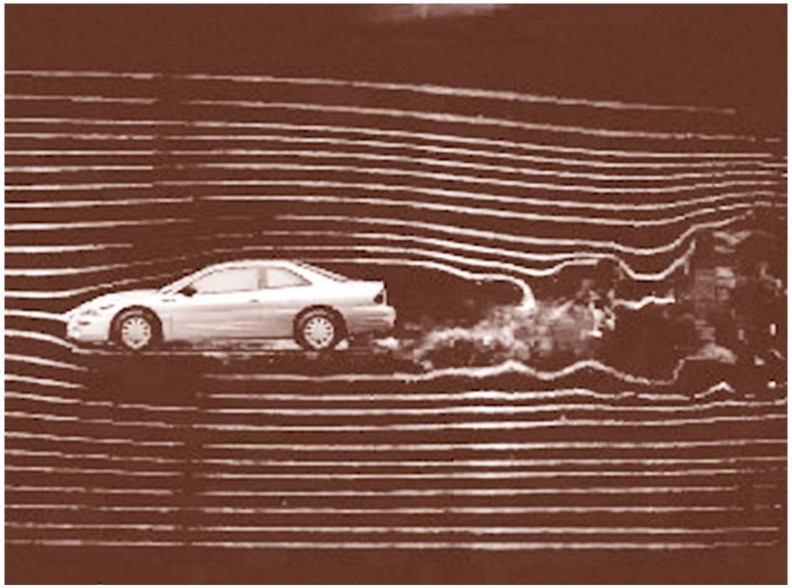
Chapter 8

Origin of Turbulence and Turbulent Shear Stress



Chapter 8 Origin of Turbulence and Turbulent Shear Stress

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In order to close the turbulent problem, theoretical assumptions are needed for the calculation of turbulent flows (Schlichting, 1979).

→ We need to have empirical hypotheses to establish a relationship between the Reynolds stresses produced by the mixing motion and the mean values of the velocity components

8.6 Mixing Length and Similarity Hypotheses in Shear flow

8.6.1 Description of turbulence problems

- Turbulent flows must instantaneously satisfy conservation of mass and momentum.
- The incompressible continuity and Navier-Stokes equations can be solved for the instantaneous flow field.
- However, to accurately simulate the turbulent field, the calculation must span from the largest geometric scales down to the Kolmogorov and Batchelor length scales.

8.6 Mixing Length and Similarity Hypotheses in Shear flow

(1) RANS vs. DNS

• Time-averaged Navier-Stokes Eq. → Reynolds Equations (RANS)

→ No. of unknowns {mean values ($\bar{u}, \bar{v}, \bar{w}, \bar{p}$) + Reynolds stress components $(\sigma_{ij} = -\rho \overline{u_i' u_j'})$ } > No. of equations

→ Closure problem:

~ The gap (deficiency of equations) can be closed only with auxiliary models and estimates based on intuition and experience.

Turbulence models

Reynolds (1895)

8.6 Mixing Length and Similarity Hypotheses in Shear flow

- **Reynolds-Averaged Navier-Stokes (RANS) equations**

$$\frac{\partial \overline{u_i}}{\partial x_i} = 0$$

$$\rho\left(\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j}\right) = -\frac{\partial \overline{p}}{\partial x_i} + \mu\frac{\partial^2 \overline{u_i}}{\partial {x_j}^2} - \rho\frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'}\right)$$

where $\rho\frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'}\right)$ = Reynolds stress tensor; it physically corresponds to the <u>transport of momentum due to the turbulent fluctuations</u>.

