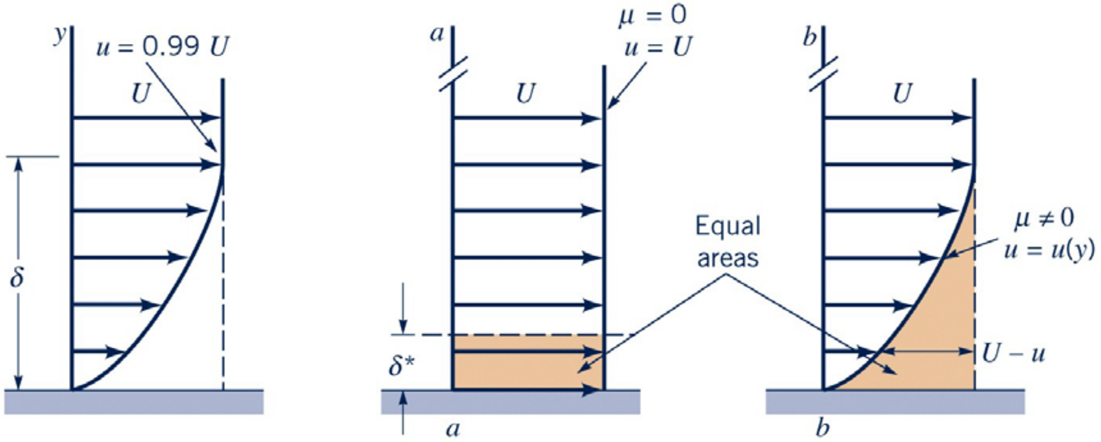
Chapter 9

Turbulent Boundary-Layer Flows



Chapter 9 Turbulent Boundary-Layer Flows

Contents

Objectives

- Study wall turbulence
- Derive equations of velocity distribution and friction coefficient for both smooth and rough walls

9.3 Mean-flow characteristics for turbulent boundary layer

9.3.2. Power-law formulas: Smooth walls

- Logarithmic equations for velocity profile and shear-stress coeff .

~ universal

~ applicable over almost entire range of Reynolds numbers

- Power-law equations

~ applicable over only limited range of Reynolds numbers

~ simpler

~ explicit relations for $\bar{u}/U$ and c_f

~ explicit relations for δ in terms of *Re* and distance x

9.3 Mean-flow characteristics for turbulent boundary layer

- **Assumptions of power-law formulas**

The power laws stem from two facts that hold for turbulent boundary layers with negligible pressure gradients when $\mathrm{Re}_\delta = \frac{U\delta}{\nu} < 5 \times 10^5$

i) Except very near the wall, mean velocity is closely proportional to a root of the distance *y* from the wall.

$$\bar{u} \propto y^{\frac{1}{n}} \quad \text{(A)}$$

ii) Shear stress coeff. c_f is inversely proportional to a root of Re_δ

$$c_f \propto \frac{1}{\mathrm{Re}_\delta^m} \quad , \quad \mathrm{Re}_\delta = \frac{U\delta}{\nu}$$

$$c_f = \frac{A}{\left(\frac{U\delta}{\nu}\right)^m} \quad (9.29)$$

where *n, A* = constants; *m* = fraction

9.3 Mean-flow characteristics for turbulent boundary layer

[Cf] Eq. (9.29) is similar to equation for laminar boundary layer, $c_f = \dfrac{3.32}{\mathrm{Re}_\delta}$

- Derivation of power equation

Combine Eqs. (9.18) and (9.29)

$$c_f = \frac{A}{\left(\dfrac{U\delta}{\nu}\right)^m}$$

(9.18): $u_* = U\sqrt{\dfrac{c_f}{2}}$

$$\therefore \frac{U}{u_*} = \sqrt{\frac{2}{c_f}} = \frac{\sqrt{2}}{\sqrt{A}}\left(\frac{U\delta}{\nu}\right)^{\frac{m}{2}}$$

$$\frac{U^{1-\frac{m}{2}}}{u_*} = \frac{\sqrt{2}}{\sqrt{A}}\left(\frac{\delta}{\nu}\right)^{\frac{m}{2}}$$

$$\left(\frac{U}{u^*}\right)^{1-\frac{m}{2}} = \sqrt{\frac{2}{A}}\left(\frac{u_*\delta}{\nu}\right)^{\frac{m}{2}}$$

$$\therefore \frac{U}{u_*} = B\left(\frac{u_*\delta}{\nu}\right)^{\frac{m}{2-m}} \qquad (9.30)$$

