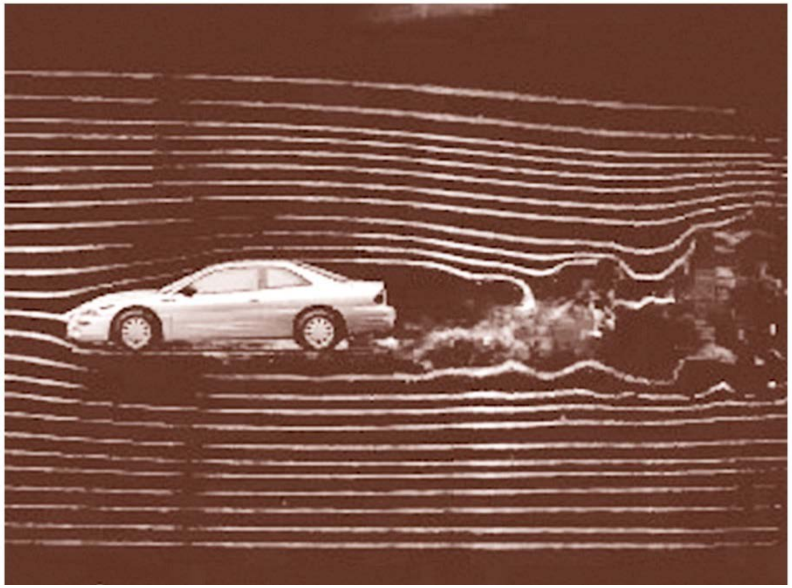
Chapter 8

Origin of Turbulence and Turbulent Shear Stress



Chapter 8 Origin of Turbulence and Turbulent Shear Stress

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8.4 Turbulent Shear Stress and Eddy Viscosity

8.4.1 Fall of pressure drop due to shear stress

shear stress = resistance to motion

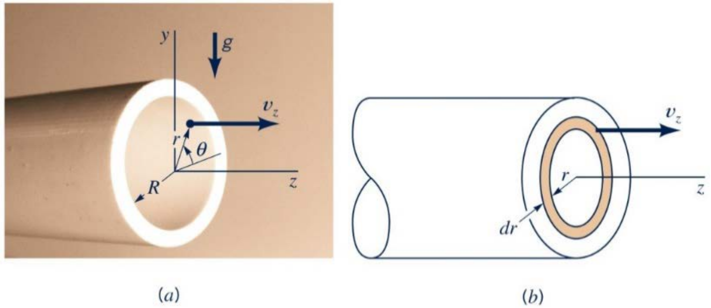
→ dissipate flow energy → fall of pressure drop along a pipe → head loss

($h_L = \dfrac{\tau_0 l}{\gamma R_h}$)

laminar flow; $\dfrac{d(p+\gamma h)}{dz} \propto V_z$

turbulent flow: $\dfrac{d(p+\gamma h)}{dz} \propto V_z^n$ ($n \approx 2$)

where V_z = average velocity



8.4 Turbulent Shear Stress and Eddy Viscosity

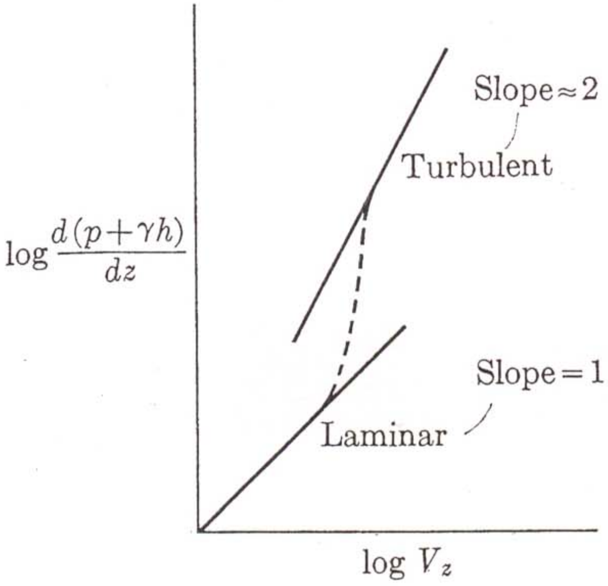


FIG. 11–6. Pressure gradient with laminar and turbulent flow in a conduit.

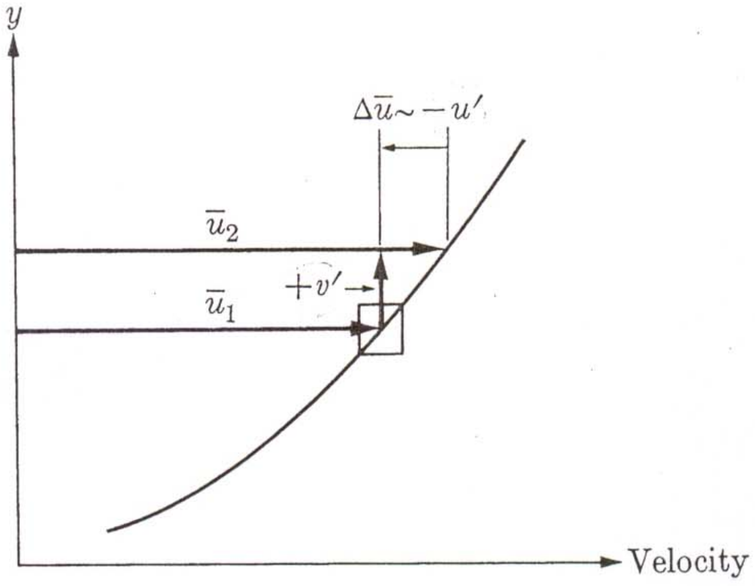


FIG. 11–7. Momentum transport by turbulent velocity fluctuation.