8.6 Mixing Length and Similarity Hypotheses in Shear flow

(2) Methods of analysis

1) Phynomenological concepts of turbulence

~ based on a superficial resemblance between molecular motion and turbulent motion

~ crucial assumptions at an early stage in the analysis

- *Eddy viscosity model (Boussinesq, 1877)*

~ turbulence-generated viscosity is modeled using analogy with molecular viscosity

~ characteristics of flow

$$\tau_t = -\rho\overline{u'v'} = \eta \frac{d\bar{u}}{dy}$$

8.6 Mixing Length and Similarity Hypotheses in Shear flow

- *Mixing length model (Prandtl, 1925)*

~ analogy with mean free path of molecules

in the kinetic theory of gases

$$\tau = -\rho \overline{u'v'} = \rho l^2 \left| \frac{d\bar{u}}{dy} \right| \frac{d\bar{u}}{dy}$$

Prandtl (1925)

2) Dimensional analysis

~ one of the most powerful tools

~ result in the relation between the dependent and independent variables

[Ex] form of the spectrum of turbulent kinetic energy

8.6 Mixing Length and Similarity Hypotheses in Shear flow

3) Asymptotic theory

~ based on asymptotic invariance

~ exploit asymptotic properties of turbulent flows as *Re* approaches infinity (or very high).

[Ex)] Theory of turbulent boundary layers

Reynolds-number similarity

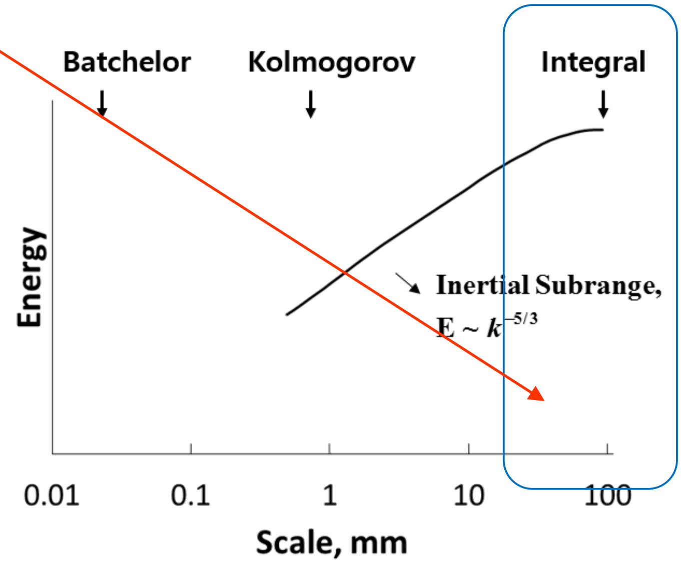
4) Stochastic approach

8.6 Mixing Length and Similarity Hypotheses in Shear flow

(3) Large Eddy Simulation (L.E.S)

- DNS

- model only large fluctuations



8.6 Mixing Length and Similarity Hypotheses in Shear flow

8.6.2 Boussinesq's eddy viscosity model

For laminar flow;

$$\tau_l = \mu \frac{d\bar{u}}{dy}$$

For turbulent flow, use analogy with laminar flow;

$$\tau_t = -\rho \overline{u'v'} = \eta \frac{d\bar{u}}{dy} \quad (8.30)$$

where η = apparent (virtual) eddy viscosity

→ turbulent mixing coefficient

~ not a property of the fluid

~ depends on $\bar{u}$; $\eta \propto \bar{u}$

8.6 Mixing Length and Similarity Hypotheses in Shear flow

8.6.3 Prandtl's mixing length theory

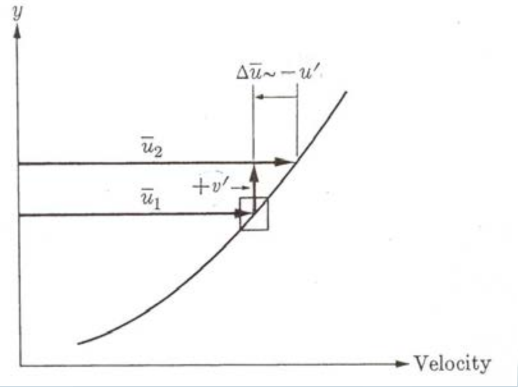
- Prandtl (1925) express momentum shear stresses in terms of mean velocity
- Originally proposed by Taylor (1915)

■ Assumptions

1) Average distance traversed by a fluctuating fluid element before it acquired the velocity of new region is related to an average (absolute) magnitude of the fluctuating velocity.