9.3 Mean-flow characteristics for turbulent boundary layer

Assume $\overline{u}$ depends on y by the same relation, Eq. (9.30), replacing δ with y

$$\frac{\overline{u}}{u_*} = B\left(\frac{u_* y}{\nu}\right)^{\frac{m}{2-m}} \quad (9.31)$$

Divide (9.31) by (9.30)

$$\frac{\overline{u}}{U} = \left(\frac{y}{\delta}\right)^{\frac{m}{2-m}} \quad (9.32)$$

This result indicates that all profiles are similar and can be represented by a single dimensionless curve like laminar boundary layer profile shown in Fig. 10.4.

However, turbulent profiles are not truly similar, so Eq. (9.32) will apply for different Reynolds number ranges only if the constant m is varied.

9.3 Mean-flow characteristics for turbulent boundary layer

Boundary-layer measurements shows that

For 3,000 < Re_δ < 70,000; $m = \frac{1}{4}$, $A = 0.0466$, $B = 8.74$

$$\frac{\bar{u}}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}} \quad (9.33)$$

$$\frac{U}{u_*} = 8.74\left(\frac{u_*\delta}{\nu}\right)^{\frac{1}{7}} \quad (9.34)$$

$$\frac{\bar{u}}{u_*} = 8.74\left(\frac{u_* y}{\nu}\right)^{\frac{1}{7}} \quad (9.35)$$

$$c_f = \frac{0.0466}{\left(Re_\delta\right)^{\frac{1}{4}}} \quad (9.36)$$

9.3 Mean-flow characteristics for turbulent boundary layer

[Re] The Blasius solution for laminar boundary layer flows

For steady laminar flow over a flat plate with zero pressure gradient, Prandtl's (1904) 2-D boundary-layer equations become as follows:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (1)$$

Blasius (1908) obtained the solution to above PDE by assuming similar profiles along the plate at every *x*

$$\frac{u}{U}=F\left(\frac{y}{\delta}\right) \qquad (2)$$

$$\delta \sim \frac{x}{\mathrm{Re}_x^{\ 1/2}} \qquad (3)$$

Blasius obtained the solution in the form of power series after he introduced a stream function for $\frac{y}{\delta}$

Blasius, H. (1883-1970): Prandtl's student

9.3 Mean-flow characteristics for turbulent boundary layer

Laminar boundary layer equations

$$\delta_{lam} = \frac{5x}{\mathrm{Re}_x^{1/2}} \quad at \frac{u}{U} = 0.992$$

$$\delta^*_{lam} = \frac{1.73x}{\mathrm{Re}_x^{1/2}}$$

$$\theta_{lam} = \frac{0.664x}{\mathrm{Re}_x^{1/2}}$$

$$c_f = \frac{0.664}{\left(Re_x\right)^{\frac{1}{2}}}$$

$$c_f = \frac{3.32}{Re_\delta}$$

$$C_f = \frac{1.328}{\left(Re_l\right)^{\frac{1}{2}}}$$

9.3 Mean-flow characteristics for turbulent boundary layer

- Relation for δ

Adopt integral-momentum eq. for steady motion with $\frac{\partial p}{\partial x} = 0$

~ integration of Prandtl's 2-D boundary-layer equations

$$\frac{\tau_0}{\rho} = U^2 \frac{\partial \theta}{\partial x} = c_f \frac{U^2}{2} \leftarrow \tau_0 = \frac{\rho}{2} c_f U^2 \qquad (9.37)$$

where θ = momentum thickness

$$\theta = \int_0^h \frac{\overline{u}}{U}\left(1 - \frac{\overline{u}}{U}\right) dy \qquad (A)$$

Substitute Eq. (9.33) into (A) and integrate

9.3 Mean-flow characteristics for turbulent boundary layer

$$\theta = \int_0^h \left(\frac{y}{\delta}\right)^{\frac{1}{7}} \left\{1 - \left(\frac{y}{\delta}\right)^{\frac{1}{7}}\right\} dy \qquad (9.38)$$

$$= \frac{7}{72}\delta \approx 0.1\delta$$

Substitute Eqs. (9.36) and (9.38) into (9.37) and integrate w.r.t. *x*

$$\delta = \frac{0.318x}{\left(\frac{Ux}{\nu}\right)^{\frac{1}{5}}} = \frac{0.318x}{\left(\mathrm{Re}_x\right)^{\frac{1}{5}}} \quad , \ \mathrm{Re}_x < 10^7 \qquad (9.39)$$

$$c_f = \frac{0.059}{\left(\mathrm{Re}_x\right)^{\frac{1}{5}}} \quad , \ \mathrm{Re}_x < 10^7 \qquad (9.40)$$

