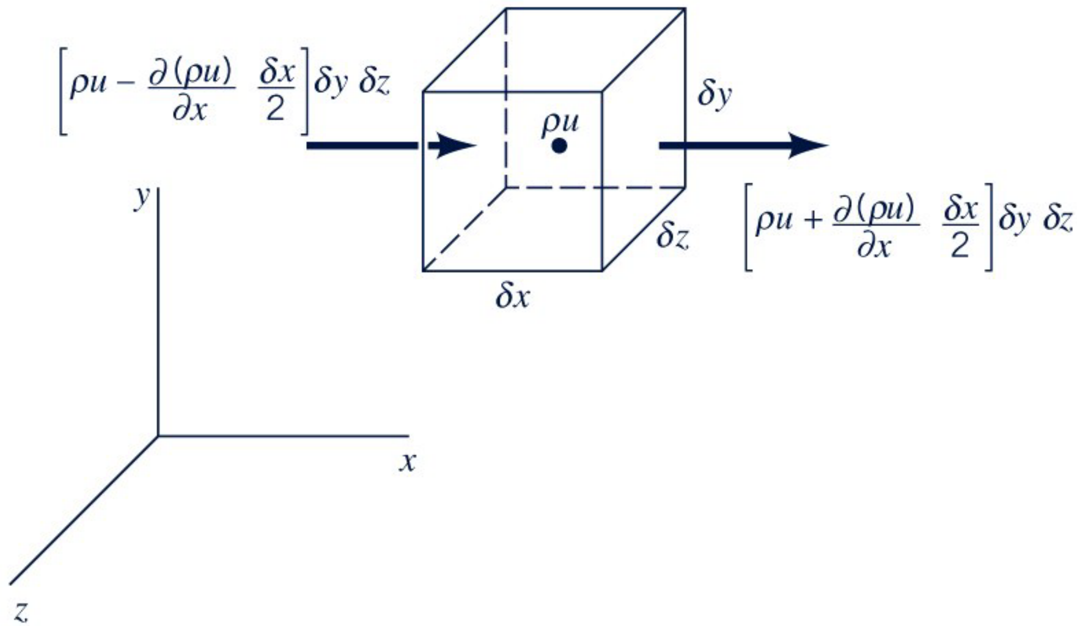
Chapter 6 Equations of Continuity and Motion

Session 6-1 Continuity equation



Chapter 6 Equations of Continuity and Motion

Contents

Chapter 6 Equations of Continuity and Motion

Objectives

- Derive 3D equations of continuity and motion
- Derive Navier-Stokes equation for Newtonian fluid
- Study solutions for simplified cases of laminar flow
- Derive Bernoulli equation for irrotational motion and frictionless flow
- Study solutions for vortex motions

6.1 Continuity Equation

- Derivation of 3-D Eq.

  conservation of mass → continuity eq.

  conservation of momentum → eq. of motion → Navier-Strokes eq.

  Consider infinitesimal control volume ($\Delta x \Delta y \Delta z$) → point form

