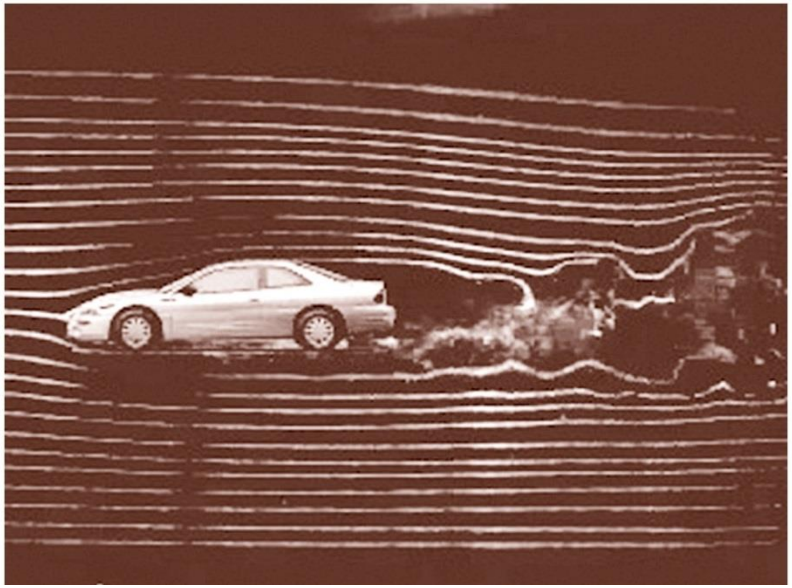
Lecture 22

Origin of Turbulence (5)



Lecture 22 Origin of Turbulence (5)

Contents

Objectives

- Learn fundamental concept of turbulence
- Study Reynolds decomposition
- Derive Reynolds equation from Navier-Stokes equation
- Study eddy viscosity model and mixing length model

22.1 Description of Turbulence Problems

22.1.1 Solutions for turbulence problems

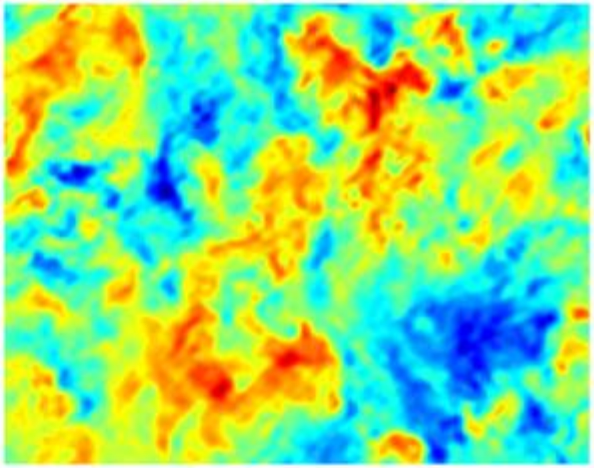
1) DNS of Navier-Stokes equations

- Turbulent flows must instantaneously satisfy conservation of mass and momentum.
- The incompressible continuity and Navier-Stokes equations can be solved for the instantaneous flow field.
- However, to accurately simulate the turbulent field, the calculation must span from the largest geometric scales down to the Kolmogorov length scales, all of which affect the flow field.

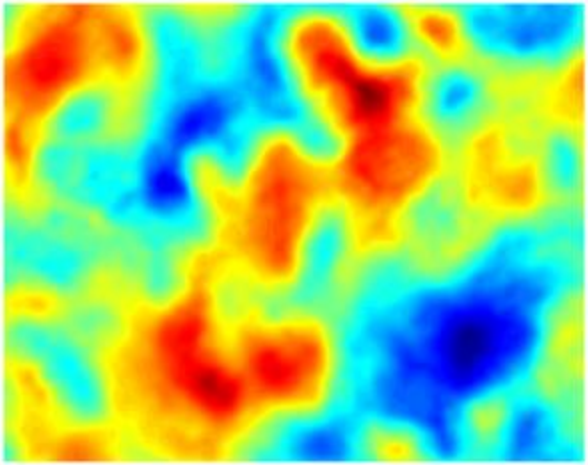
22.1 Description of Turbulence Problems

- **Large Eddy Simulation (LES)**

- reduce the computational cost by neglecting the small scale turbulences, via low pass filtering of the N-S equations