$$\overline{|v'|} \propto l \left| \frac{d\bar{u}}{dy} \right| \qquad (8.31a)$$

where $l = l(y)$ = mixing length



8.6 Mixing Length and Similarity Hypotheses in Shear flow

2) Two orthogonal fluctuating velocities are proportional to each other.

$$\overline{|u'|} \propto \overline{|v'|} \propto l\left|\frac{d\bar{u}}{dy}\right| \qquad (8.31b)$$

Substituting (8.31) into (8.13) leads to

$$\tau = -\rho\overline{u'v'} = \rho l^2 \left|\frac{d\bar{u}}{dy}\right| \frac{d\bar{u}}{dy} \qquad (8.32)$$

Therefore, combining (8.30) and (8.32), dynamic eddy viscosity can be expressed as

$$\eta = \rho l^2 \left|\frac{d\bar{u}}{dy}\right| \qquad (8.33)$$

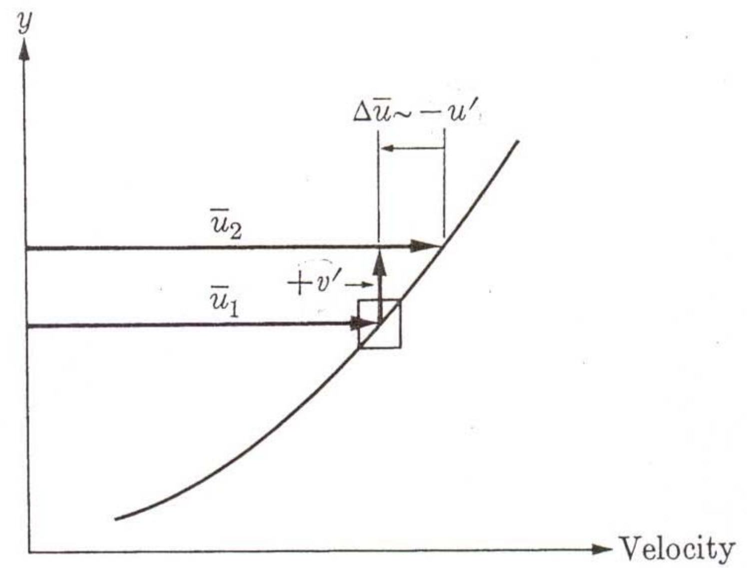
→ Prandtl's formulation has a restricted usefulness because it is <u>not possible to predict mixing length function </u>for flows in general.

8.6 Mixing Length and Similarity Hypotheses in Shear flow

[Re] Mixing-length theory (Schlichting, 1979)

Consider simplest case of parallel flow in which the velocity varies only from streamline to streamline.

$$\rightarrow \quad \begin{cases} \bar{u} = \bar{u}(y) \\ \bar{v} = \bar{w} = 0 \end{cases}$$

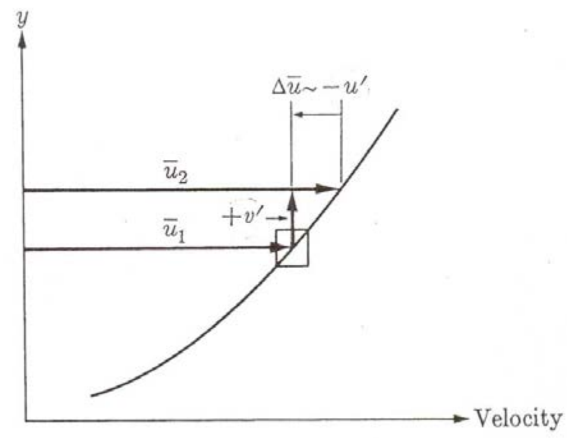


Shearing stress is given as

$$\tau'_{xy} = \tau_t = -\rho\overline{u'v'} = \eta\frac{d\bar{u}}{dy}$$

8.6 Mixing Length and Similarity Hypotheses in Shear flow

○ Simplified mechanism of the motion



1) Fluid particles move in lump both in longitudinal and in the transverse direction.

