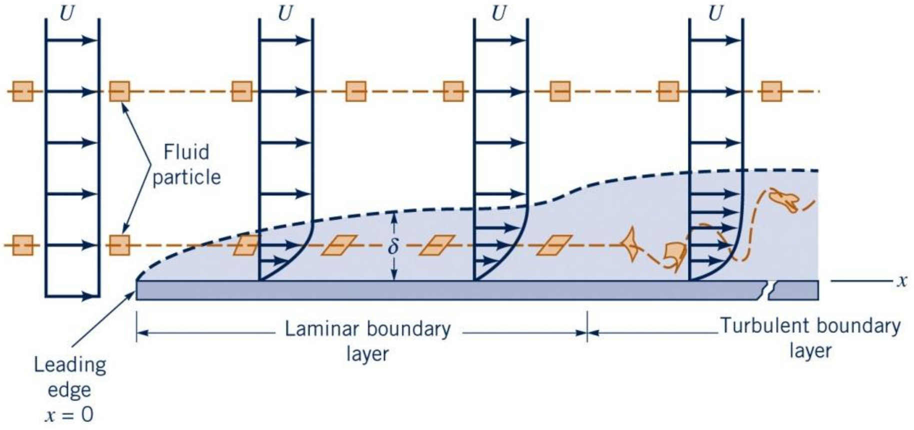
Lecture 17

Boundary Layer Flows (2)



Lecture 17 Boundary Layer Flows (2)

Contents

Objectives

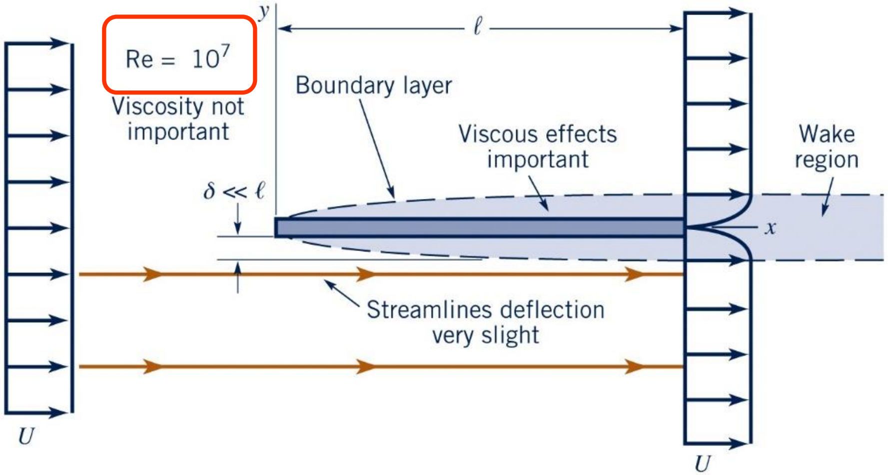
- Derive the equation for 2-D boundary layers and integral equation
- Study the solution for laminar boundary layer flows

17.1 Equation for 2-D Boundary Layers

▪ Boundary Layer Theory

• Prandtl (1904) suggested the boundary layer concept in which viscous effects are important only in the boundary layer regions, and outside boundary layer, the fluid acts as if it were inviscid.

A necessary condition for this structure of the flow is that the Reynolds number be large.



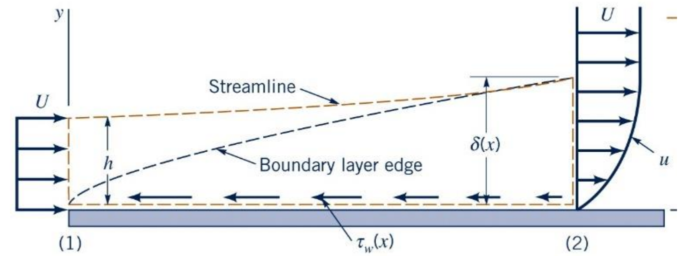
17.1 Equation for 2-D Boundary Layers

17.1.1 Boundary-layer thickness definitions

- In the boundary layer, the fluid change its velocity from the upstream value of *U* to zero on the solid surface (no slip).

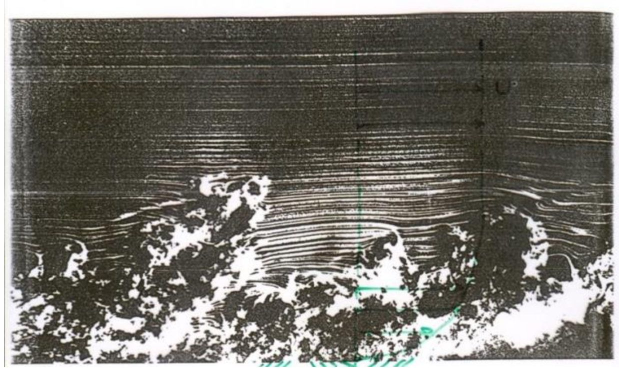
- The velocity profile is given as

$$u = u(x, y) \qquad (17.1)$$



- In actuality, there is no sharp edge to the boundary layer, that is, $u \rightarrow U$.