  [Cf] Finite control volume – arbitrary CV → integral form

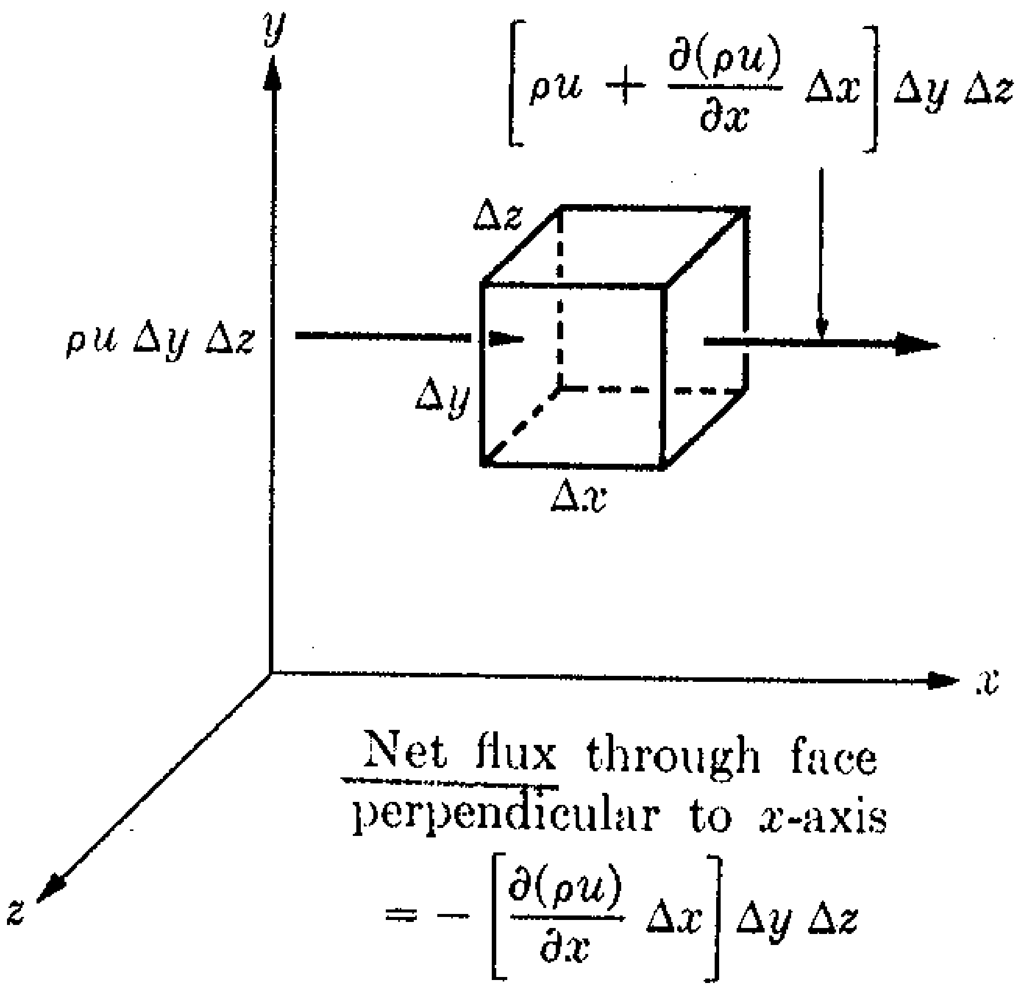
Apply principle of conservation of matter to the CV

→ sum of net flux = time rate change of mass inside C.V.

1) mass flux per unit time (mass flow)

$$= \frac{mass}{time} = \rho \frac{vol}{time} = \rho Q = \rho u \Delta A$$

6.1 Continuity Equation



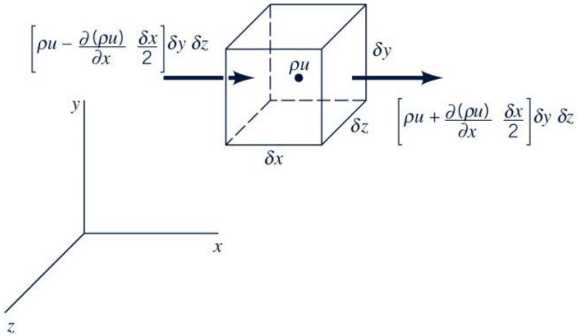
6.1 Continuity Equation

- net flux through face perpendicular to x-axis

= flux in –flux out

$$= \rho u \Delta y \Delta z - \left(\rho u + \frac{\partial(\rho u)}{\partial x} \Delta x \right) \Delta y \Delta z = -\frac{\partial(\rho u)}{\partial x} \Delta x \Delta y \Delta z$$

- net flux through face perpendicular to y-axis



$$= -\frac{\partial(\rho v)}{\partial y} \Delta x \Delta y \Delta z$$

- net flux through face perpendicular to z-axis

$$= -\frac{\partial(\rho w)}{\partial z} \Delta x \Delta y \Delta z \qquad \text{(A)}$$

6.1 Continuity Equation

2) time rate change of mass inside C.V.

