes equation for this flow can be written in terms of the vorticity as $\frac{d\vec{\zeta}}{dt} = \nu \nabla^2 \vec{\zeta}$. (This is a "diffusion" equation and indicates that vorticity is diffused into a fluid at a rate which depends on the magnitude of the kinematic viscosity.) Note that $d\vec{\zeta}/dt$ is the substantial derivative.

$$\vec{\zeta} = \nabla \times \vec{q} = \vec{i}\,\xi + \vec{j}\,\eta + \vec{k}\,\zeta \qquad \xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \qquad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$

Problems

2. (6-5) Consider a steady, incompressible laminar flow between parallel plates as shown in Fig. 6-4 for the following conditions: a =0.03 m, U =0.3 m/sec, $\mu = 0.476$ N·sec/m^2, $\partial p / \partial x$=625 N/m^3 (pressure increases in + x-direction). (a) Plot the velocity distribution, $u(z)$, in the z-direction. Use Eq. (6.24) (b) In which direction is the net fluid motion? (c) Plot the distribution of shear stress τ_{zx} in the z-direction.

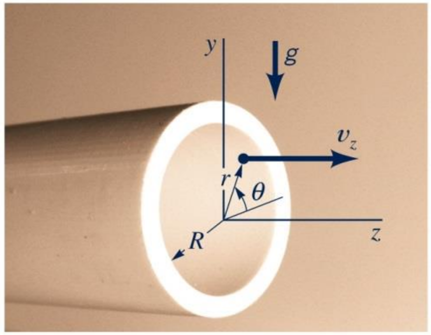


Problems

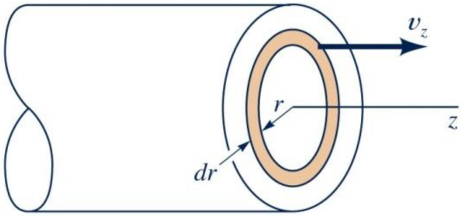
3. (6-7) An incompressible liquid of density ρ and viscosity μ flows in a thin film down glass plate inclined at an angle α to the horizontal. The thickness, a , of the liquid film normal to the plate is constant, the velocity is everywhere parallel to the plate, and the flow is steady. Neglect viscous shear between the air and the moving liquid at the free surface. Determine the variation in longitudinal velocity in the direction normal to the plate, the shear stress at the plate, and the average velocity of flow.

Problems

4. (6-11) Consider steady *laminar flow* in the horizontal axial direction through the annular space between two concentric circular tubes. The radii of the inner and outer tube are r_1 and r_2, respectively. Derive the expression for the velocity distribution in the direction as a function of viscosity, pressure gradient $\partial p / \partial x$, and tube dimensions.



(a)



(b)

Problems

5. (6-15) The velocity potential for a steady incompressible flow is given by $\Phi = (-a/2)(x^2 + 2y - z^2)$, where a is an arbitrary constant greater than zero. (a) Find the equation for the velocity vector $\vec{q} = \vec{i}u + \vec{j}v + \vec{k}w$.

(b) Find the equation for the streamlines in the xz ($y = 0$) plane.

(c) Prove that the continuity equation is satisfied.

6. (6-21) The velocity variation across the radius of a rectangular bend (Fig. 6-22) may be approximated by a free vortex distribution $v_\theta r = const.$ Derive an expression for the pressure difference between the inside and outside of the bend as a function of the discharge Q, the fluid density ρ, and the geometric parameters R and b, assuming frictionless flow.

Problems

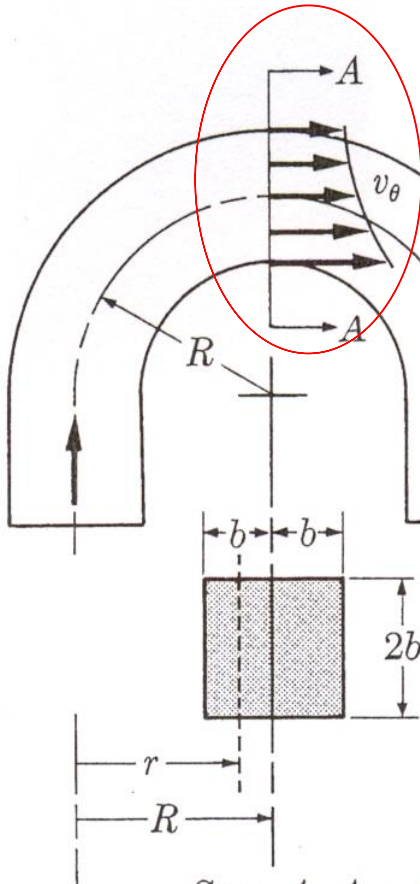


FIGURE 6–22