8.4 Turbulent Shear Stress and Eddy Viscosity

8.4.2 Shear stress resisting to motion

(1) Boussinesq's eddy viscosity concept

$$\tau_{total} = \underbrace{\mu \frac{d\bar{u}}{dy}}_{\text{laminar flow}} + \underbrace{\eta \frac{d\bar{u}}{dy}}_{\text{turbulent flow}} \qquad (8.10)$$

where

$\bar{u}$ = mean local velocity (time - averaged)

μ = dynamic molecular viscosity → property of the fluid

8.4 Turbulent Shear Stress and Eddy Viscosity

η = dynamic eddy viscosity that depends on the state of the turbulent motion ← turbulent intensity

($\varepsilon = \dfrac{\eta}{\rho}$ = kinematic eddy viscosity)

$\mu \dfrac{d\bar{u}}{dy}$ - apparent stress computed from the velocity gradient of mean motion.

$\eta \dfrac{d\bar{u}}{dy}$ - additional apparent stress associated with the turbulence

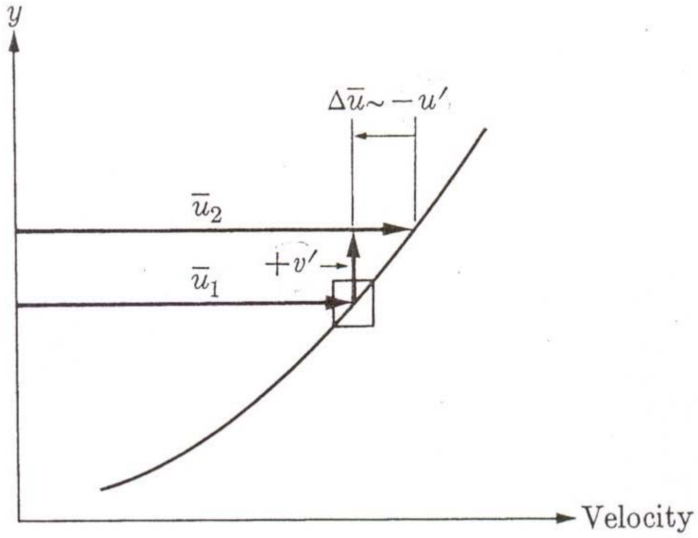
For laminar flow, $\eta = 0$

For turbulent flow, $\eta \gg \mu \quad \rightarrow \quad \tau_{turb} > \tau_{lam}$

8.4 Turbulent Shear Stress and Eddy Viscosity

(2) Physical model of momentum transport (exchange)

~ momentum transport by turbulent velocity fluctuation (Ch. 3)



Step 1: lower-velocity fluid parcel in layer 1 fluctuates with a v'-velocity into layer 2

8.4 Turbulent Shear Stress and Eddy Viscosity

Step 2: its velocity in the direction of the stream is less than mean velocity of the layer 2 by an amount -u'

Step 3: drag of the faster moving surroundings accelerates the fluid element and increases its momentum

Step 4: The mass flux $\left(\frac{mass}{time \cdot area}\right)$ crossing from layer 1 to layer 2

$$= \rho v'$$

Step 5: Flow-direction momentum change = mass flux × velocity

$$= \rho v' \times (-u') = -\rho u' v'$$

Step 6: Average over a time period

$$= -\rho \overline{u' v'}$$

= effective resistance to motion

= effective shearing stress

8.4 Turbulent Shear Stress and Eddy Viscosity

(3) Reynolds stress

$$= -\rho\overline{u'v'} \qquad (8.11)$$

= time rate change of momentum per unit area

= effective resistance to motion

~ actually acceleration terms

~ instantaneous viscous stresses due to turbulent motion = $\eta\frac{d\bar{u}}{dy}$

$$\tau_{total} = \mu\left(\frac{d\bar{u}}{dy}\right) - \rho\overline{u'v'} = \tau_{yx} \qquad (8.12)$$

↑	↖
shear stress due to transverse molecular momentum transport	shear stress due to transverse momentum transport of macroscopic fluid particles by turbulent motion

8.4 Turbulent Shear Stress and Eddy Viscosity

For fully developed turbulence,

$$\boxed{\tau_{yx} \cong \eta\left(\frac{d\bar{u}}{dy}\right) \approx -\rho\overline{u'v'} \propto V_z^2} \qquad (8.13)$$

[Re] $\boxed{\text{Reynolds stress} = -\rho\overline{u'v'}}$

~ If u' and v' are uncorrelated, there would be no turbulent momentum transport.

