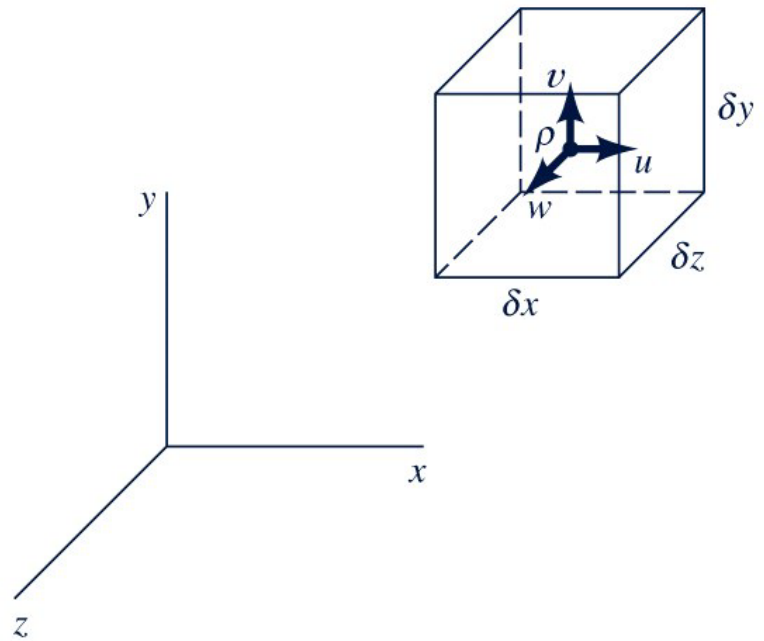
Chapter 6 Equations of Continuity and Motion

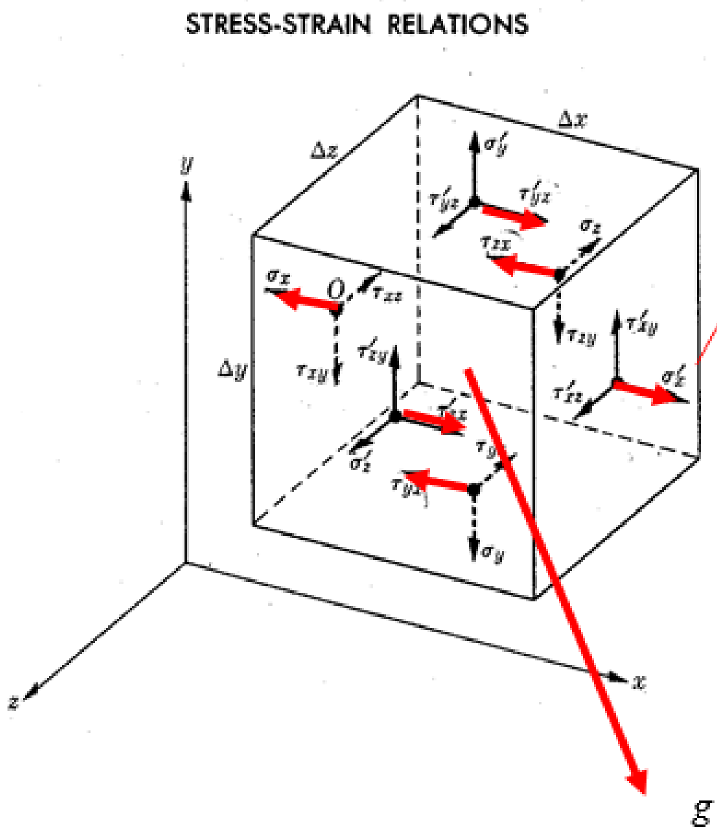
Session 6-2 Equation of motion



Chapter 6 Equations of Continuity and Motion

Contents

6.4 Equations of Motion



$$\sigma_x + \frac{\partial \sigma_x}{\partial x} \Delta x$$

6.4 Equations of Motion

- Apply Newton's 2nd law of motion

$$\sum \vec{F} = m\vec{a} \qquad \text{(A)}$$

$$\Delta F_x = \Delta m a_x$$

- External forces = surface force + body force
  - Surface force:

~ normal force + tangential force

  - Body forces:

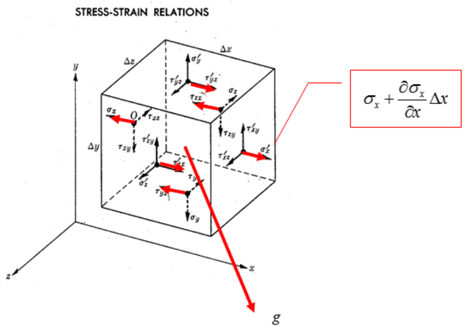
~ due to gravitational or electromagnetic fields, <u>no contact</u>