2) If a lump of fluid is displaced from a layer at to a new layer, then, the difference in velocities is expressed as (use Taylor series and neglect high-order terms)

8.6 Mixing Length and Similarity Hypotheses in Shear flow

$$\Delta u_1 = \bar{u}(y_1) - \bar{u}(y_1 - l) \approx l\left(\frac{d\bar{u}}{dy}\right)_{y=y_1} \qquad ; v' > 0$$

where *l* = Prandtl's mixing length (mixture length)

For a lump of fluid which arrives at upper layer from the lower laminar

$$\Delta u_2 = \bar{u}(y_1 + l) - \bar{u}(y_1) \approx l\left(\frac{d\bar{u}}{dy}\right)_{y=y_1} \qquad ; v' < 0$$

3) These velocity differences caused by the transverse motion can be regarded as the turbulent velocity fluctuation at

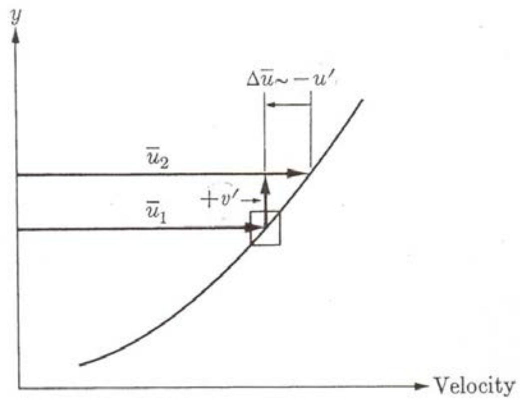
$$\overline{|u'|} = \frac{1}{2}(|\Delta u_1| + |\Delta u_2|) = l\left|\left(\frac{d\bar{u}}{dy}\right)_{y_1}\right| \qquad (2)$$

8.6 Mixing Length and Similarity Hypotheses in Shear flow

○ Physical interpretation of the mixing length l.

= distance in the transverse direction which must be covered by an agglomeration of fluid particles travelling with its mean velocity in order to make the difference between it's velocity and the velocity in the new laminar equal to the mean transverse fluctuation in turbulent flow.

4) Transverse velocity fluctuation originates in two ways.



8.6 Mixing Length and Similarity Hypotheses in Shear flow

5) Transverse component is the same order of magnitude as

$$\overline{|v'|} = const \cdot \overline{|u'|} = const \cdot l \frac{d\bar{u}}{dy} \quad (3)$$

6) Fluid lumps which arrive at layer with a <u>positive value of *v'*</u> (upwards from lower layer) give rise mostly to a <u>negative *u'*.</u>

$$\therefore\ u'v' < 0$$

$$\overline{u'v'} = -c\overline{|u'|}\,\overline{|v'|} \quad (4)$$

where 0 < c < 1

8.6 Mixing Length and Similarity Hypotheses in Shear flow

Combine Eqs. 2-4

$$\overline{u'v'} = -constant \cdot -l^2 \left| \frac{d\bar{u}}{dy} \right| \frac{d\bar{u}}{dy}$$

Include constant into *l* (mixing length)

$$\overline{u'v'} = -l^2 \left| \frac{d\bar{u}}{dy} \right| \frac{d\bar{u}}{dy} \qquad (5)$$

Therefore, shear stress is given as

$$\tau = -\rho\overline{u'v'} = \rho l^2 \left| \frac{d\bar{u}}{dy} \right| \frac{d\bar{u}}{dy} \qquad (6)$$

→ Prandtl's mixing-length hypothesis

8.6 Mixing Length and Similarity Hypotheses in Shear flow

8.6.4 Von Karman's similarity hypothesis

- Von Karman (1930), a student of Prandtl, attempted to remove the mixing length l
- Relate the mixing length to velocity gradient using the similarity rule
- Turbulent fluctuations are similar at all point of the field of flow
- Velocity is characteristics of the turbulent fluctuating motion.
- For 2-D mean flow in the x- direction, a necessary condition to secure compatibility between the similarity hypothesis and the vorticity transport equation is