DNS



LES

22.1 Description of Turbulence Problems

2) Reynolds-Averaged Navier-Stokes (RANS) equations

- Reynolds equation can be solved to acquire the average characteristics of instantaneous turbulent motion.
- However, using Reynolds equations, in order to close the turbulent problem, theoretical assumptions are needed for the calculation of turbulent flows (Schlichting, 1979).
- We need to have empirical hypotheses to establish a relationship between the Reynolds stresses produced by the mixing motion and the mean values of the velocity components.

22.1 Description of Turbulence Problems

$$\frac{\partial \overline{u_i}}{\partial x_i} = 0 \qquad (22.1)$$

$$\rho\left(\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j}\right) = -\frac{\partial \overline{p}}{\partial x_i} + \mu \frac{\partial^2 \overline{u_i}}{\partial {x_j}^2} - \rho \frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'}\right) \qquad (22.2)$$

where $\rho \frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'}\right)$ = Reynolds stress tensor;

- It physically corresponds to the transport of momentum due to the turbulent fluctuations.

→ no. of unknowns > no. of equations

- The gap (deficiency of equations) can be closed only with auxiliary models and estimates based on intuition and experience. → Turbulence models

22.1 Description of Turbulence Problems

22.1.2 Method of analysis

- **Phynomenological concepts of turbulence**

~ based on a superficial resemblance between molecular motion and turbulent motion

~ crucial assumptions at an early stage in the analysis

- Eddy viscosity model (Boussinesq, 1877)

~ turbulence-generated viscosity is modeled using analogy with molecular viscosity

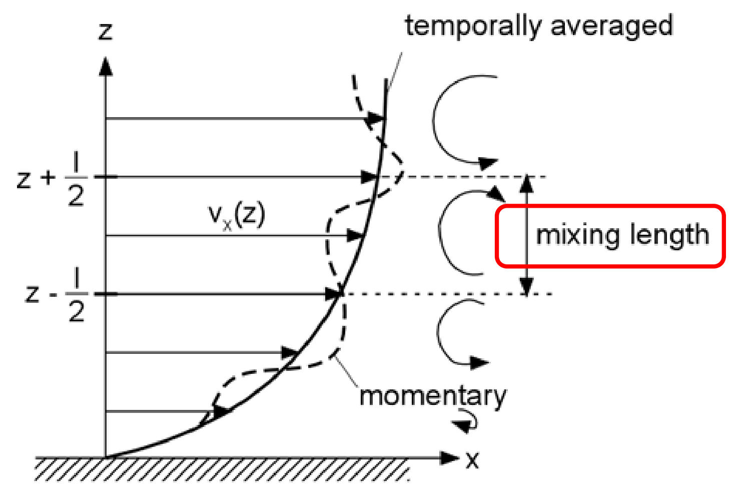
$$\tau_t = -\rho \overline{u'v'} = \eta \frac{d\bar{u}}{dy} \qquad (22.3)$$

22.1 Description of Turbulence Problems

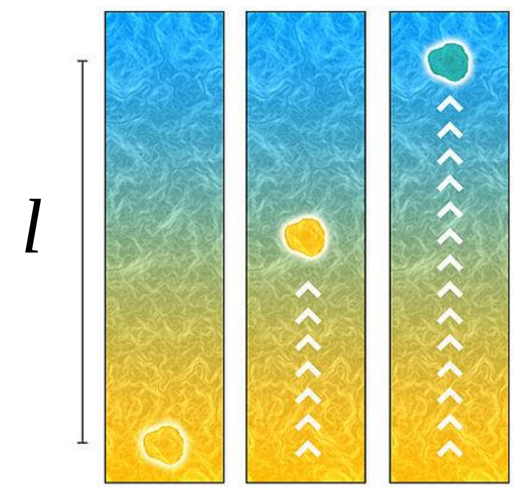
- Mixing length model (Prandtl, 1925)

~ analogy with mean free path of molecules in the kinetic theory of gases

$$\tau = -\rho\overline{u'v'} = \rho l^2 \left| \frac{d\bar{u}}{dy} \right| \frac{d\bar{u}}{dy} \qquad (22.4)$$



- The mixing length is a distance that a fluid parcel will keep its original characteristics before dispersing them into the surrounding fluid.