Integrate (9.40) over whole length, *l*, to get average coefficient

$$C_f = \frac{0.074}{\left(\mathrm{Re}_l\right)^{\frac{1}{5}}} \quad , \ \mathrm{Re}_l < 10^7 \qquad (9.41)$$

9.3 Mean-flow characteristics for turbulent boundary layer

[Re] Derivation of (9.39) and (9.40)

$$U^2 \frac{\partial \theta}{\partial x} = c_f \frac{U^2}{2}$$

$$\frac{\partial \theta}{\partial x} = \frac{c_f}{2} \qquad \text{(B)}$$

Substitute (9.38) and (9.36) into (B)

$$\frac{\partial}{\partial x}\left(\frac{7}{72}\delta\right) = \frac{1}{2}\left(0.0466 / (\mathrm{Re}_\delta)^{\frac{1}{4}}\right)$$

$$\frac{7}{72}\ \frac{\partial \delta}{\partial x} = \frac{0.0233}{(\mathrm{Re}_\delta)^{\frac{1}{4}}} = \frac{0.0233}{\left(\frac{U\delta}{\nu}\right)^{\frac{1}{4}}}$$

$$\frac{\partial \delta}{\partial x} = \frac{0.2397}{\left(\frac{U\delta}{\nu}\right)^{\frac{1}{4}}}$$

9.3 Mean-flow characteristics for turbulent boundary layer

Integrate once w.r.t. *x*

$$\delta = \frac{0.2397}{\left(\frac{U\delta}{\nu}\right)^{\frac{1}{4}}} x + C$$

B.C.: $\delta \cong 0$ at $x = 0$

$$0 = \frac{0.2397}{\left(\frac{U\delta}{\nu}\right)^{\frac{1}{4}}} 0 + C \quad \rightarrow \quad C = 0$$

$$\delta = \frac{0.2397}{\left(\frac{U\delta}{\nu}\right)^{\frac{1}{4}}} x$$

$$\delta^{\frac{5}{4}} = \frac{0.2397}{\left(\frac{Ux}{\nu}\right)^{\frac{1}{4}}} x^{\frac{5}{4}}$$

9.3 Mean-flow characteristics for turbulent boundary layer

$$\therefore \delta = \left[\frac{0.2397}{\left(\frac{Ux}{\nu}\right)^{\frac{1}{4}}} x^{\frac{5}{4}} \right]^{\frac{4}{5}} = \frac{(0.2397)^{\frac{4}{5}}}{\left(\frac{Ux}{\nu}\right)^{\frac{1}{5}}} x$$

$$\therefore \delta = \frac{0.318}{\left(\frac{Ux}{\nu}\right)^{\frac{1}{5}}} x = \frac{0.318}{\left(\mathrm{Re}_x\right)^{\frac{1}{5}}} x$$

→ Eq. (C)(9.39)

(9-36): $$c_f = \frac{0.0466}{\left(\mathrm{Re}_\delta\right)^{\frac{1}{4}}} = \frac{0.0466}{\left(\frac{U\delta}{\nu}\right)^{\frac{1}{4}}}$$ (C)

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Substitute (9.39) into (C)

$$\therefore c_f = \frac{0.0466}{\left(\frac{U}{\nu}\right)^{\frac{1}{4}} \left[\frac{0.318}{\left(\frac{Ux}{\nu}\right)^{\frac{1}{5}}}\right]^{\frac{1}{4}}} = \frac{0.062}{\left(\frac{U}{\nu}\right)^{\frac{1}{4}} \left[\frac{x^{\frac{4}{5}}}{\left(\frac{U}{\nu}\right)^{\frac{1}{5}}}\right]^{\frac{1}{4}}}$$

$$= \frac{0.062}{\left(\frac{U}{\nu}\right)^{\frac{1}{4}} \frac{x^{\frac{1}{5}}}{\left(\frac{U}{\nu}\right)^{\frac{1}{20}}}} = \frac{0.062}{\left(\frac{U}{\nu}\right)^{\frac{1}{5}} x^{\frac{1}{5}}} = \frac{0.062}{\left(\frac{Ux}{\nu}\right)^{\frac{1}{5}}} = \frac{0.062}{\left(\mathrm{Re}_x\right)^{\frac{1}{5}}}$$