$$= \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z) \qquad \text{(B)}$$

Thus, equating (A) and (B) gives

$$\frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z) = -\frac{\partial(\rho u)}{\partial x}\Delta x \Delta y \Delta z - \frac{\partial(\rho v)}{\partial y}\Delta x \Delta y \Delta z - \frac{\partial(\rho w)}{\partial z}\Delta x \Delta y \Delta z$$

$$LHS = \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z) = \rho \frac{\partial}{\partial t}(\Delta x \Delta y \Delta z) + \Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t}$$

6.1 Continuity Equation

Since C.V. is fixed → $\dfrac{\partial(\Delta x \Delta y \Delta z)}{\partial t} = 0$

$$\therefore \quad LHS = \Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t}$$

Cancelling terms makes

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{q} = 0} \qquad (6.1)$$

→ Continuity Eq. for compressible fluid in unsteady flow (point form)

6.1 Continuity Equation

The 2nd term of Eq. (6.1) can be expressed as

$$\nabla \cdot (\rho \vec{q}) = \underbrace{\vec{q}\nabla\rho}_{\text{I}} + \underbrace{\rho\nabla \cdot \vec{q}}_{\text{II}}$$

(I): $\vec{q}\nabla\rho = (u\vec{i} + v\vec{j} + w\vec{k})\left(\frac{\partial\rho}{\partial x}\vec{i} + \frac{\partial\rho}{\partial y}\vec{j} + \frac{\partial\rho}{\partial z}\vec{k}\right)$ — gradient

$$= u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z}$$

(II): $\rho\nabla \cdot \vec{q} = \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$ — divergence

$$\therefore \nabla \cdot (\rho\vec{q}) = u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \qquad \text{(i)}$$

6.1 Continuity Equation

Substituting (i) into Eq (6.1) yields

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0$$

$$\frac{d\rho}{dt}$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot\rho\vec{q}=0 \qquad (6.1)$$

$$\frac{d\rho}{dt}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0 \qquad (6.2a)$$

$$\frac{d\rho}{dt}+\rho(\nabla\cdot\vec{q})=0 \qquad (6.2b)$$

6.1 Continuity Equation

[Re] Total derivative (total rate of density change)

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{dx}{dt} + \frac{\partial \rho}{\partial y}\frac{dy}{dt} + \frac{\partial \rho}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}$$

1) For steady-state conditions

$$\rightarrow \frac{\partial \rho}{\partial t} = 0$$

6.1 Continuity Equation

Then (6.1) becomes

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=\nabla\cdot(\rho\vec{q})=0 \tag{6.3}$$

2) For incompressible fluid (whether or not flow is steady)

$$\rightarrow \frac{d\rho}{dt}=0 \tag{6.4}$$

Then (6.2) becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\nabla\cdot\vec{q}=0 \tag{6.5}$$

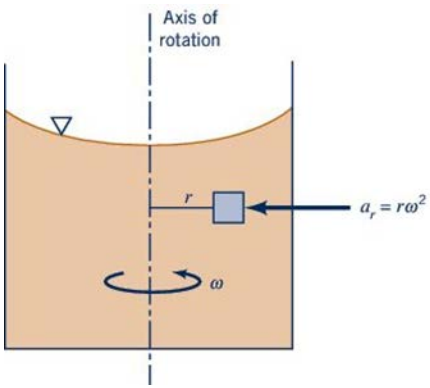
6.1 Continuity Equation

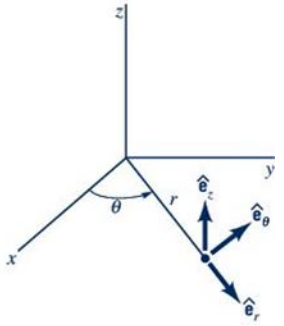
[Re] Continuity equation in polar (cylindrical) coordinates

u, r - radial

v, θ - azimuthal

w, z - axial





For compressible fluid of unsteady flow

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho u r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v)}{\partial \theta} + \frac{\partial(\rho w)}{\partial z} = 0$$