22.2 Basic Turbulence Models

22.2.1 Boussinesq's eddy viscosity model

For laminar flow;

$$\tau_l = \mu \frac{d\bar{u}}{dy} \qquad (22.5)$$

For turbulent flow, use analogy with laminar flow;

$$\tau_t = -\rho \overline{u'v'} = \eta \frac{d\bar{u}}{dy} \qquad (22.6)$$

where η = apparent (virtual) eddy viscosity

→ turbulent mixing coefficient

~ not a property of the fluid

~ depends on $\bar{u}$; $\eta \propto \bar{u}$

22.2 Basic Turbulence Models

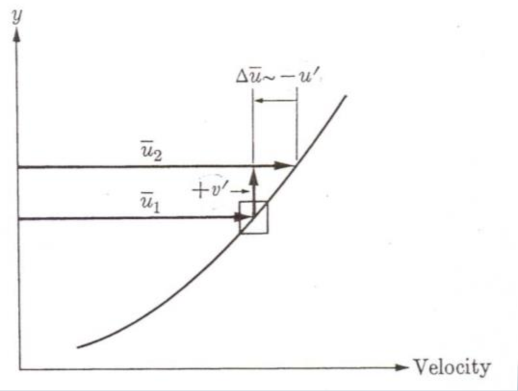
22.2.2 Prandtl's mixing length theory

- Prandtl (1925) express momentum shear stresses in terms of mean velocity
- Originally proposed by Taylor (1915)

■ Assumptions

1) Average distance traversed by a fluctuating fluid element before it acquired the velocity of new region is related to an average (absolute) magnitude of the fluctuating velocity.

$$\overline{|v'|} \propto l \left| \frac{d\bar{u}}{dy} \right| \qquad (22.7a)$$



where $l = l(y)$ = mixing length

22.2 Basic Turbulence Models

2) Two orthogonal fluctuating velocities are proportional to each other.

$$\overline{|u'|} \propto \overline{|v'|} \propto l\left|\frac{d\bar{u}}{dy}\right| \qquad (22.7b)$$

Substituting (22.7) into (22.3) leads to

$$\tau = -\rho\overline{u'v'} = \rho l^2 \left|\frac{d\bar{u}}{dy}\right| \frac{d\bar{u}}{dy} \qquad (22.8)$$

Therefore, combining (22.3) and (22.8), dynamic eddy viscosity can be expressed as

$$\eta = \rho l^2 \left|\frac{d\bar{u}}{dy}\right| \qquad (22.9)$$

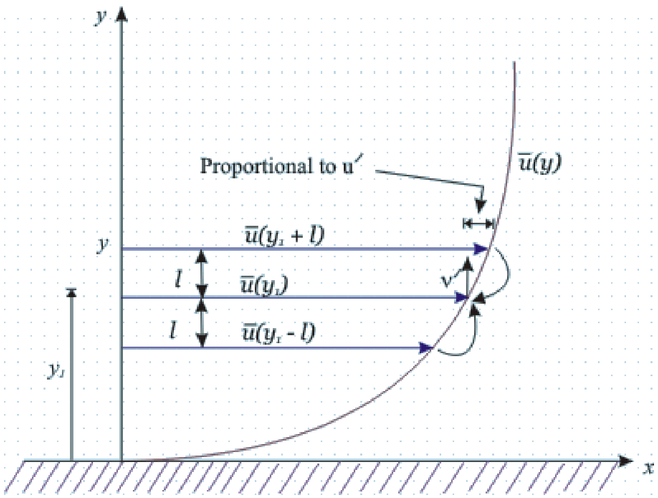
→ Prandtl's formulation has a restricted usefulness because it is <u>not possible to predict mixing length function</u> for flows in general.