~ usually not zero (correlated)

~ may exchange momentum of mean motion

~ exchanges momentum between turbulence and mean flow

8.4 Turbulent Shear Stress and Eddy Viscosity

[Re] Effective addition to the normal pressure intensity acting in the flow direction

$$= -\overline{\rho u'u'} = -\overline{\rho u'^2} \tag{8.14}$$

[Re] Momentum transport

Eq. (3.2): $$\boxed{\frac{d(\Delta m \mathrm{v})}{dt}\frac{1}{area}} = K\frac{d}{dy}\left(\frac{\Delta m \mathrm{v}}{vol}\right)$$

Newton's 2nd law of motion

$$F = ma = m\frac{d\mathrm{v}}{dt} = \frac{d(m\mathrm{v})}{dt} \tag{A}$$

$$\frac{F}{area} = \frac{d(m\mathrm{v})}{dt}\frac{1}{area} \tag{B}$$

8.4 Turbulent Shear Stress and Eddy Viscosity

Assume only shear stresses exist,

Then LHS of (B) = τ

Combine (3. 2) & (B)

$$\tau = \eta \frac{d\bar{u}}{dy} \qquad \text{(C)}$$

By the way, for the turbulent motion

RHS of (B) = time rate change of momentum per unit area = $-\rho\overline{u'v'}$

$$\therefore \; -\rho\overline{u'v'} = \tau \qquad \text{(D)}$$

Combine (C) and (D)

$$\tau_t = -\rho\overline{u'v'} = \eta \frac{d\bar{u}}{dy} \qquad \text{(E)}$$

8.5 Reynolds Equations for Incompressible Fluids

8.5.1 Reynolds Equation

Navier-Stokes Eq. = equations of motion of a viscous fluid

~ applicable to both turbulent and non-turbulent flows

~ very difficult to obtain exact solution because of complexity of turbulence

~ Alternative is to consider the pattern of the mean turbulent motion even through we cannot establish the true details of fluctuations.

→ average Navier-Stokes Eq. over time to derive Reynolds Eq.

8.5 Reynolds Equations for Incompressible Fluids

N-S Eq. in x-dir.:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z} \qquad (8.15)$$

Continuity Eq. for incompressible fluid:

$$\rho\left(u\frac{\partial u}{\partial x}+u\frac{\partial v}{\partial y}+u\frac{\partial w}{\partial z}\right)=0 \qquad (A)$$

Add (A) to (8.15), then LHS becomes

$$\text{LHS} = \frac{\partial u}{\partial t}+2u\frac{\partial u}{\partial x}+\left(v\frac{\partial u}{\partial y}+u\frac{\partial v}{\partial y}\right)+\left(w\frac{\partial u}{\partial z}+u\frac{\partial w}{\partial z}\right)=\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}+\frac{\partial uw}{\partial z}$$

8.5 Reynolds Equations for Incompressible Fluids

Whole equation is

$$\rho\left(\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}+\frac{\partial uw}{\partial z}\right)=\rho g_x-\frac{\partial p_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z} \qquad (8.16)$$

Decomposition:

$$\begin{aligned} u &= \overline{u}+u' \\ v &= \overline{v}+v' \\ w &= \overline{w}+w' \\ p_x &= \overline{p_x}+p_x' \end{aligned} \qquad (8.17)$$

Substitute (8.17) into (8.16), and average over time

$$\rho\left\{\frac{\overline{\partial(\overline{u}+u')}}{\partial t}+\frac{\overline{\partial(\overline{u}+u')^2}}{\partial x}+\frac{\overline{\partial(\overline{u}+u')(\overline{v}+v')}}{\partial y}+\frac{\overline{\partial(\overline{u}+u')(\overline{w}+w')}}{\partial z}\right\}$$

$$=\rho g_x-\frac{\overline{\partial(\overline{p_x}+p_x')}}{\partial x}+\frac{\overline{\partial\tau_{yx}}}{\partial y}+\frac{\overline{\partial\tau_{zx}}}{\partial z}$$

8.5 Reynolds Equations for Incompressible Fluids

Rearrange according to the Reynolds average rule

$$\frac{\overline{\partial(\bar{u}+u')}}{\partial t} = \frac{\partial \bar{u}}{\partial t} + \cancel{\frac{\partial \overline{u'}}{\partial t}} = \frac{\partial \bar{u}}{\partial t}$$

$$\frac{\overline{\partial(\bar{u}+u')^2}}{\partial x} = \frac{\partial}{\partial x}\overline{(\bar{u}^2 + \cancel{2\bar{u}u'} + u'^2)} = \frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \overline{u'^2}}{\partial x}$$

$$\frac{\overline{\partial(\bar{u}+u')(\bar{v}+v')}}{\partial y} = \frac{\partial}{\partial y}\overline{(\bar{u}\,\bar{v} + \cancel{\bar{u}v'} + \cancel{u'\bar{v}} + u'v')} = \frac{\partial \bar{u}\,\bar{v}}{\partial y} + \frac{\partial \overline{u'v'}}{\partial y}$$

$$\frac{\overline{\partial(\bar{u}+u')(\bar{w}+w')}}{\partial z} = \frac{\partial}{\partial z}\overline{(\bar{u}\,\bar{w} + \cancel{\bar{u}w'} + \cancel{u'\bar{w}} + u'w')} = \frac{\partial \bar{u}\,\bar{w}}{\partial z} + \frac{\partial \overline{u'w'}}{\partial z}$$