  → very intermittent

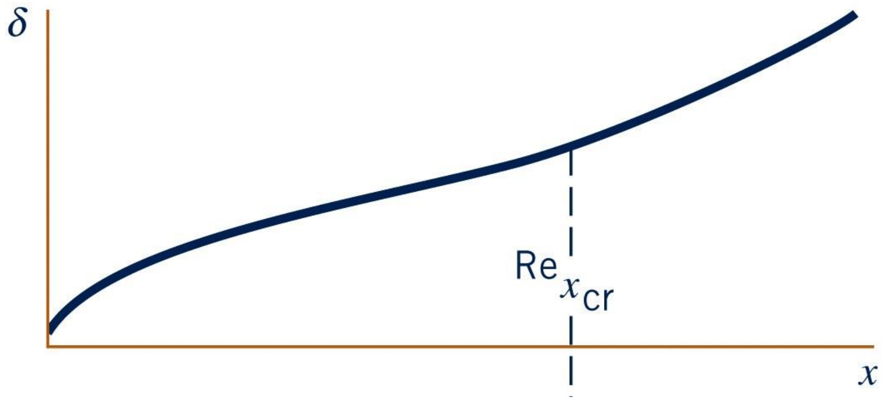


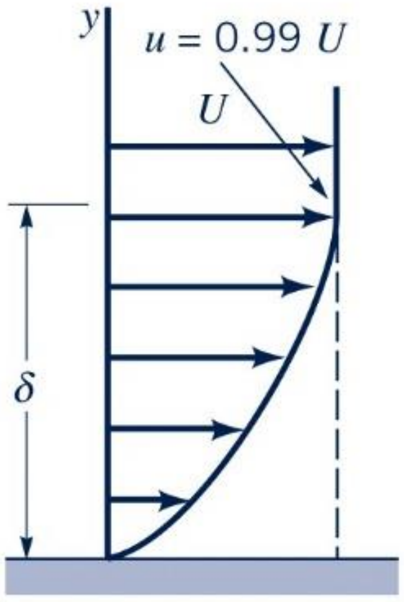
17.1 Equation for 2-D Boundary Layers

(1) Boundary-layer thickness, δ

- Define δ as the distance to the point where the velocity is within 1% of the free-stream velocity, U

$$@\ y=\delta \rightarrow u_{\delta}=0.99U \qquad (17.2)$$





17.1 Equation for 2-D Boundary Layers

(2) Mass displacement thickness, $\delta^* \, (\delta_1)$

~ is the thickness of an imaginary layer of fluid of velocity U.

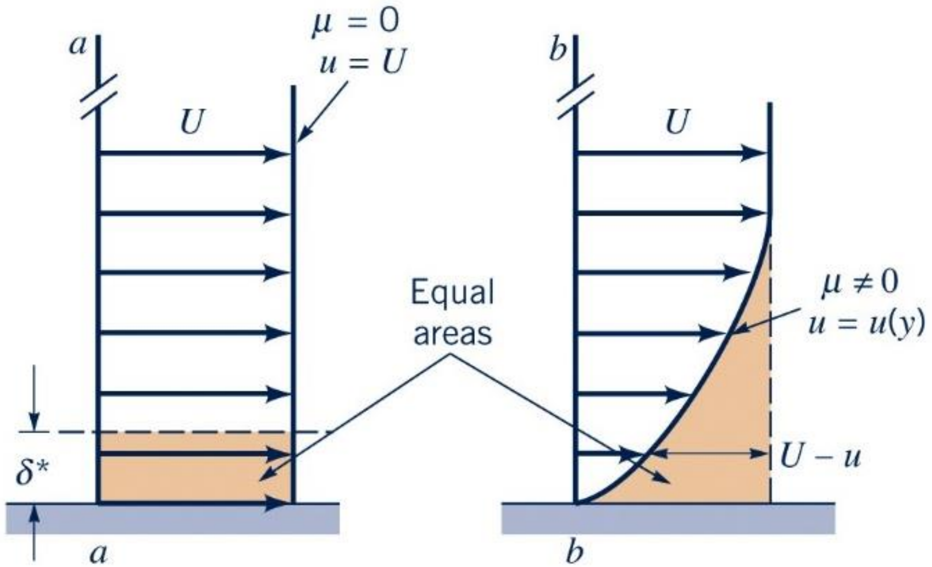
~ is the thickness of mass flux rate (flowrate) equal to the amount of defect

$$\rho U \delta^* = \underbrace{\rho \int_0^h (U-u)dy}_{mass\ defect} \qquad h \geq \delta$$

$$\delta^* = \int_0^h \left(1 - \frac{u}{U}\right)dy \qquad (17.3)$$

[Re] mass flux = mass/time

$$= \rho Q = \rho U A = \rho U \delta^* \times 1$$



17.1 Equation for 2-D Boundary Layers

(3) Momentum thickness, $\theta\,(\delta_2)$

- Velocity retardation within δ causes a reduction in the rate of momentum flux.