22.2 Basic Turbulence Models

[Re] Mixing-length theory (Schlichting, 1979)

Consider simplest case of parallel flow in which the velocity varies only from streamline to streamline.

$$\rightarrow \begin{cases} \bar{u} = \bar{u}(y) \\ \bar{v} = \bar{w} = 0 \end{cases}$$



Shearing stress is given as

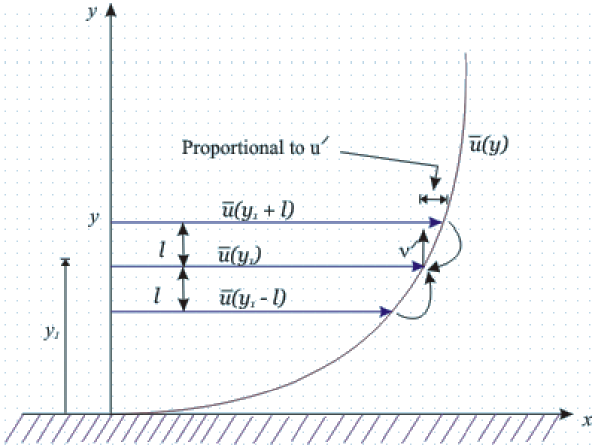
$$\tau'_{xy} = \tau_t = -\rho \overline{u'v'}$$

22.2 Basic Turbulence Models

1) Fluid particles move in lump both in longitudinal and in the transverse direction.

2) If a lump of fluid is displaced from a layer at to a new layer, then, the <u>difference in velocities </u>is expressed as (use Taylor series and neglect high-order terms)

$$\Delta u_1 = \bar{u}(y_1) - \bar{u}(y_1 - l) \approx l \left(\frac{d\bar{u}}{dy} \right)_{y=y_1} \qquad ; v' > 0 \qquad (1)$$



where l = <u>Prandtl's mixing length (mixture length)</u>

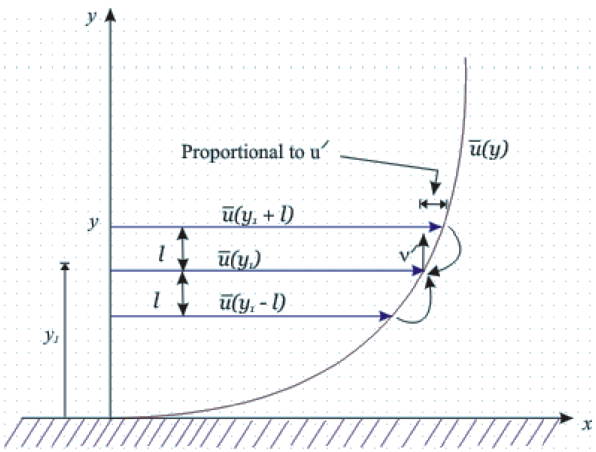
22.2 Basic Turbulence Models

For a lump of fluid which arrives at upper layer from the lower laminar

$$\Delta u_2 = \bar{u}(y_1 + l) - \bar{u}(y_1) \approx l\left(\frac{d\bar{u}}{dy}\right)_{y=y_1} \qquad ; v' < 0 \qquad (2)$$

3) These velocity differences caused by the transverse motion can be regarded as the turbulent velocity fluctuation at

$$\overline{|u'|} = \frac{1}{2}(|\Delta u_1| + |\Delta u_2|) = l\left|\left(\frac{d\bar{u}}{dy}\right)_{y_1}\right| \qquad (3)$$