For incompressible fluid

$$\frac{1}{r}\frac{\partial(ur)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0$$

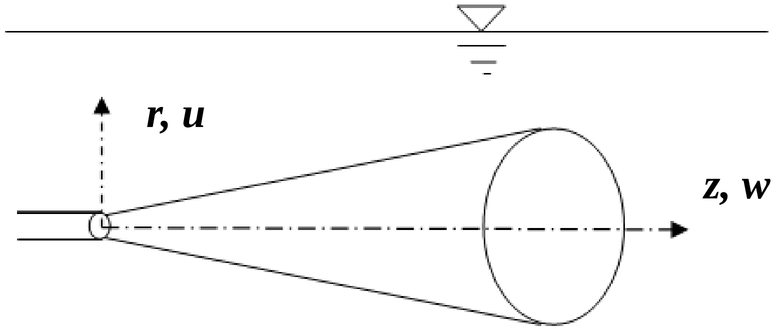
6.1 Continuity Equation

For incompressible fluid and flow of axial symmetry

$$\frac{\partial \rho}{\partial t} = 0, \quad \frac{\partial \rho}{\partial r} = \frac{\partial \rho}{\partial \theta} = \frac{\partial \rho}{\partial z} = 0, \qquad \frac{\partial (\rho v)}{\partial \theta} = 0$$

$$\therefore \frac{1}{r}\frac{\partial (ur)}{\partial r} + \frac{\partial w}{\partial z} = 0$$ → 2-D boundary layer flow

Example: submerged jet



6.2 Stream Function in 2-D, Incompressible Flows

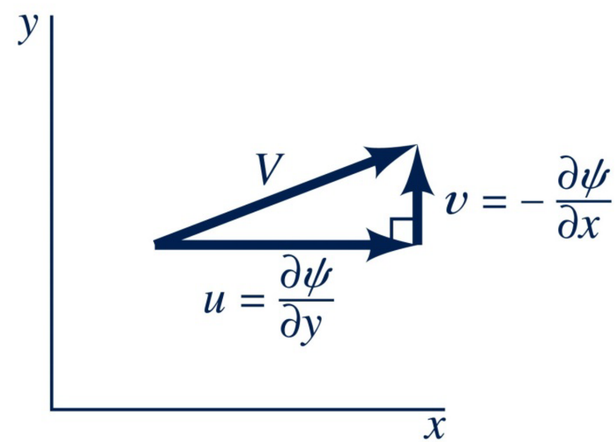
- 2-D incompressible continuity eq. is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (6.7)$$

In 2D flowfield, define stream function $\psi(x, y)$ as

$$u = -\frac{\partial \psi}{\partial y} \qquad (6.8)$$

$$v = \frac{\partial \psi}{\partial x} \qquad (6.9)$$



- We can have a simplified equation to determine only one unknown function.

6.2 Stream Function in 2-D, Incompressible Flows

Then LHS of Eq. (6.7) becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)=-\frac{\partial^2 \psi}{\partial x \partial y}+\frac{\partial^2 \psi}{\partial x \partial y}=0 \qquad (6.10)$$

→ Thus, continuity equation is satisfied.

1) Apply stream function to the equation for a stream line in 2-D flow

Eq. (2.10): $vdx-udy=0$ (6.11)

Substitute (6.8) into (6.11)

$$\frac{\partial \psi}{\partial x}dx+\frac{\partial \psi}{\partial y}dy=d\psi=0 \qquad (6.12)$$

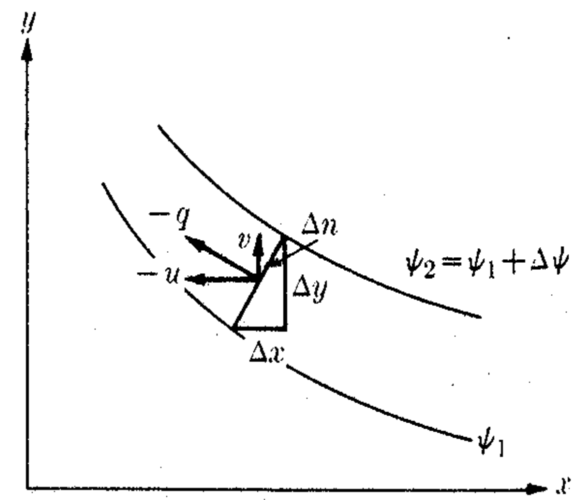
6.2 Stream Function in 2-D, Incompressible Flows

$$\psi = \text{constant} \tag{6.13}$$

→ The stream function is constant along a streamline.