→ θ is the thickness of an imaginary layer of fluid of velocity U for which the momentum flux rate equals the reduction caused by the velocity profile.

$$\rho\theta U^2 = \rho\int_0^h (U-u)u\,dy = \rho\int_0^h (Uu-u^2)\,dy$$

$$\theta = \int_0^h \frac{u}{U}\left(1-\frac{u}{U}\right)dy \qquad (17.4)$$

17.1 Equation for 2-D Boundary Layers

[Re] momentum in θ = $mass \times velocity = \rho\theta U \times U = \rho\theta U^2$

momentum in shaded area = $\int \left[\rho(U-u) \times u \right] dy$

$$\boxed{\delta > \delta^* > \theta}$$

(4) Energy thickness, δ_3

$$\frac{1}{2}\rho U^3 \delta_3 = \frac{1}{2}\int_0^h \rho u(U^2 - u^2) dy$$

$$\boxed{\delta_3 = \int_0^h \frac{u}{U}(1 - \frac{u^2}{U^2}) dy} \qquad (17.5)$$

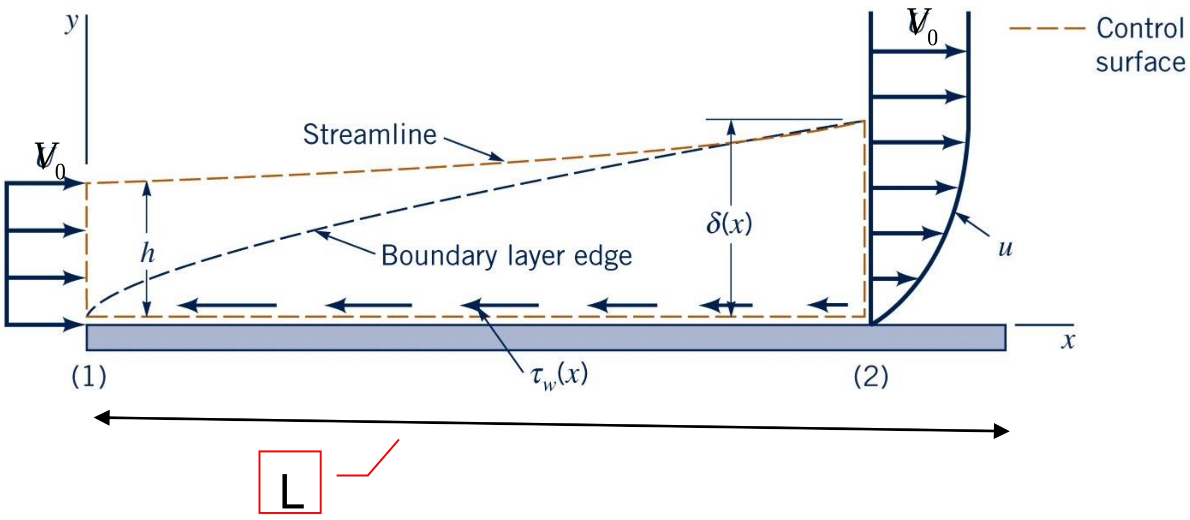
17.1 Equation for 2-D Boundary Layers

17.1.2 Equation for 2-D boundary layers

- Two-dimensional boundary layer equations by Prandtl

→ simplification of the N-S Eq. using order-of-magnitude arguments

- Start with 2D dimensionless N-S eq. for incompressible fluid with negligible gravitational effects



L, V_0 - constant reference values

17.1 Equation for 2-D Boundary Layers

Within thin and small curvature boundary layer

$$u \gg v, \qquad x \gg y$$

$$\frac{\partial u}{\partial y} \gg \frac{\partial u}{\partial x}$$

$$\frac{\partial p}{\partial y} \; is \; small \sim may \; be \; neglected$$

dimensionless boundary-layer thickness δ°

$$\delta^{\circ} = \frac{\delta}{L} \rightarrow \boxed{\delta^{\circ} \ll 1}$$

$\therefore$ scale for decreasing order

$$\frac{1}{\delta^{\circ 2}} > \frac{1}{\delta^{\circ}} > 1 > \delta^{\circ} > \delta^{\circ 2}$$

17.1 Equation for 2-D Boundary Layers

Order of magnitude

$$x^\circ \sim O\ (1)$$

$$y^\circ \sim O\ (\delta^\circ)$$

$$u^\circ \sim O\ (1)$$

$$v^\circ \sim O\ (\delta^\circ)$$

$$\frac{\partial u^\circ}{\partial x^\circ} \sim O\ (1)$$

$$\frac{\partial v^\circ}{\partial y^\circ} \sim O\ (1) \leftarrow continuity \left(\frac{\partial v^\circ}{\partial y^\circ} = -\frac{\partial u^\circ}{\partial x^\circ} \right)$$