$$\therefore \rho\left(\frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \bar{u}\,\bar{v}}{\partial y} + \frac{\partial \bar{u}\,\bar{w}}{\partial z}\right) = \rho g_x - \frac{\partial \bar{p}_x}{\partial x} + \frac{\partial \bar{\tau}_{yx}}{\partial y} + \frac{\partial \bar{\tau}_{zx}}{\partial z}$$

$$-\rho\left(\frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z}\right)$$

8.5 Reynolds Equations for Incompressible Fluids

Subtract Continuity Eq. of mean motion ($\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{u}\frac{\partial \bar{v}}{\partial y}+\bar{u}\frac{\partial \bar{w}}{\partial z}=0$)

$$\rho\left(\frac{\partial \bar{u}}{\partial t}+\frac{\partial \bar{u}^2}{\partial x}+\frac{\partial \bar{u}\,\bar{v}}{\partial y}+\frac{\partial \bar{u}\,\bar{w}}{\partial z}\right)-\left(\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{u}\frac{\partial \bar{v}}{\partial y}+\bar{u}\frac{\partial \bar{w}}{\partial z}\right)$$

$$=\rho g_x-\frac{\partial \bar{p}_x}{\partial x}+\frac{\partial \bar{\tau}_{yx}}{\partial y}+\frac{\partial \bar{\tau}_{zx}}{\partial z}-\rho\left(\frac{\partial \overline{u'^2}}{\partial x}+\frac{\partial \overline{u'v'}}{\partial y}+\frac{\partial \overline{u'w'}}{\partial z}\right)$$

$$\therefore \rho\left(\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right)$$

$$=\rho g_x-\frac{\partial \bar{p}_x}{\partial x}+\frac{\partial \bar{\tau}_{yx}}{\partial y}+\frac{\partial \bar{\tau}_{zx}}{\partial z}-\rho\underline{\left(\frac{\partial \overline{u'^2}}{\partial x}+\frac{\partial \overline{u'v'}}{\partial y}+\frac{\partial \overline{u'w'}}{\partial z}\right)} \qquad (8.18)$$

- turbulence acceleration terms
- mean transport of fluctuating momentum by turbulent velocity fluctuations

8.5 Reynolds Equations for Incompressible Fluids

y-direction:

$$\rho\left(\frac{\partial\bar{v}}{\partial t}+\bar{u}\frac{\partial\bar{v}}{\partial x}+\bar{v}\frac{\partial\bar{v}}{\partial y}+\bar{w}\frac{\partial\bar{v}}{\partial z}\right)=\rho g_y+\frac{\partial\bar{\tau}_{xy}}{\partial x}-\frac{\partial\bar{p}_y}{\partial y}+\frac{\partial\bar{\tau}_{zy}}{\partial z}$$

$$\underline{-\rho\left(\frac{\partial\overline{v'u'}}{\partial x}+\frac{\partial\overline{v'^2}}{\partial y}+\frac{\partial\overline{v'w'}}{\partial z}\right)}$$

z-direction:

$$\rho\left(\frac{\partial\bar{w}}{\partial t}+\bar{u}\frac{\partial\bar{w}}{\partial x}+\bar{v}\frac{\partial\bar{w}}{\partial y}+\bar{w}\frac{\partial\bar{w}}{\partial z}\right)=\rho g_z+\frac{\partial\bar{\tau}_{xz}}{\partial x}+\frac{\partial\bar{\tau}_{yz}}{\partial y}-\frac{\partial\bar{p}_z}{\partial z}$$

$$\underline{-\rho\left(\frac{\partial\overline{w'u'}}{\partial x}+\frac{\partial\overline{w'v'}}{\partial y}+\frac{\partial\overline{w'^2}}{\partial z}\right)}$$

8.5 Reynolds Equations for Incompressible Fluids

Rearrange (8.18)

$$\rho\left(\frac{\partial\bar{u}}{\partial t}+\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}+\bar{w}\frac{\partial\bar{u}}{\partial z}\right)$$

$$=\rho g_x+\frac{\partial}{\partial x}\underline{\left(-\bar{p}_x-\rho\overline{u'^2}\right)}+\frac{\partial}{\partial y}\underline{(\bar{\tau}_{yx}-\rho\overline{u'v'})}+\frac{\partial}{\partial z}\underline{(\bar{\tau}_{zx}-\rho\overline{u'w'})}$$

Sum of apparent stress of the mean motion and additional apparent stress due to turbulent fluctuations

8.5 Reynolds Equations for Incompressible Fluids

Introduce Newtonian stress relations: Eqs. 5.29 & 5.30

$$\sigma_x = -p + 2\mu \frac{\partial u}{\partial x} - \frac{2}{3}\mu \nabla \cdot \vec{q}$$

$$\sigma_y = -p + 2\mu \frac{\partial v}{\partial y} - \frac{2}{3}\mu \nabla \cdot \vec{q}$$

$$\sigma_z = -p + 2\mu \frac{\partial w}{\partial z} - \frac{2}{3}\mu \nabla \cdot \vec{q}$$

$$\tau_{yx} = \tau_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

Substitute velocity decomposition, Eqs (8.17) into Eqs. (5.29) &

(5.30) and average over time for incompressible fluid ($\nabla \cdot \vec{q} = 0$)