→ (9.40)

Integrate (9.40) over l

$$C_f = \frac{1}{l}\int_0^l \frac{0.062}{\left(\mathrm{Re}_x\right)^{\frac{1}{5}}}dx = \frac{0.062}{l}\int_0^l \frac{1}{\left(\frac{Ux}{\nu}\right)^{\frac{1}{5}}}dx \, \frac{0.062}{l\left(\frac{U}{\nu}\right)^{\frac{1}{5}}}\int_0^l \frac{1}{x^{\frac{1}{5}}}dx = \frac{0.076}{\left(\mathrm{Re}_l\right)^{\frac{1}{5}}}$$

9.3 Mean-flow characteristics for turbulent boundary layer

9.3.3. Laws for rough walls

(1) Effects of roughness

- Rough walls:

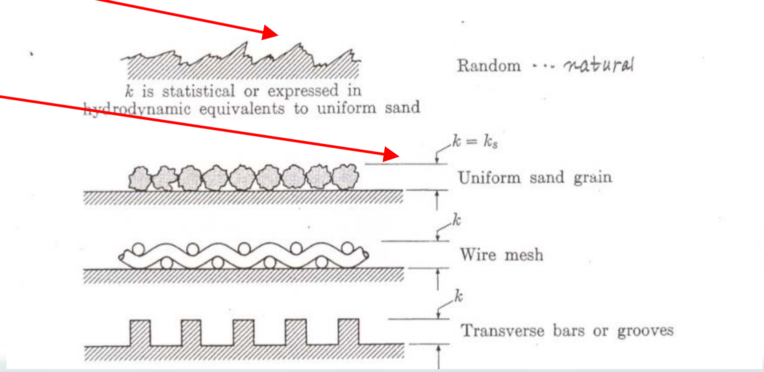
- Velocity distribution and resistance = *f* (Reynolds number , roughness)

- Smooth walls:

- velocity distribution and resistance = *f* (Reynolds number)

- For natural roughness, k is random, and statistical quantity

→ $k = k_s$ = uniform sand grain



9.3 Mean-flow characteristics for turbulent boundary layer

▪ Measurement of roughness effects

a) experiments with sand grains cemented to smooth surfaces - Nikuradse

b) evaluate roughness value ≡ height k_s

c) compare hydrodynamic behavior with other types and magnitude of roughness

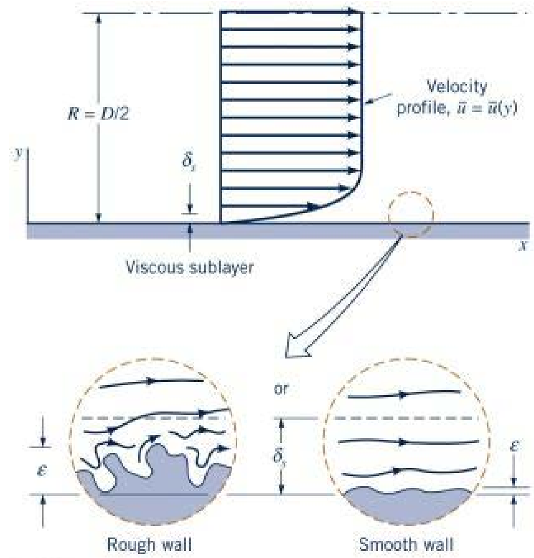
▪ Effects of roughness

i) $\frac{k_s}{\delta'} < 1$

~ roughness has negligible effect on the wall shear

→ hydrodynamically smooth

$\delta' = \frac{4\nu}{u_*}$ = laminar sublayer thickness



9.3 Mean-flow characteristics for turbulent boundary layer

ii) $\frac{k_s}{\delta'} > 1$

~ roughness effects appear

~ roughness disrupts the laminar sublayer

~ smooth-wall relations for velocity and C_f no longer hold

→ hydrodynamically rough

iii) $\frac{k_s}{\delta'}$ > 15 ~25

~ friction and velocity distribution depend only on roughness rather than Reynolds number