$$\frac{\partial u^\circ}{\partial y^\circ} \sim O\ (\frac{1}{\delta^\circ})$$

$$\frac{\partial v^\circ}{\partial x^\circ} \sim O\ (\delta^\circ)$$

$$x^\circ = \frac{x}{L}$$

$$y^\circ = \frac{y}{L}$$

$$u^\circ = \frac{u}{V_0}$$

$$v^\circ = \frac{v}{V_0}$$

$$p^\circ = \frac{p}{\rho V_0^2}$$

17.1 Equation for 2-D Boundary Layers

$$\frac{\partial^2 u^\circ}{\partial (x^\circ)^2} = \frac{\partial}{\partial x^\circ}\left(\frac{\partial u^\circ}{\partial x^\circ}\right) \sim O(1)$$

$$\frac{\partial^2 v^\circ}{\partial (y^\circ)^2} = \frac{\partial}{\partial y^\circ}\left(\frac{\partial v^\circ}{\partial y^\circ}\right) \sim O(\frac{1}{\delta^\circ})$$

$$\frac{\partial u^\circ}{\partial t^\circ} = \frac{\partial u^\circ}{\partial x^\circ}\frac{\partial x^\circ}{\partial t^\circ} = u^\circ \frac{\partial u^\circ}{\partial x^\circ} \sim O(1)$$

$$\frac{\partial v^\circ}{\partial t^\circ} = \frac{\partial v^\circ}{\partial x^\circ}\frac{\partial x^\circ}{\partial t^\circ} = u^\circ \frac{\partial v^\circ}{\partial x^\circ} \sim O(\delta^\circ)$$

$$\mathrm{Re} = \frac{\rho u x}{\mu} \sim O(1/\delta^{\circ 2})$$

17.1 Equation for 2-D Boundary Layers

$$x:\frac{\partial u^\circ}{\partial t^\circ}+u^\circ\frac{\partial u^\circ}{\partial x^\circ}+v^\circ\frac{\partial u^\circ}{\partial y^\circ}=-\frac{\partial p^\circ}{\partial x^\circ}+\frac{1}{\mathrm{Re}}\left(\cancel{\frac{\partial^2 u^\circ}{\partial x^{\circ 2}}}+\frac{\partial^2 u^\circ}{\partial y^{\circ 2}}\right) \qquad (17.6)$$

$1 \qquad 1\mathrm{x}1 \qquad \delta^\circ\times 1/\delta^\circ \qquad \delta^{\circ 2}(1+1/\delta^{\circ 2})\rightarrow 1$

$$y:\cancel{\frac{\partial v^\circ}{\partial t^\circ}}+u^\circ\cancel{\frac{\partial v^\circ}{\partial x^\circ}}+v^\circ\cancel{\frac{\partial v^\circ}{\partial y^\circ}}=-\cancel{\frac{\partial p^\circ}{\partial y^\circ}}+\frac{1}{\mathrm{Re}}\left(\cancel{\frac{\partial^2 v^\circ}{\partial x^{\circ 2}}}+\cancel{\frac{\partial^2 v^\circ}{\partial y^{\circ 2}}}\right)$$

$\delta^\circ \qquad 1\times\delta^\circ \qquad \delta^\circ\times 1 \qquad \delta^{\circ 2}(\delta^\circ+1/\delta^\circ) \quad \rightarrow \quad \delta^\circ$

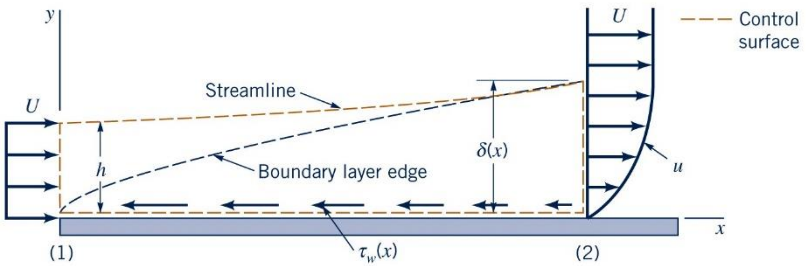
$$Continuity:\frac{\partial u^\circ}{\partial x^\circ}+\frac{\partial v^\circ}{\partial y^\circ}=0$$

$1 \qquad 1$

17.1 Equation for 2-D Boundary Layers

Therefore, eliminate all terms of order less than unity in Eq. (17.6) and revert to dimensional terms

$$\boxed{\begin{aligned}&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}\\&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\end{aligned}}$$



(17.7)

→ Prandtl's 2-D boundary-layer equation

$$BC: 1)\quad y=0\,;\, u=0, v=0$$

$$2)\quad y=\infty\,;\, u=U(x) \qquad (17.8)$$

Unknowns: *u, v, p*; Eqs. = 2 → needs assumptions for *p*

17.1 Equation for 2-D Boundary Layers

17.1.3 Integral momentum equation for 2-D boundary layers

- Prandtl's equation can be integrated to obtain the relation between boundary shear stress and velocity distribution for steady motion of incompressible fluid.