8.5 Reynolds Equations for Incompressible Fluids

1) x-direction:

$$\bar{\sigma}_x = -\bar{p}_x = -\overline{(\bar{p} \cancel{+ p'})} + 2\mu \frac{\partial \overline{(\bar{u} \cancel{+ u'})}}{\partial x} = -\bar{p} + 2\mu \frac{\partial \bar{u}}{\partial x}$$

$$\bar{\tau}_{yx} = \mu \left\{ \frac{\partial \overline{(\bar{v} \cancel{+ v'})}}{\partial x} + \frac{\partial \overline{(\bar{u} \cancel{+ u'})}}{\partial y} \right\} = \mu \left(\frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial y} \right)$$

$$\bar{\tau}_{zx} = \mu \left\{ \frac{\partial \overline{(\bar{u} \cancel{+ u'})}}{\partial z} + \frac{\partial \overline{(\bar{w} \cancel{+ w'})}}{\partial x} \right\} = \mu \left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x} \right)$$

(8.20 a)

(2) y-direction:

$$-\bar{p}_y = -\bar{p} + 2\mu \frac{\partial \bar{v}}{\partial y}$$

$$\bar{\tau}_{xy} = \mu \left(\frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial y} \right)$$

$$\bar{\tau}_{zy} = \mu \left(\frac{\partial \bar{w}}{\partial y} + \frac{\partial \bar{v}}{\partial z} \right)$$

(8.20 b)

8.5 Reynolds Equations for Incompressible Fluids

(3) z-direction:

$$-\bar{p}_z = -\bar{p} + 2\mu \frac{\partial \bar{w}}{\partial z}$$

$$\bar{\tau}_{xz} = \mu \left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x} \right)$$

$$\bar{\tau}_{yz} = \mu \left(\frac{\partial \bar{w}}{\partial y} + \frac{\partial \bar{v}}{\partial z} \right) \qquad (8.20\ c)$$

Substitute Eq. (8.20) into Eq. (8.18)

$$\rho \left(\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} + \bar{w} \frac{\partial \bar{u}}{\partial z} \right)$$

$$= \rho g_x - \frac{\partial}{\partial x} \left(\bar{p} - 2\mu \frac{\partial \bar{u}}{\partial x} \right) + \frac{\partial}{\partial y} \left(\mu \frac{\partial \bar{v}}{\partial x} + \mu \frac{\partial \bar{u}}{\partial y} \right) + \frac{\partial}{\partial z} \left(\mu \frac{\partial \bar{u}}{\partial z} + \mu \frac{\partial \bar{w}}{\partial x} \right)$$

$$- \rho \left(\frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z} \right)$$

8.5 Reynolds Equations for Incompressible Fluids

$$= \rho g_x - \frac{\partial \overline{p}}{\partial x} + \mu\left(2\frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{v}}{\partial y \partial x} + \frac{\partial^2 \overline{u}}{\partial y^2} + \frac{\partial^2 \overline{u}}{\partial z^2} + \frac{\partial^2 \overline{w}}{\partial z \partial x}\right)$$

$$- \rho\left(\frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z}\right)$$

$$= \rho g_x - \frac{\partial \overline{p}}{\partial x} + \underbrace{\mu\left(\frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{u}}{\partial y^2} + \frac{\partial^2 \overline{u}}{\partial z^2}\right)}_{= \mu \nabla^2 \overline{u}} + \underbrace{\mu\left(\frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{v}}{\partial y \partial x} + \frac{\partial^2 \overline{w}}{\partial z \partial x}\right)}_{(\mathrm{I})}$$

$$- \rho\left(\frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z}\right)$$

By the way,

(I) = $\frac{\partial}{\partial x}\left(\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z}\right) = 0$ ($\because$ Continuity Eq. for incompressible fluid)

Therefore, substituting this relation yields

8.5 Reynolds Equations for Incompressible Fluids

x-dir.:

$$\rho\left(\frac{\partial\overline{u}}{\partial t}+\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{v}\frac{\partial\overline{u}}{\partial y}+\overline{w}\frac{\partial\overline{u}}{\partial z}\right)$$

$$=\rho g_x-\frac{\partial\overline{p}}{\partial x}+\mu\nabla^2\overline{u}-\rho\left(\frac{\partial\overline{u'^2}}{\partial x}+\frac{\partial\overline{u'v'}}{\partial y}+\frac{\partial\overline{u'w'}}{\partial z}\right) \qquad (8.22\text{ a})$$

y-dir.:

$$\rho\left(\frac{\partial\overline{v}}{\partial t}+\overline{u}\frac{\partial\overline{v}}{\partial x}+\overline{v}\frac{\partial\overline{v}}{\partial y}+\overline{w}\frac{\partial\overline{v}}{\partial z}\right)$$

$$=\rho g_y-\frac{\partial\overline{p}}{\partial y}+\mu\nabla^2\overline{v}-\rho\left(\frac{\partial\overline{v'u'}}{\partial x}+\frac{\partial\overline{v'^2}}{\partial y}+\frac{\partial\overline{v'w'}}{\partial z}\right) \qquad (8.22\text{ b})$$

z-dir.:

$$\rho\left(\frac{\partial\overline{w}}{\partial t}+\overline{u}\frac{\partial\overline{w}}{\partial x}+\overline{v}\frac{\partial\overline{w}}{\partial y}+\overline{w}\frac{\partial\overline{w}}{\partial z}\right)$$

$$=\rho g_z-\frac{\partial\overline{p}}{\partial z}+\mu\nabla^2\overline{w}-\rho\left(\frac{\partial\overline{w'u'}}{\partial x}+\frac{\partial\overline{w'v'}}{\partial y}+\frac{\partial\overline{w'^2}}{\partial z}\right) \qquad (8.22\text{ c})$$