2) Apply stream function to the law of conservation of mass

$$-qdn = -udy + vdx \tag{6.14}$$

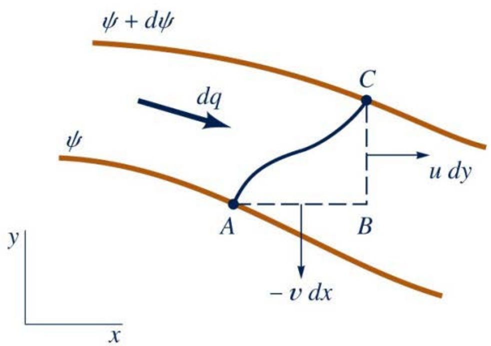


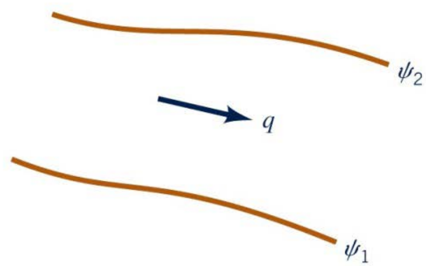
Substitute (6.8) into (6.14)

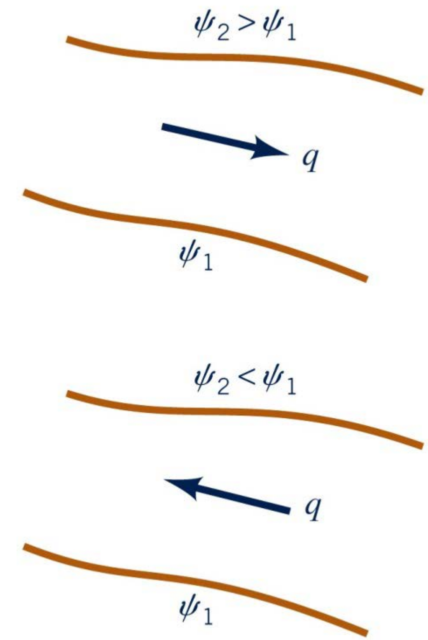
$$-qdn = \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx = d\psi \tag{6.15}$$

6.2 Stream Function in 2-D, Incompressible Flows

→ Change in ψ ($d\psi$) between adjacent streamlines is equal to the volume rate of flow per unit width.







- Stream function in cylindrical coordinates

$$v_r = -\frac{\partial \psi}{r \partial \theta} \quad \text{radial} \qquad v_\theta = \frac{\partial \psi}{\partial r} \quad \text{azimuthal}$$

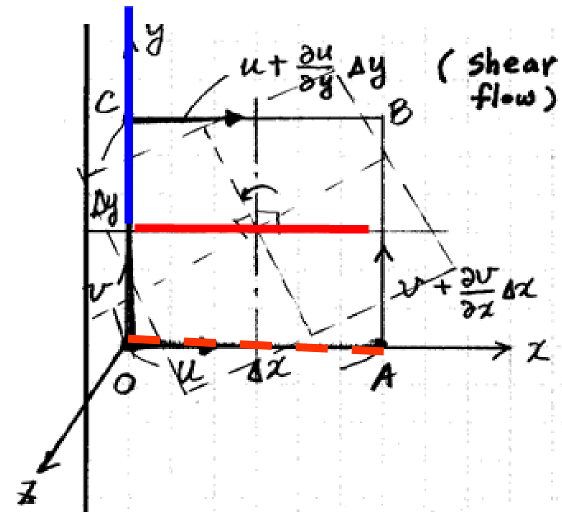
6.3 Rotational and Irrotational Motion

6.3.1 Rotation and vorticity

Assume the rate of rotation of fluid element Δx and Δy about z-axis is positive when it rotates counterclockwise.