Assumptions:

constant density: $d\rho = 0$

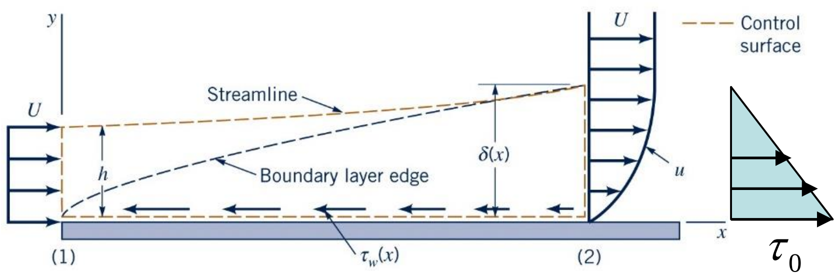
steady motion: $\dfrac{\partial()}{\partial t} = 0$

pressure gradient = 0: $\dfrac{\partial p}{\partial x} = 0$

BC's:

$@\, y = h\,;\quad \tau = 0,\quad u = U$

$@\, y = 0\,;\quad \tau = \tau_0,\quad u = 0$



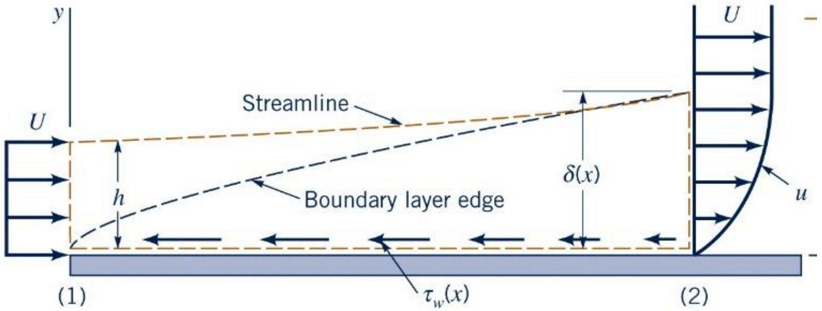
17.1 Equation for 2-D Boundary Layers

- Prandtl's 2-D boundary-layer equations become as follows:

$$\cancel{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\cancel{\frac{1}{\rho}\frac{\partial p}{\partial x}} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} \qquad (17.7)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} \qquad \text{(A)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{(B)}$$



Integrate Eq. (A) w.r.t. y

$$\int_{y=0}^{y=h\geq\delta}\Big(\underbrace{u\frac{\partial u}{\partial x}}_{①} + \underbrace{v\frac{\partial u}{\partial y}}_{②}\Big)dy = \underbrace{\frac{\mu}{\rho}\int_{y=0}^{y=h}\frac{\partial^2 u}{\partial y^2}\,dy}_{③} \qquad \text{(C)}$$

17.1 Equation for 2-D Boundary Layers

$$③ = \mu \int_0^h \frac{\partial^2 u}{\partial y^2} dy = \int_0^h \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) dy = \int_0^h \frac{\partial \tau}{\partial y} dy = [\tau]_0^h$$

$$= \tau\big|_{y=h} - \tau\big|_{y=0} = 0 - \tau_0 = -\tau_0$$

$$② = \int_0^h v \frac{\partial u}{\partial y} dy = \underbrace{\int_0^h \frac{\partial uv}{\partial y} dy}_{④} - \underbrace{\int_0^h u \frac{\partial v}{\partial y} dy}_{⑤} \qquad \text{(D)}$$

[Re] Integration by parts: $\int v u' dy = uv - \int u v' dy$

$$④ = \int_0^h \frac{\partial uv}{\partial y} dy = [uv]_0^h = Uv_h - 0 = Uv$$

17.1 Equation for 2-D Boundary Layers

Continuity Eq.: $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$ (i)

$$\rightarrow \quad v = -\int_0^h \frac{\partial u}{\partial x} dy \qquad \text{(ii)}$$

Substitute (i) into ⑤

$$⑤ = \int_0^h u\left(-\frac{\partial u}{\partial x}\right) dy = -\int_0^h u \frac{\partial u}{\partial x} dy \qquad \text{(iii)}$$

Substitute (ii) into ④

$$④ = Uv = -U\int_0^h \frac{\partial u}{\partial x} dy$$

17.1 Equation for 2-D Boundary Layers

Eq. (D) becomes

$$\int_0^h v\frac{\partial u}{\partial y}dy = -U\int_0^h \frac{\partial u}{\partial x}dy + \int_0^h u\frac{\partial u}{\partial x}dy \tag{E}$$

Then, (C) becomes

$$\int_0^h u\frac{\partial u}{\partial x}dy - U\int_0^h \frac{\partial u}{\partial x}dy + \int_0^h u\frac{\partial u}{\partial x}dy = -\frac{\tau_0}{\rho} \tag{F}$$

For steady motion with and U = const., (F) becomes

$$\frac{\tau_0}{\rho} = U\int_0^h \frac{\partial u}{\partial x}dy - 2\int_0^h u\frac{\partial u}{\partial x}dy = \int_0^h \frac{\partial Uu}{\partial x}dy - \int_0^h \frac{\partial u^2}{\partial x}dy$$

$$= \int_0^h \frac{\partial}{\partial x}[u(U-u)]dy = \frac{\partial}{\partial x}\underbrace{\int_0^h u(U-u)\,dy}_{\theta U^2} = \frac{\partial}{\partial x}(\theta U^2)$$