8.5 Reynolds Equations for Incompressible Fluids

[Re]

1) Reynolds Equation of motion → solve for mean motion

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial}{\partial x_j}(\overline{u'_i u'_j}) = \bar{X}_i - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{\mu}{\rho}\frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j}$$

time rate of change of momentum

rate of convection of the momentum

rate of diffusion of momentum by turbulence

body force

force due to mean pressure

rate of molecular diffusion of momentum by viscosity

2) Navier-Stokes Eq. → apply to instantaneous motion

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = X_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\mu}{\rho}\frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

8.5 Reynolds Equations for Incompressible Fluids

- Reynolds Equations (temporal mean eq. of motion)

→ Navier-Stokes form for incompressible fluid (RANS)

[Re] No. of Equations = 4

No. of Unknowns: 4 + 9 (turbulence fluctuating terms)

→ 9 products of one-point double correlation of velocity fluctuation

$(\overline{u'_i u'_j})$

8.5 Reynolds Equations for Incompressible Fluids

8.5.2 Closure Model

Assumptions are needed to close the gap between No. of equations and No. unknowns.

→ Turbulence modeling: Ch. 10

■ Boussinesq's eddy viscosity model – the simplest model

$$-\overline{u'^2} = \varepsilon_x \frac{\partial \overline{u}}{\partial x}$$

$$-\overline{u'v'} = \varepsilon_y \frac{\partial \overline{u}}{\partial y}$$

$$\boxed{-\overline{u'v'} \propto \frac{\partial \overline{u}}{\partial y}}$$

$$-\overline{u'w'} = \varepsilon_z \frac{\partial \overline{u}}{\partial z} \qquad \text{(A)}$$

8.5 Reynolds Equations for Incompressible Fluids

Reynolds Equation in *x*-dir.:

$$\rho\left(\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right)$$

$$=\rho g_x-\frac{\partial \bar{p}}{\partial x}+\mu\left(\frac{\partial^2 \bar{u}}{\partial x^2}+\frac{\partial^2 \bar{u}}{\partial y^2}+\frac{\overline{\partial^2 u}}{\partial z^2}\right)-\rho\left(\frac{\partial \overline{u'^2}}{\partial x}+\frac{\partial \overline{u'v'}}{\partial y}+\frac{\partial \overline{u'w'}}{\partial z}\right)$$

$$=\rho g_x-\frac{\partial \bar{p}}{\partial x}+\rho\underline{\left[\frac{\partial}{\partial x}\left(\frac{\mu}{\rho}\frac{\partial \bar{u}}{\partial x}-\overline{u'^2}\right)+\frac{\partial}{\partial y}\left(\frac{\mu}{\rho}\frac{\partial \bar{u}}{\partial y}-\overline{u'v'}\right)+\frac{\partial}{\partial z}\left(\frac{\mu}{\rho}\frac{\partial \bar{u}}{\partial z}-\overline{u'w'}\right)\right]} \qquad \text{(B)}$$

8.5 Reynolds Equations for Incompressible Fluids

Substitute (A) and $\frac{\mu}{\rho} = \nu$ into (B)

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = \varepsilon$$

$$\rho\left(\frac{\partial \overline{u}}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} + \overline{w}\frac{\partial \overline{u}}{\partial z}\right)$$

$$= \rho g_x - \frac{\partial \overline{p}}{\partial x} + \rho\left[\frac{\partial}{\partial x}\left\{(\nu + \varepsilon_x)\frac{\partial \overline{u}}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{(\nu + \varepsilon_y)\frac{\partial \overline{u}}{\partial y}\right\} + \frac{\partial}{\partial z}\left\{(\nu + \varepsilon_z)\frac{\partial \overline{u}}{\partial z}\right\}\right]$$

$$= \rho g_x - \frac{\partial \overline{p}}{\partial x} + \rho(\nu + \varepsilon)\left(\frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{u}}{\partial y^2} + \frac{\partial^2 \overline{u}}{\partial z^2}\right)$$

$$= \rho g_x - \frac{\partial \overline{p}}{\partial x} + \rho(\nu + \varepsilon)\nabla^2 \overline{u}$$

$$= \rho g_x - \frac{\partial \overline{p}}{\partial x} + (\mu + \eta)\nabla^2 \overline{u}$$

where ν = kinematic molecular viscosity; ε = kinematic eddy viscosity;

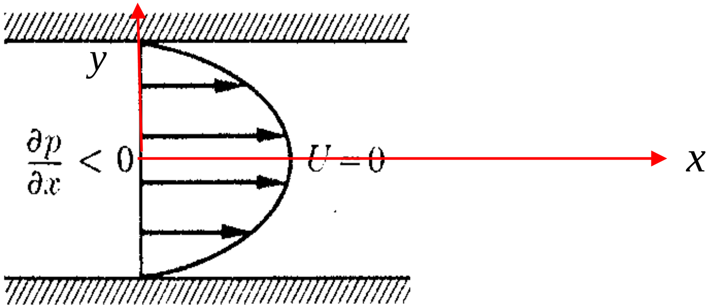
μ = dynamic molecular viscosity; η = dynamic eddy viscosity