22.2 Basic Turbulence Models

4) Transverse component is the same order of magnitude as

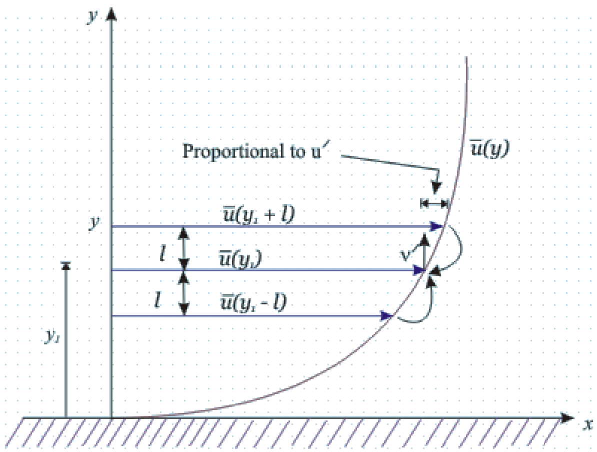
$$\overline{|v'|} = const \cdot \overline{|u'|} = const \cdot l \frac{d\bar{u}}{dy} \quad (4)$$

5) Fluid lumps which arrive at layer with a positive value of *v'* (upwards from lower layer) give rise mostly to a negative *u'*.

$$\therefore\ u'v' < 0$$

$$\overline{u'v'} = -c\,\overline{|u'|}\,\overline{|v'|} \quad (5)$$

where 0 < *c* < 1



22.2 Basic Turbulence Models

6) Combine Eqs. (3)~(5)

$$\overline{u'v'} = -constant \cdot l^2 \left| \frac{d\bar{u}}{dy} \right| \frac{d\bar{u}}{dy}$$

Include constant into *l* (mixing length)

$$\overline{u'v'} = -l^2 \left| \frac{d\bar{u}}{dy} \right| \frac{d\bar{u}}{dy} \qquad (6)$$

Therefore, shear stress is given as

$$\tau = -\rho\overline{u'v'} = \rho l^2 \left| \frac{d\bar{u}}{dy} \right| \frac{d\bar{u}}{dy} \qquad (7)$$

→ Prandtl's mixing-length hypothesis

22.2 Basic Turbulence Models

22.2.3 Von Karman's similarity hypothesis

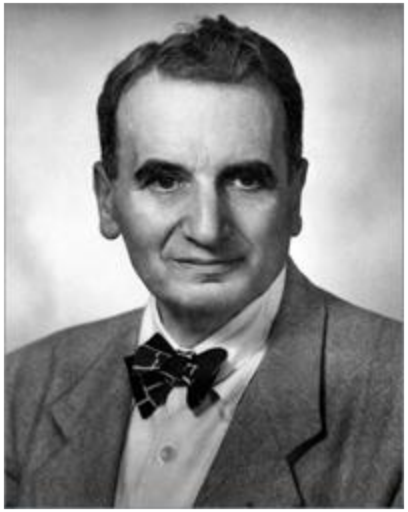
- Von Karman (1930), a student of Prandtl, attempted to remove the mixing length l
- Relate the mixing length to velocity gradient using the similarity rule
- Turbulent fluctuations are similar at all point of the field of flow
- Velocity is characteristics of the turbulent fluctuating motion.
- For 2-D mean flow in the x- direction, a necessary condition to secure compatibility between the similarity hypothesis and the vorticity transport equation is

22.2 Basic Turbulence Models

$$l \sim \frac{\frac{d\overline{u}}{dy}}{\frac{d^2\overline{u}}{dy^2}}$$

$$l = \kappa \left| \frac{d\overline{u} / dy}{d^2\overline{u} / dy^2} \right| \qquad \text{(A)}$$

where κ = empirical dimensionless constant



Theodore von Karman (1881~1963)

Substituting (A) into (22.8) gives

$$\tau = \rho\kappa^2 \frac{(d\overline{u} / dy)^4}{(d^2\overline{u} / dy^2)^2} \qquad (22.10)$$