~ act at the centroid of the element → <u>centroidal force</u>

6.4 Equations of Motion

Consider only gravitational force

$$\vec{g} = \vec{i}g_x + \vec{j}g_y + \vec{k}g_z$$



LHS of (A):

$$\Delta F_x = \left(\rho \Delta x \Delta y \Delta z\right) g_x$$ body force (B)

$$-\sigma_x \Delta y \Delta z + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x}\Delta x\right)\Delta y \Delta z$$ normal force

$$-\tau_{yx} \Delta x \Delta z + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\Delta y\right)\Delta x \Delta z$$ tangential force

$$-\tau_{zx} \Delta x \Delta y + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\Delta z\right)\Delta x \Delta y$$ tangential force

6.4 Equations of Motion

Divide (B) by volume of element

$$\frac{\Delta F_x}{\Delta x \Delta y \Delta z} = \rho g_x + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \qquad \text{(C)}$$

RHS of (A):

$$\frac{\Delta m a_x}{\Delta x \Delta y \Delta z} = \rho a_x \qquad \text{(D)}$$

6.4 Equations of Motion

Combine (C) and (D)

$$\boxed{\begin{aligned}
&\rho g_x + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho a_x \\
&\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho a_y \\
&\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = \rho a_z
\end{aligned}} \qquad (6.21)$$

6.4 Equations of Motion

6.4.1 Navier-Stokes equations

– Eq (6.21) ~ general equation of motion containing 9 unknowns

– For Newtonian fluids (with single viscosity coeff.), use stress-strain relation given in (5.29) and (5.30) to reduce the number of unknowns

→ Navier-Stokes equations

Eq. (5.29):

$$\sigma_x = \underbrace{-p}_{\text{pressure}} + \underbrace{2\mu\frac{\partial u}{\partial x} - \left(\frac{2}{3}\right)\mu(\nabla\cdot\vec{q})}_{\text{normal stress due to fluid deformation and viscosity}}$$

6.4 Equations of Motion

$$\sigma_y = -p + 2\mu \frac{\partial v}{\partial y} - \left(\frac{2}{3}\right)\mu\left(\nabla \cdot \vec{q}\right) \tag{6.22}$$

$$\sigma_z = -p + 2\mu \frac{\partial w}{\partial z} - \left(\frac{2}{3}\right)\mu\left(\nabla \cdot \vec{q}\right) \tag{6.23}$$

Eq. (5.30):

$$\tau_{yx} = \tau_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

6.4 Equations of Motion

Substitute Eqs. (5.29) & (5.30) into (6.21)

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu(\nabla\cdot\vec{q})\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] = \rho a_x$$

Assume <u>constant viscosity</u> (neglect effect of pressure and temperature on viscosity variation)

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\frac{\partial}{\partial x}\left[2\frac{\partial u}{\partial x} - \frac{2}{3}(\nabla\cdot\vec{q})\right] + \mu\frac{\partial}{\partial y}\left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \mu\frac{\partial}{\partial z}\left[\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] = \rho a_x$$

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}}$$

6.4 Equations of Motion

Expand and simplify

normal stress

shear stress

$$L.H.S = \rho g_x - \frac{\partial p}{\partial x} + 2\mu \frac{\partial^2 u}{\partial x^2} - \frac{2}{3}\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z}\right) + \mu\left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z}\right)$$

$$= \rho g_x - \frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + \frac{1}{3}\mu\left(\underline{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z}}\right)$$

$$\boxed{\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)}$$

$$= \rho g_x - \frac{\partial p}{\partial x} + \boxed{\mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + \frac{1}{3}\mu \frac{\partial}{\partial x}(\nabla \cdot \vec{q})}$$

normal stress + shear stress

6.4 Equations of Motion

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + \frac{1}{3}\mu\frac{\partial}{\partial x}\left(\nabla\cdot\vec{q}\right) = \rho a_x$$

$$\rho g_y - \frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] + \frac{1}{3}\mu\frac{\partial}{\partial y}\left(\nabla\cdot\vec{q}\right) = \rho a_y$$

$$\rho g_z - \frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right] + \frac{1}{3}\mu\frac{\partial}{\partial z}\left(\nabla\cdot\vec{q}\right) = \rho a_z \qquad (6.24)$$