8.5 Reynolds Equations for Incompressible Fluids

8.5.3 Examples

(1) Turbulent flow between parallel plates

Apply Reynolds equations to steady uniform motion in the x-direction between parallel horizon walls



$$\frac{\partial (\;)}{\partial t} = 0$$ ← steady motion

$$\frac{\partial (vel)}{\partial x} = 0$$ ← uniform motion $\left(\begin{array}{l} \frac{\partial u}{\partial x} = 0 \\ \frac{\partial u'}{\partial x} = 0 \end{array} \right.$

$$\frac{\partial (\;)}{\partial z} = 0, \; w = 0$$ ← 2-D motion

$$\bar{v} = \frac{1}{T}\int_0^T v\,dt = 0$$ ← unidirectional mean flow

$$v' \neq 0$$

8.5 Reynolds Equations for Incompressible Fluids

Incorporate these assumptions into Eqs. (8.22)

$$x\ :\ \rho\left(\cancel{\frac{\partial\overline{u}}{\partial t}}+\overline{u}\cancel{\frac{\partial\overline{u}}{\partial x}}+\cancel{\overline{v}}\frac{\partial\overline{u}}{\partial y}+\cancel{\overline{w}}\frac{\partial\overline{u}}{\partial z}\right)$$

$$=\rho g_x-\frac{\partial\overline{p}}{\partial x}+\mu\nabla^2\overline{u}-\rho\left(\cancel{\frac{\partial\overline{u'^2}}{\partial x}}+\frac{\partial\overline{u'v'}}{\partial y}+\cancel{\frac{\partial\overline{u'w'}}{\partial z}}\right)$$

$$\therefore 0=\rho g_x-\frac{\partial\overline{p}}{\partial x}+\mu\frac{\partial^2\overline{u}}{\partial y^2}-\rho\frac{\partial\overline{u'v'}}{\partial y}$$

$$=-\rho g\frac{\partial h}{\partial x}-\frac{\partial\overline{p}}{\partial x}+\mu\frac{\partial^2\overline{u}}{\partial y^2}-\rho\frac{\partial\overline{u'v'}}{\partial y}\qquad\text{(A)}$$

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$$y\ :\ \rho\left(\cancel{\frac{\partial \bar{v}}{\partial t}}+\bar{u}\cancel{\frac{\partial \bar{v}}{\partial x}}+\cancel{\bar{v}}\frac{\partial \bar{v}}{\partial y}+\cancel{\bar{w}}\frac{\partial \bar{v}}{\partial z}\right)$$

$$=\rho g_y-\frac{\partial \bar{p}}{\partial y}+\mu\nabla^2\cancel{\bar{v}}-\rho\left(\cancel{\frac{\partial \overline{v'u'}}{\partial x}}+\frac{\partial \overline{v'^2}}{\partial y}+\cancel{\frac{\partial \overline{v'w'}}{\partial z}}\right)$$

$$\boxed{g_y=-g\frac{\partial h}{\partial y}}$$

$$\therefore 0=\rho g_y-\frac{\partial \bar{p}}{\partial y}-\rho\frac{\partial \overline{v'^2}}{\partial y}$$

$$\frac{\partial}{\partial y}(\bar{p}+\gamma h)+\rho\frac{\partial \overline{v'^2}}{\partial y}=0 \qquad (8.25)$$

Integrate (8.25)

$$\bar{p}+\gamma h+\rho\overline{v'^2}=const. \qquad (8.26)$$

→ In turbulent flow, static pressure distribution in planes perpendicular to flow direction differs from the hydrostatic pressure by $\rho\overline{v'^2}$

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Rearrange (A)

$$\frac{\partial}{\partial x}(\bar{p}+\gamma h) = -\rho\frac{\partial \overline{u'v'}}{\partial y} + \mu\frac{\partial^2 \bar{u}}{\partial y^2} = \frac{\partial}{\partial y}\left(-\rho\overline{u'v'} + \cancel{\mu\frac{\partial \bar{u}}{\partial y}}\right) \qquad \text{(D)}$$

neglect since
turbulence contribution
to shear is dominant

Integrate (D) w.r.t. y (measured from centerline between the plates)

$$\frac{d}{dx}(\bar{p}+\gamma h)y = -\rho\overline{u'v'} = \tau$$

$$\boxed{\tau_{tur} = -\rho\overline{u'v'} \propto y}$$

→ τ distribution is linear with distance from the wall for both laminar and turbulent flows.