→ fully rough flow condition

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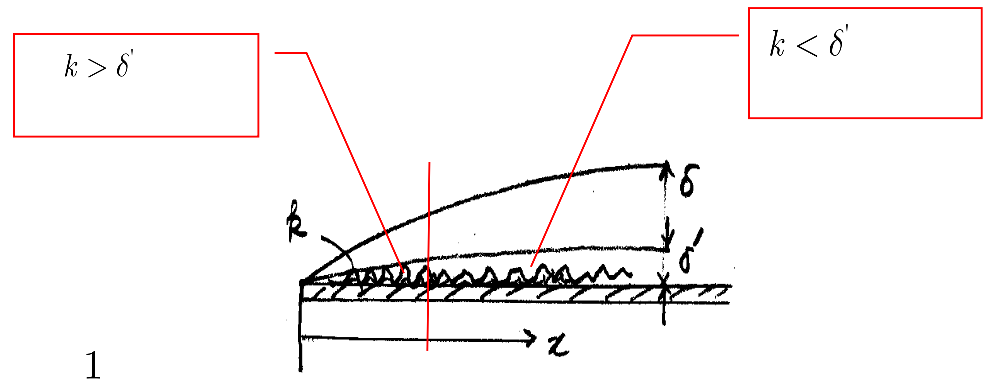
- Critical roughness, k_{crit}



$$k_{crit} = \delta'$$

$$= \frac{4\nu}{u_*} = \frac{4\nu}{U\sqrt{c_f/2}} \propto \mathrm{Re}_x \propto x$$

$$c_f \propto \frac{1}{\mathrm{Re}_x}$$

If x increases, then c_f decreases, and δ' increases.

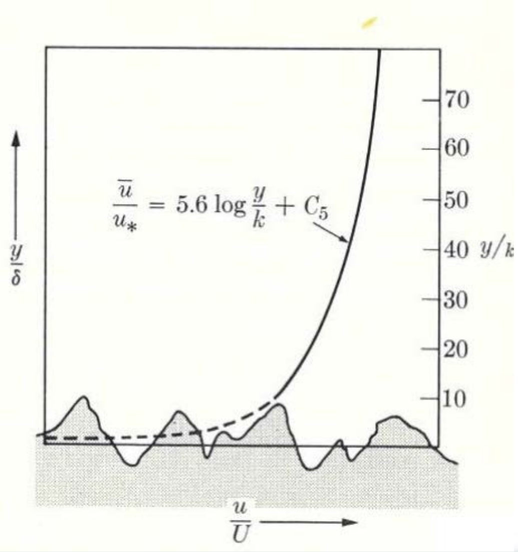
Therefore, for a surface of uniform roughness, it is possible to be hydrodynamically rough upstream, and hydrodynamically smooth downstream.

9.3 Mean-flow characteristics for turbulent boundary layer

(2) Rough-wall velocity profiles

Assume roughness height *k* accounts for magnitude, form, and distribution of the roughness. Then

$$\frac{\overline{u}}{u_*} = f\left(\frac{y}{k}\right) \tag{9.42}$$



9.3 Mean-flow characteristics for turbulent boundary layer

Make *f* in Eq. (9.42) be a logarithmic function to overlap the velocity-defect law, Eq. (9.16), which is applicable for both rough and smooth boundaries.

(9.16): $\dfrac{U-\bar{u}}{u_*} = 5.6\log\left(\dfrac{y}{\delta}\right) + 2.5, \quad \dfrac{y}{\delta} < 0.15$

i) For rough walls, in the wall region

$$\frac{\bar{u}_{rough}}{u_*} = -5.6\log\left(\frac{k}{y}\right) + C_5, \quad \frac{u_* y}{\nu} > 50 \sim 100, \quad \frac{y}{\delta} < 0.15 \qquad (9.43)$$

where C_5 = constant = *f* (size, shape, distribution of the roughness)

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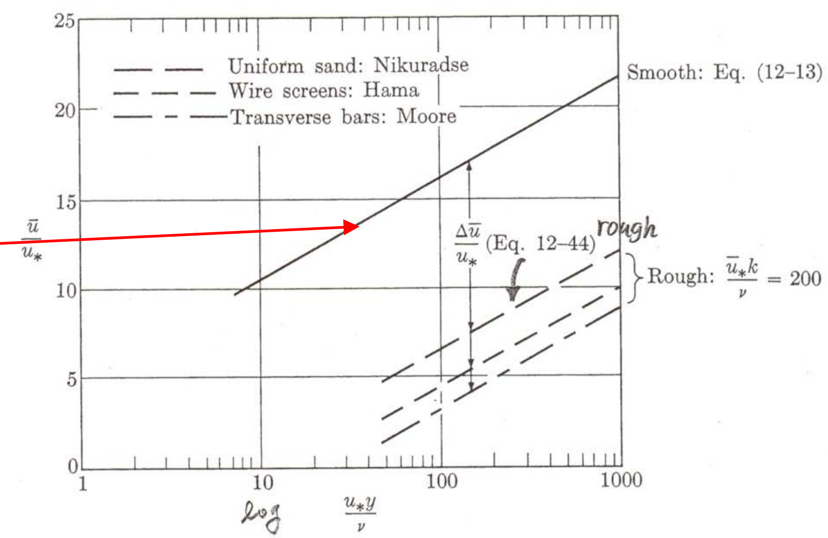
ii) For smooth walls, in the wall region