→ Von Karman's similarity rule

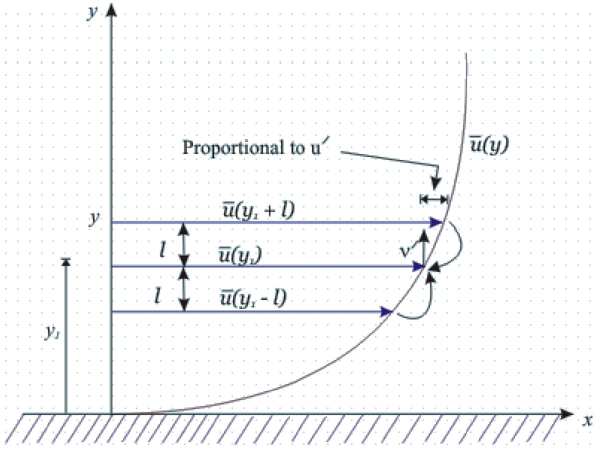
22.2 Basic Turbulence Models

22.2.4 Prandtl's velocity-distribution law

For wall turbulence (immediate neighborhood of the wall),

$$l = \kappa y \qquad (1)$$

where κ = von Karman const ≈ 0.4



Substitute (1) into (22.8)

$$\tau = \rho \kappa^2 y^2 \left(\frac{d\bar{u}}{dy} \right)^2 \qquad (2)$$

Arrange (2) in terms of $\bar{u}$

$$\frac{d\bar{u}}{dy} = \frac{1}{\kappa y} \sqrt{\frac{\tau}{\rho}} = \frac{u_*}{\kappa y}$$

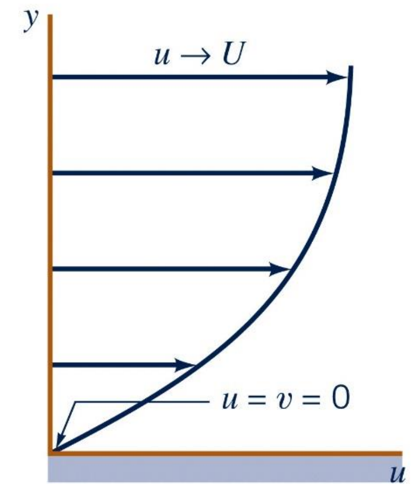
$u_* = \sqrt{\tau / \rho}$ = shear velocity (3)

22.2 Basic Turbulence Models

Integrate (3) w.r.t. *y*

$$\bar{u} = \frac{u_*}{\kappa} \ln y + C \qquad (4)$$

→ **Prandtl's velocity distribution law**



Apply Prandtl's velocity distribution law to whole region

$$\bar{u} = \bar{u}_{\max} \quad at \quad y = h$$

$$\bar{u}_{\max} = \frac{u_*}{\kappa} \ln h + C \qquad (5)$$

Subtract (4) from (5) to eliminate constant of integration

$$\frac{\bar{u}_{\max} - \bar{u}}{u_*} = \frac{1}{\kappa} \ln \frac{h}{y} \qquad (6)$$

→ **Prandtl's universal velocity-defect law**

Problems

Homework Assignment # 6

Due: 1 week from today

22-1. The velocity data listed in Table were obtained at a point in a turbulent flow of sea water.

1) Compute the energy of turbulence per unit volume.

2) Determine the mean velocity in the x-direction, $\overline{u}$, and verify that $\overline{u'} = 0$.

3) Determine the magnitude of the three independent turbulent shear stresses in Eq. (21-11).

* Include units in your answer

Problems

time, sec	u cm/s	u' cm/s	v' cm/s	w' cm/s
0.0	89.92	−4.57	1.52	0.91
0.1	95.10	0.61	0.00	−0.30
0.2	103.02	8.53	−3.66	−2.13
0.3	99.67	5.18	−1.22	−0.61
0.4	92.05	−2.44	−0.61	0.30
0.5	87.78	−6.71	2.44	0.91
0.6	92.96	−1.52	0.91	−0.61
0.7	90.83	−3.66	1.83	0.61
0.8	96.01	1.52	0.61	0.91
0.9	93.57	−0.91	0.30	−0.61
1.0	98.45	3.96	−1.52	−1.22