8.6 Mixing Length and Similarity Hypotheses in Shear flow

$$l \sim \frac{\dfrac{d\overline{u}}{dy}}{\dfrac{d^2\overline{u}}{dy^2}}$$

$$l = \kappa \left| \frac{d\overline{u} \,/\, dy}{d^2\overline{u} \,/\, dy^2} \right|$$

where κ = empirical dimensionless constant

Substituting (A) into (8.32) gives

$$\tau = \rho\kappa^2 \frac{(d\overline{u} / dy)^4}{(d^2\overline{u} / dy^2)^2} \qquad (8.35)$$

→ Von Karman's similarity rule

8.6 Mixing Length and Similarity Hypotheses in Shear flow

8.6.5 Prandtl's velocity-distribution law

For wall turbulence (immediate neighborhood of the wall),

$$l = \kappa y$$

$$\tau = \rho \kappa^2 y^2 \left(\frac{d\bar{u}}{dy} \right)^2 \qquad (1)$$

$$\frac{d\bar{u}}{dy} = \frac{1}{\kappa y} \sqrt{\frac{\tau}{\rho}} = \frac{u_*}{\kappa y} \qquad (2)$$

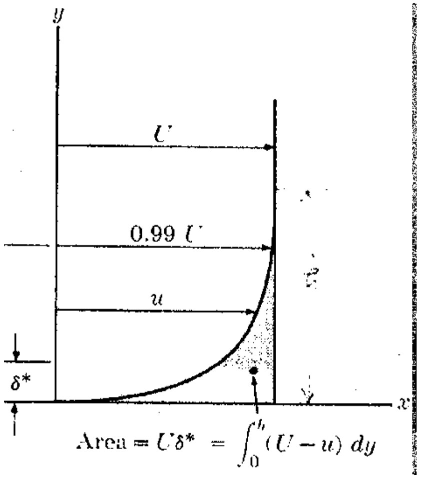
where $u_* = \sqrt{\tau / \rho}$ = shear velocity; κ = von Karman const ≈ 0.4

8.6 Mixing Length and Similarity Hypotheses in Shear flow

Integrate (2) w.r.t. y

$$\bar{u} = \frac{u_*}{\kappa} \ln y + C \qquad (3)$$

→ **Prandtl's velocity distribution law**



Apply Prandtl's velocity distribution law to whole region

$$\bar{u} = \bar{u}_{\max} \quad at \quad y = h$$

$$\bar{u}_{\max} = \frac{u_*}{\kappa} \ln h + C \qquad (4)$$

Subtract (3) from (4) to eliminate constant of integration

$$\frac{\bar{u}_{\max} - \bar{u}}{u_*} = \frac{1}{\kappa} \ln \frac{h}{y} \qquad (5)$$

→ **Prandtl's universal velocity-defect law**

Homework Assignment # 8

Due: 1 week from today

8-1. The velocity data listed in Table were obtained at a point in a turbulent flow of sea water.

1) Compute the energy of turbulence per unit volume.

2) Determine the mean velocity in the x-direction, $\overline{u}$, and verify that $\overline{u'} = 0$.

3) Determine the magnitude of the three independent turbulent shear stresses in Eq. (8-21).

* Include units in your answer

8.6 Mixing Length and Similarity Hypotheses in Shear flow

time, sec	u cm/s	u' cm/s	v' cm/s	w' cm/s
0.0	89.92	−4.57	1.52	0.91
0.1	95.10	0.61	0.00	−0.30
0.2	103.02	8.53	−3.66	−2.13
0.3	99.67	5.18	−1.22	−0.61
0.4	92.05	−2.44	−0.61	0.30
0.5	87.78	−6.71	2.44	0.91
0.6	92.96	−1.52	0.91	−0.61
0.7	90.83	−3.66	1.83	0.61
0.8	96.01	1.52	0.61	0.91
0.9	93.57	−0.91	0.30	−0.61
1.0	98.45	3.96	−1.52	−1.22