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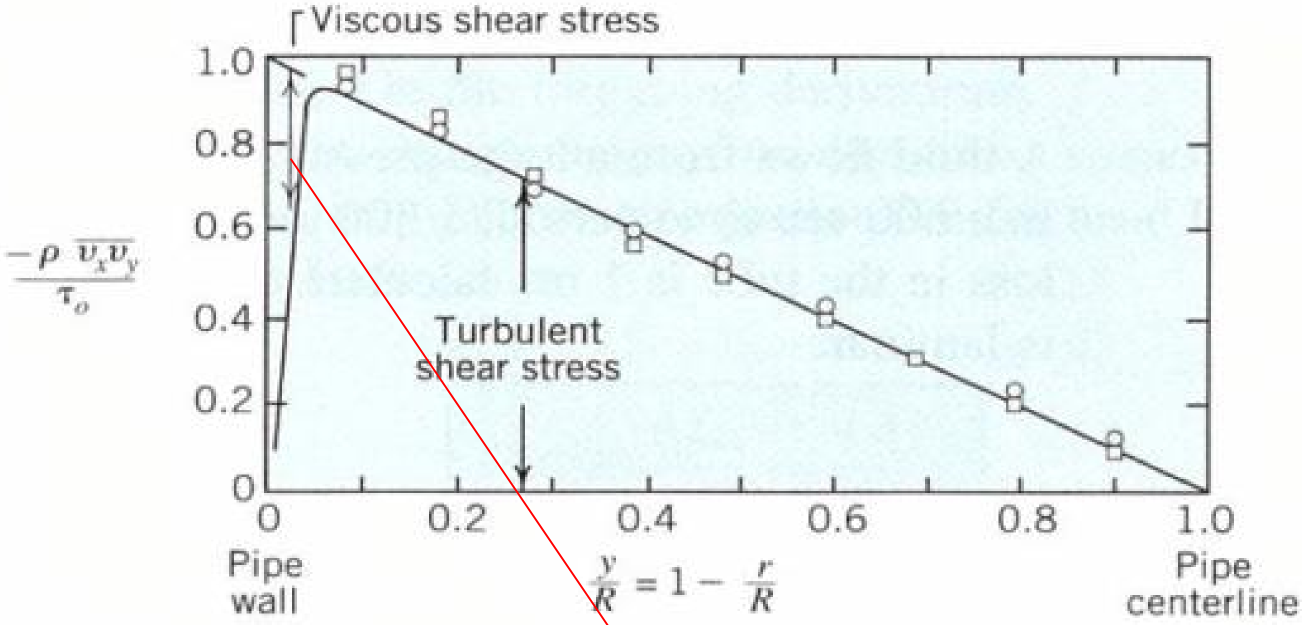

Near wall, viscous shear is dominant.

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(2) Equations for a turbulent boundary layer

Apply Prandtl's 2-D boundary-layer equations

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} \qquad (8.7a)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\rightarrow\ u\frac{\partial u}{\partial x}+u\frac{\partial v}{\partial y}=0 \qquad (8.7b)$$

Add Continuity Eq. and Eq. (8.7a)

$$\frac{\partial u}{\partial t}+\underbrace{2u\frac{\partial u}{\partial x}}_{\displaystyle\downarrow\ \frac{\partial u^2}{\partial x}}+\underbrace{\left(v\frac{\partial u}{\partial y}+u\frac{\partial v}{\partial y}\right)}_{\displaystyle\downarrow\ \frac{\partial uv}{\partial y}}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} \qquad (A)$$

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Substitute velocity decomposition into (A) and average over time

$$\frac{\overline{\partial(\bar{u}+u')}}{\partial t} = \frac{\partial \bar{u}}{dt}$$

$$\frac{\overline{\partial(\bar{u}+u')^2}}{\partial x} = \frac{\partial \bar{u}^2}{dx} + \frac{\partial \overline{u'^2}}{\partial x}$$

$$\frac{\overline{\partial(\bar{u}+u')(\bar{v}+v')}}{\partial y} = \frac{\partial \bar{u}\,\bar{v}}{\partial y} + \frac{\partial \overline{u'v'}}{\partial y}$$

$$-\frac{1}{\rho}\frac{\partial}{\partial x}\overline{(\bar{p}+p')} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}$$

$$\frac{\mu}{\rho}\frac{\partial^2}{\partial y^2}(\bar{u}+u') = \frac{\mu}{\rho}\frac{\partial^2 \bar{u}}{\partial y^2}$$

Thus, (A) becomes

$$\therefore \frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \bar{u}\,\bar{v}}{\partial y} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\partial \overline{u'^2}}{\partial x} - \frac{\partial \overline{u'v'}}{\partial y} \qquad \text{(B)}$$

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Subtract Continuity eq. from (B)

$$\frac{\partial \overline{u}}{\partial t}+\overline{u}\frac{\partial \overline{u}}{\partial x}+\overline{v}\frac{\partial \overline{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x}+\frac{\mu}{\rho}\frac{\partial^2 \overline{u}}{\partial y^2}-\frac{\partial \overline{u'^2}}{\partial x}-\frac{\partial \overline{u'v'}}{\partial y}$$

$$\boxed{\rho\left(\frac{\partial \overline{u}}{\partial t}+\overline{u}\frac{\partial \overline{u}}{\partial x}+\overline{v}\frac{\partial \overline{u}}{\partial y}\right)=-\frac{\partial \overline{p}}{\partial x}-\rho\frac{\partial \overline{u'^2}}{\partial x}+\mu\frac{\partial^2 \overline{u}}{\partial y^2}-\rho\frac{\partial \overline{u'v'}}{\partial y}} \tag{8.8a}$$

→ *x*-eq.

Adopt similar equation as Eq. (8.25) for *y*-eq.

$$\boxed{0=-\frac{\partial}{\partial y}(\overline{p}+\rho\overline{v'^2})} \tag{8.8b}$$

Continuity eq.:

$$\boxed{\frac{\partial \overline{u}}{\partial x}+\frac{\partial \overline{v}}{\partial y}=0} \tag{8.8c}$$