17.1 Equation for 2-D Boundary Layers

where θ = momentum thickness

$$\boxed{\frac{\tau_0}{\rho} = \frac{\partial}{\partial x}(U^2\theta) = U^2\frac{\partial\theta}{\partial x}} \qquad (17.9)$$

$$\frac{\partial\theta}{\partial x} = \frac{\tau_0}{\rho U^2} = \left(\frac{u^*}{U}\right)^2$$

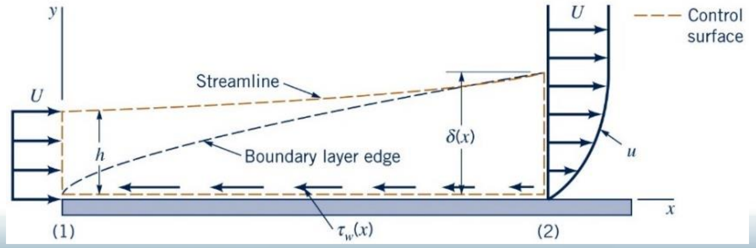
Introduce surface (frictional) resistance coefficient c_f

$$\tau_0 = \frac{\rho}{2}c_f U^2$$

$$c_f = \frac{\tau_0}{\frac{\rho}{2}U^2} \qquad (17.10)$$

Combine (17.9) with (17.10)

$$\boxed{c_f = 2\frac{\partial\theta}{\partial x}} \qquad (17.11)$$



17.1 Equation for 2-D Boundary Layers

- Integral momentum equation for unsteady motion

→ unsteady motion: $\dfrac{\partial ()}{\partial t} \neq 0$

→ pressure gradient, $\dfrac{\partial p}{\partial x} \neq 0$

First, simplify Eq. (17.7) for external flow where viscous influence is negligible.

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + v\cancel{\frac{\partial U}{\partial y}} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\cancel{\frac{\partial^2 U}{\partial y^2}}$$

$$\rho\frac{\partial U}{\partial t} + \rho U\frac{\partial U}{\partial x} = -\frac{\partial p}{\partial x} \qquad \text{(A)}$$

17.1 Equation for 2-D Boundary Layers

Substitute (A) into (17.7)

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\underbrace{\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}}_{\boxed{-\frac{1}{\rho}\frac{\partial p}{\partial x}}}+\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}$$

Integrate

$$\int_0^h \underbrace{\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}dy}_{①}=\int_0^h\left\{\underbrace{\frac{\partial u}{\partial t}-\frac{\partial U}{\partial t}}_{②}+\underbrace{u\frac{\partial u}{\partial x}-U\frac{\partial U}{\partial x}}_{③}+\underbrace{v\frac{\partial u}{\partial y}}_{④}\right\}dy \qquad \text{(B)}$$

$$① : \int_0^h \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}\,dy=-\frac{\tau_0}{\rho}$$

$$② : \int_0^h\left(\frac{\partial u}{\partial t}-\frac{\partial U}{\partial t}\right)dy=\int_0^h\frac{\partial}{\partial t}(u-U)dy=\frac{\partial}{\partial t}\underbrace{\int_0^h(u-U)dy}_{\boxed{-U\delta^*}}=-\frac{\partial}{\partial t}U\delta^*$$

17.1 Equation for 2-D Boundary Layers

$$③ = \underbrace{\int_0^h \left(u\frac{\partial u}{\partial x} - u\frac{\partial U}{\partial x} \right) dy}_{③\text{-}1} + \underbrace{\int \left(u\frac{\partial U}{\partial x} - U\frac{\partial U}{\partial x} \right) dy}_{③\text{-}2}$$

$$③\text{-}1 = \int_0^h \left\{ u\frac{\partial}{\partial x}(u-U) \right\} dy$$

$$③\text{-}2 = \int_0^h \left\{ (u-U)\frac{\partial U}{\partial x} \right\} dy = \frac{\partial U}{\partial x}\underline{\int_0^h (u-U)dy} = \frac{\partial U}{\partial x}(-U\delta^*)$$

$$④ = \int_0^h v\frac{\partial u}{\partial y}dy = -U\int_0^h \frac{\partial u}{\partial x}dy + \int u\frac{\partial u}{\partial x}dy = \int_0^h (u-U)\frac{\partial u}{\partial x}dy$$

$$\boxed{Eq.(E)}$$

17.1 Equation for 2-D Boundary Layers

Combine ③-1 and ④

$$\int_0^h u\frac{\partial}{\partial x}(u-U)dy+\int_0^h (u-U)\frac{\partial u}{\partial x}dy=\int_0^h \left[u\frac{\partial}{\partial x}(u-U)+(u-U)\frac{\partial u}{\partial x}\right]dy$$

$$=\int_0^h \frac{\partial}{\partial x}\{u(u-U)\}dy=\frac{\partial}{\partial x}\underline{\int_0^h u(u-U)dy}=\frac{\partial}{\partial x}(-\theta U^2)$$