- time rate of rotation of Δx -face about z -axis

$$= \frac{1}{\Delta t} \frac{\left[\left\{ v + \left(\frac{\partial v}{\partial x} \right) \Delta x \right\} - v \right] \Delta t}{\Delta x} = \frac{\partial v}{\partial x}$$



6.3 Rotational and Irrotational Motion

- time rate of rotation of Δy-face about z- axis

$$= -\frac{1}{\Delta t}\frac{\left[u + \left(\frac{\partial u}{\partial y}\Delta y\right) - u\right]\Delta t}{\Delta y} = -\frac{\partial u}{\partial y}$$

net rate of rotation = average of rotations of Δx-and Δy-face

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

6.3 Rotational and Irrotational Motion

Doing the same way for x-axis, and y-axis

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \qquad (6.16a)$$

1) Rotation

$$\vec{\omega} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

$$= \frac{1}{2}\left(\nabla \times \vec{q}\right) = \frac{1}{2} curl\ \vec{q} \qquad (6.16b)$$

6.3 Rotational and Irrotational Motion

Magnitude:

$$\left|\vec{\omega}\right| = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$$

a) Ideal fluid → irrotational flow

$$\nabla \times \vec{q} = 0$$

$$\omega_x = \omega_y = \omega_z = 0$$

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \qquad (6.17)$$

b) Viscous fluid → rotational flow

$$\nabla \times \vec{q} \neq 0$$

6.3 Rotational and Irrotational Motion

2) Vorticity

$$\vec{\zeta} = curl\ \vec{q} = \nabla \times \vec{q} = 2\vec{\omega}$$

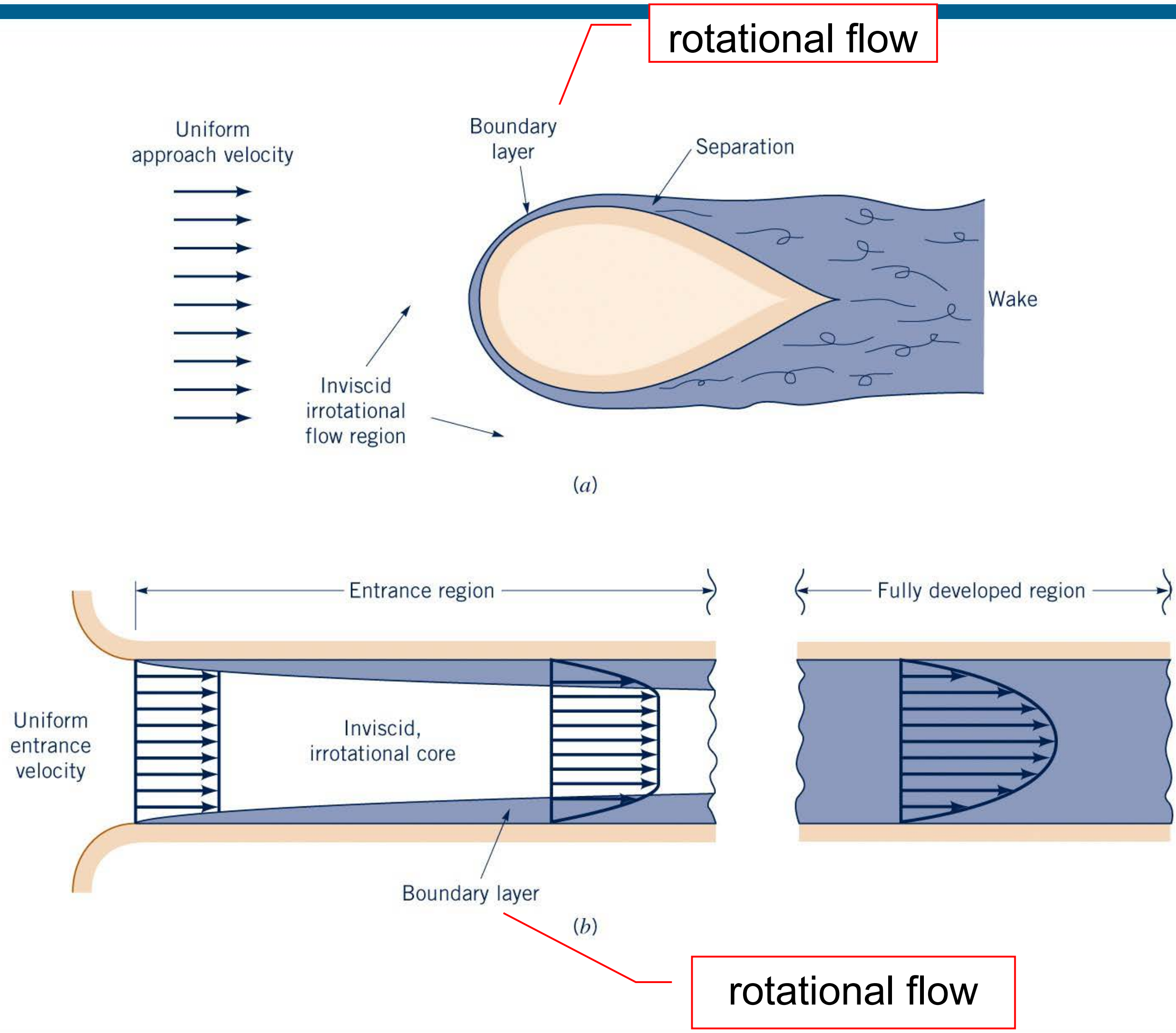
- Rotation in cylindrical coordinates

$$\omega_r = \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)$$

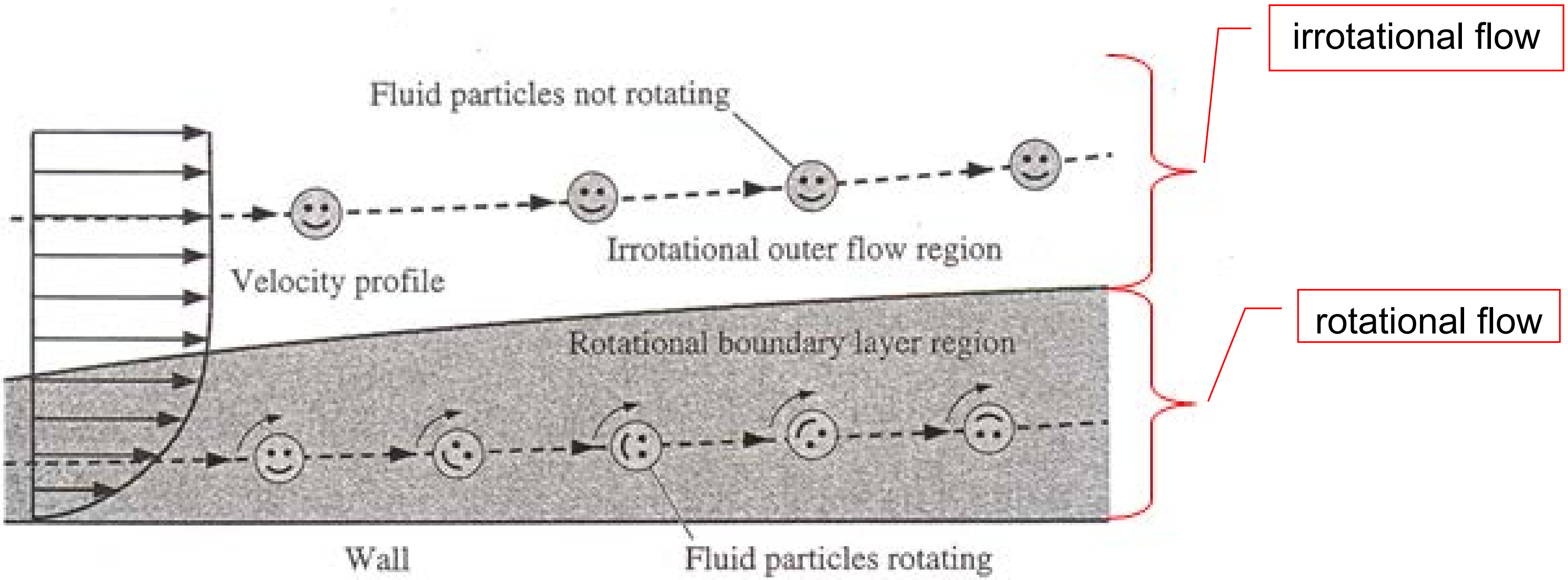
$$\omega_\theta = \frac{1}{2}\left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)$$