→ Navier-Stokes equation for compressible fluids with constant viscosity

6.4 Equations of Motion

◆ Vector form

$$\rho\vec{g} - \nabla p + \mu\nabla^2\vec{q} + \frac{\mu}{3}\nabla(\nabla\cdot\vec{q}) = \rho\frac{\partial\vec{q}}{\partial t} + \rho(\vec{q}\cdot\nabla)\vec{q}$$

where $\vec{a} = \frac{d\vec{q}}{dt} = \frac{\partial\vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q}$ --- Eq. (2.5)

1) For inviscid (ideal) fluid flow, $(\mu = 0) \rightarrow$ viscous forces are neglected.

$$\rho\vec{g} - \nabla p = \rho\frac{\partial\vec{q}}{\partial t} + \rho(\vec{q}\cdot\nabla)\vec{q}$$

→ Euler equations for ideal fluid

6.4 Equations of Motion

2) For incompressible fluids, $\nabla \cdot \vec{q} = 0$ (Continuity Eq.)

$$\boxed{\rho \vec{g} - \nabla p + \mu \nabla^2 \vec{q} = \rho \frac{\partial \vec{q}}{\partial t} + \rho \left(\vec{q} \cdot \nabla \right) \vec{q}} \qquad (6.25)$$

Define acceleration due to gravity as

$$\left.\begin{aligned} g_x &= -g \frac{\partial h}{\partial x} \\ g_y &= -g \frac{\partial h}{\partial y} \\ g_z &= -g \frac{\partial h}{\partial z} \end{aligned}\right\} \quad \vec{g} = -g \nabla h$$

6.4 Equations of Motion

where h = vertical direction measured positive upward

For Cartesian axes oriented so that h and z coincide

$$g_x = g_y = 0 \quad , \quad \frac{\partial h}{\partial z} = 1 \tag{6.26}$$

$$g_z = -g \tag{6.27}$$

→ minus sign indicates that acceleration due to gravity is in the negative h direction

Then, N-S equation for incompressible fluids and isothermal flows are

6.4 Equations of Motion

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-g\frac{\partial h}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right]$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-g\frac{\partial h}{\partial y}-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right]$$

$$\underbrace{\frac{\partial w}{\partial t}}_{\text{Local acceleration}}+\underbrace{u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}}_{\text{Convective acceleration}}=\underbrace{-g\frac{\partial h}{\partial z}}_{\text{Body force per mass}}\underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial z}}_{\text{Pressure force per mass}}+\underbrace{\frac{\mu}{\rho}\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right]}_{\text{Viscosity force per mass}} \qquad (6.28)$$

6.4 Equations of Motion

Eq. (6.28): 4 unknowns – u, v, w, p

→ We need one more equation to obtain a solution when the boundary conditions are specified.

→ Eq. of continuity for incompressible fluid

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

♦ Boundary conditions

1) kinematic BC: velocity normal to any rigid boundary (wall) equal the boundary velocity (velocity = 0 for stationary boundary)

2) physical BC: no slip condition (continuum stick to a rigid boundary)

→ tangential velocity relative to the wall vanish at the wall surface

6.4 Equations of Motion

♦ General solutions for Navier-Stocks equations are not available because of the nonlinear, 2nd-order nature of the partial differential equations.

→ Only particular solutions may be obtained by simplifications.

→ Numerical solutions are usually sought.

♦ Navier-Stocks equations in cylindrical coordinates for constant density and viscosity

r - component:

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{{v_\theta}^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)$$

$$=\rho g_r-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(rv_r\right)\right\}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right]$$

6.4 Equations of Motion

θ - component:

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}-\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)$$

$$=\rho g_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r v_\theta\right)\right\}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right]$$

z - component:

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)$$

$$=\rho g_z-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right] \qquad (6.29)$$

6.4 Equations of Motion

Continuity eq. for incompressible fluid

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta)+\frac{\partial}{\partial z}(v_z)=0 \qquad (6.30)$$

Normal & shear stresses for constant density and viscosity

$$\sigma_r=-p+2\mu\frac{\partial v_r}{\partial r}$$

$$\sigma_\theta=-p+2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}\right)$$

$$\sigma_z=-p+2\mu\frac{\partial v_z}{\partial z}$$

6.4 Equations of Motion

$$\tau_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} \right]$$

$$\tau_{\theta z} = \mu\left[\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta} \right]$$

$$\tau_{zr} = \mu\left[\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right]$$