Substituting all these into (B) yields

$$-\frac{\tau_0}{\rho}=-\frac{\partial}{\partial t}(U\delta^*)-U\frac{\partial U}{\partial x}\delta^*-\frac{\partial}{\partial x}(\theta U^2)$$

$$\frac{\tau_0}{\rho}=\frac{\partial}{\partial x}(U^2\theta)+U\frac{\partial U}{\partial x}\delta^*+\frac{\partial}{\partial t}(U\delta^*) \qquad (17.12)$$

→ Karman's integral momentum equation

17.2 Laminar Boundary Layers

17.2.1 The Blasius solution for laminar flow

- Blasius (1908) solved the Prandtl's boundary layer equations for steady laminar flow.
- Assume the pressure gradient $\frac{\partial p}{\partial x}$ is zero.
- Then, Eq. (17.7) will reduce to

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} \qquad (17.13a)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (17.13b)$$

17.2 Laminar Boundary Layers

- The boundary conditions are

At the wall

$$y = 0 : u = 0, v = 0 \qquad (17.14a)$$

Beyond the boundary layer

$$y = \infty : u = U \qquad (17.14b)$$

- Blasius obtained the solution by assuming similar velocity profiles along the plate at every *x*.

$$\frac{u}{U} = F\left(\frac{y}{\delta}\right) = F(\eta) \qquad (17.15)$$

where $F \ \eta$ is the same for all *x*, and $\eta = \dfrac{y}{x / \mathrm{Re}_x^{1/2}}$.

17.2 Laminar Boundary Layers

- Introducing a stream function makes Eq. (17-13a) into the ordinary differential equation

$$\psi = -\sqrt{\nu x U}\, f(\eta) \qquad (17.16a)$$

$$u = -\partial\psi \,/\, \partial y = Uf^{'} \qquad (17.16b)$$

$$v = \partial\psi \,/\, \partial x = \frac{1}{2}\sqrt{\nu U \,/\, x}\;\; \eta f^{'} - f \qquad (17.16c)$$

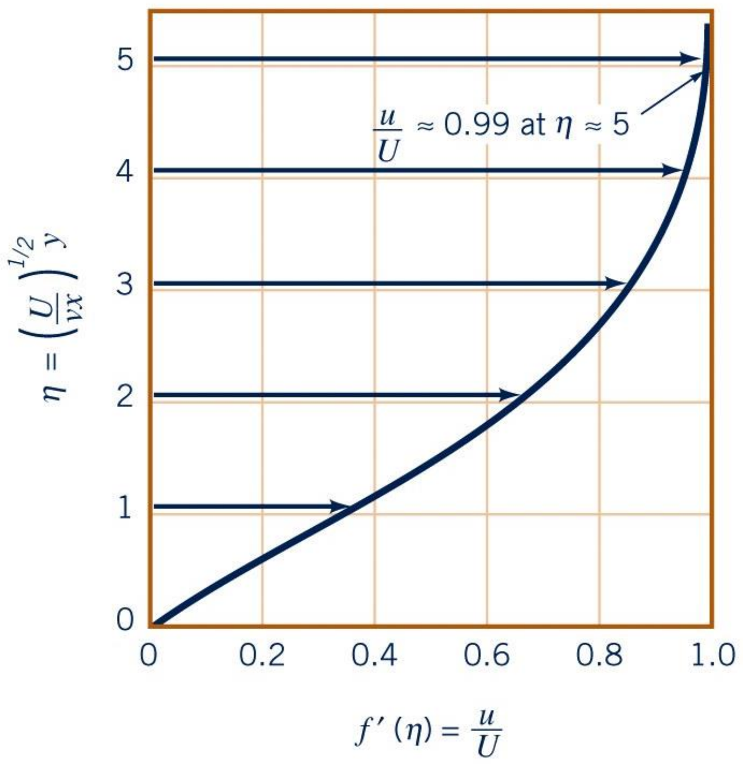
$$ff^{''} + f^{'''} = 0 \qquad (17.17)$$

- Blasius obtained the solution of Eq. (17.17) in the <u>form of a power series</u> expanded about $\eta = 0$.

17.2 Laminar Boundary Layers

Laminar Flow along a Flat Plate (the Blasius Solution)

$\eta = y(U/\nu x)^{1/2}$	$f'(\eta) = u/U$	η	$f'(\eta)$
0	0	3.6	0.9233
0.4	0.1328	4.0	0.9555
0.8	0.2647	4.4	0.9759
1.2	0.3938	4.8	0.9878
1.6	0.5168	5.0	0.9916
2.0	0.6298	5.2	0.9943
2.4	0.7290	5.6	0.9975
2.8	0.8115	6.0	0.9990
3.2	0.8761	∞	1.0000