$$\omega_z = \frac{1}{2}\left(-\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r}\right)$$

6.3 Rotational and Irrotational Motion



6.3 Rotational and Irrotational Motion

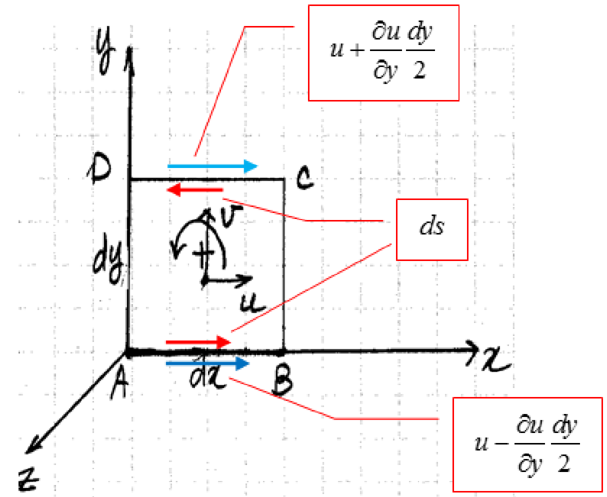


6.3 Rotational and Irrotational Motion

6.3.2 Circulation

Γ = line integral of the tangential velocity component about any closed contour *S*

$$\Gamma = \oint \vec{q} \cdot d\vec{s} \qquad (6.19)$$



6.3 Rotational and Irrotational Motion

– take line integral from A to B, C, D, A ~ infinitesimal CV

$$d\Gamma \cong \left[u - \frac{\partial u}{\partial y}\frac{dy}{2}\right]dx + \left[v + \frac{\partial v}{\partial x}\frac{dx}{2}\right]dy - \left[u + \frac{\partial u}{\partial y}\frac{dy}{2}\right]dx - \left[v - \frac{\partial v}{\partial x}\frac{dx}{2}\right]dy$$

$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy$$

$$d\Gamma \cong \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy$$

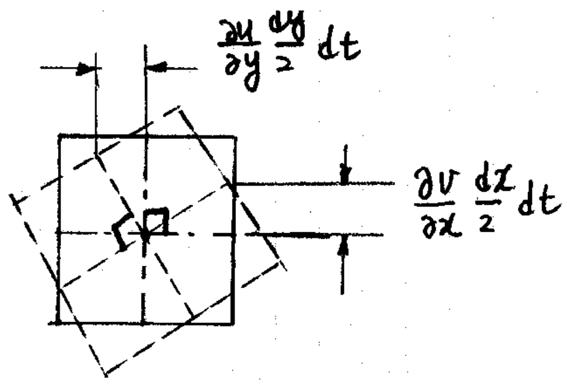
$$\Gamma = \iint_A \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dA = \iint_A 2\omega_z dA = \iint_A \left(\nabla \times \vec{q}\right)_z dA \qquad (6.20)$$

6.3 Rotational and Irrotational Motion

For irrotational flow,

circulation $\Gamma = 0$ (if there is no singularity vorticity source).

[Re] Fluid motion and deformation of fluid element



Motion
- translation
- rotation

Deformation
- linear deformation
- angular deformation