17.2 Laminar Boundary Layers

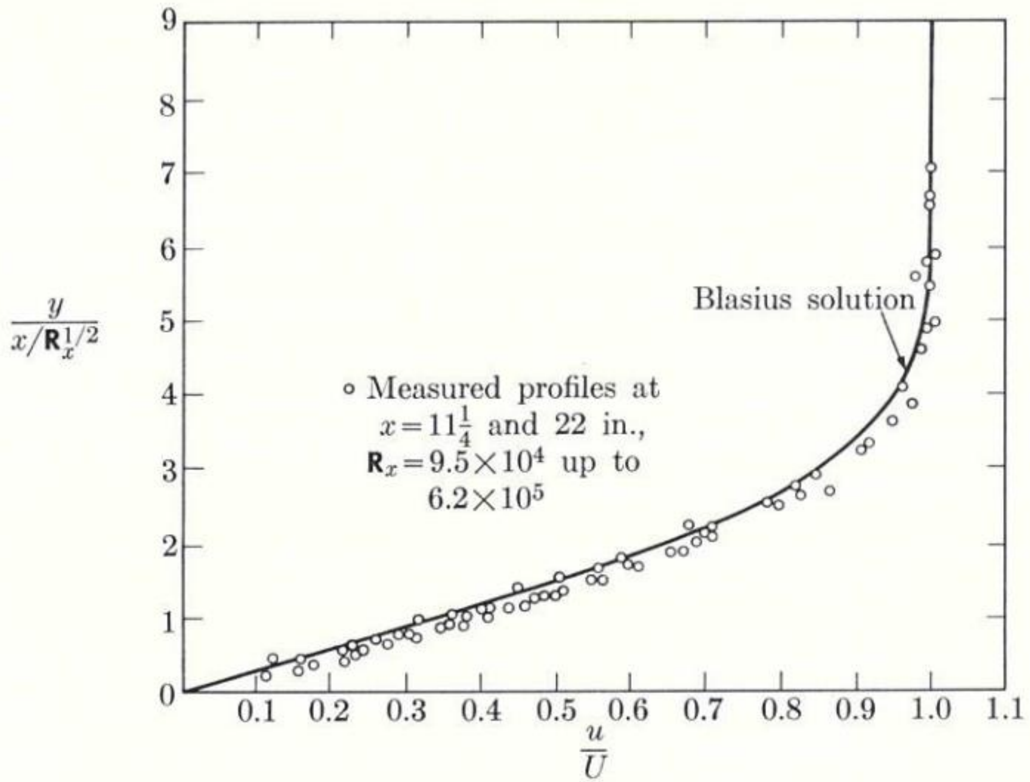


FIG. 10–4. Velocity profiles in a laminar boundary layer on a flat plate [2].

17.2 Laminar Boundary Layers

From the solution, it is found that

$$\frac{u}{U} = 0.992 \text{ when } \eta = 5 = \frac{y}{x / \mathrm{Re}_x^{1/2}}$$

Thus, we have the thickness of the boundary layer as

$$y = \delta_{lam} = \frac{5x}{\mathrm{Re}_x^{1/2}} \qquad (17.18)$$

Then, based on definition of the mass and momentum thicknesses, we have

$$\delta^*_{lam} = \frac{1.73x}{\mathrm{Re}_x^{1/2}} \qquad (17.19)$$

$$\theta_{lam} = \frac{0.664x}{\mathrm{Re}_x^{1/2}} \qquad (17.20)$$

17.2 Laminar Boundary Layers

The local wall shear stress is given by the value of the velocity gradient at the wall

$$\tau_0 = \mu \left(\frac{\partial u}{\partial y} \right)_{y=0} = 0.332 \mu Re_x^{1/2} \frac{U}{x} \qquad (17.21)$$

Thus, the local wall shear stress coefficient is given as

$$c_f = \frac{\tau_0}{\frac{\rho}{2} U^2} = \frac{0.664}{(Re_x)^{\frac{1}{2}}} \qquad (17.22)$$

$$c_f = \frac{3.32}{Re_\delta} \qquad (17.23)$$

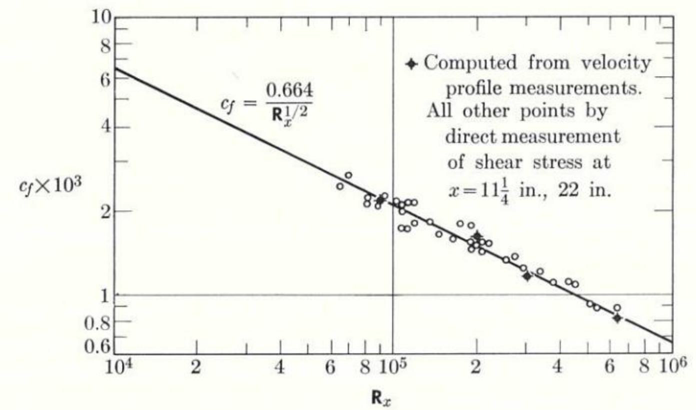


FIG. 10–5. Local shear stress coefficients for laminar boundary layer on flat plate [3].

17.2 Laminar Boundary Layers

Drag for one side of a plate of width b is computed to be

$$D = b\int_0^l \tau_0(x)dx = \frac{1.328}{\mathrm{Re}_l^{1/2}}\rho\frac{U^2}{2}bl \qquad (17.24)$$

Using the relation

$$D = C_f\frac{\rho}{2}U^2bl \qquad (17.25)$$

We get the coefficient of surface resistance as

$$C_f = \frac{1.328}{Re_l^{1/2}} \qquad (17.26)$$