(9.13):

$$\frac{\overline{u}_{smooth}}{u_*} = 5.6\log\left(\frac{u_* y}{\nu}\right) + C_2, \quad \frac{u_* y}{\nu} > 30 \sim 70, \quad \frac{y}{\delta} < 0.15$$

where $C_2 = 4.9$



Subtract Eq. (9.43) from Eq. (9.13)

$$\frac{\Delta\overline{u}}{u_*} = \frac{\overline{u}_{smooth} - \overline{u}_{rough}}{u_*} = 5.6\log\left(\frac{u_* k}{\nu}\right) + C_6 \quad (9.44)$$

→ <u>Roughness reduces the local mean velocity</u> $\overline{u}$ in the wall region

where C_5 and C_6 → Table 9-4

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TABLE 12–4

VALUES OF CONSTANTS IN ROUGH-WALL EQUATIONS FOR THE WALL REGION

($y/\delta < 0.15$; $u_* k/\nu > 50$ to 100)

Roughness type	Source of data	C_5, Eq. (12–43)	C_6, Eq. (12–44)	C_8, Eq. (12–46)
Uniform sand grains	Nikuradse [11] (pipes)	8.2	−3.3	7.55
Wire screens	Hama [12] (plates)	6.1	−1.2	6.1
Transverse bars	Moore [7] (plates)	4.9	0	5.25

Eq. (12–43): $\bar{u}/u_* = -5.6 \log (k/y) + C_5,$

Eq. (12–44): $\Delta\bar{u}/u_* = 5.6 \log (u_* k/\nu) + C_6,$

Eq. (12–46): $1/\sqrt{c_f} = 3.96 \log (\delta/k) + C_8.$

(Constants in this table were evaluated graphically from Fig. 12–12.)

9.3 Mean-flow characteristics for turbulent boundary layer

(3) Surface-resistance formulas: rough walls

Combine Eqs. (9.43) and (9.16)

$$\frac{U-\overline{u}}{u_*} = -5.6\log\left(\frac{y}{\delta}\right) + 2.5 \ , \quad \frac{y}{\delta} < 0.15$$

$$+\left| \ \frac{\overline{u}}{u_*} = -5.6\log\left(\frac{k}{y}\right) + C_5 \right.$$

$$\rightarrow \quad \frac{U}{u_*} = 5.6\log\left(\frac{\delta}{k}\right) + C_7$$

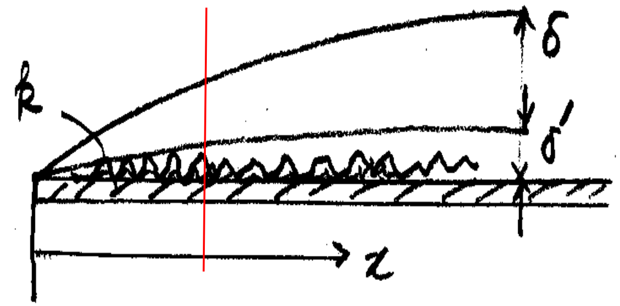
$$\therefore \frac{U}{u_*} = \sqrt{\frac{2}{c_f}} = 5.6\log\left(\frac{\delta}{k}\right) + C_7 \qquad (9.45)$$

$$\frac{1}{\sqrt{c_f}} = 3.96\log\left(\frac{\delta}{k}\right) + C_8 \qquad (9.46)$$

9.3 Mean-flow characteristics for turbulent boundary layer

[Ex. 9.3] Rough wall velocity distribution and local skin friction coefficient

- Comparison of the boundary layers on a smooth plate and a plate roughened by sand grains



- Given: $\tau_0 = 0.485\ lb/ft^2$ on both plates

  U = 10 ft/sec past the <u>rough plate</u>

  k_s = 0.001 ft

Water temp. = 58 °F on both plates

9.3 Mean-flow characteristics for turbulent boundary layer

(a) Velocity reduction Δu due to roughness

From Table 1-3:

$$\rho = 1.938 \;\; slug / ft^3 \; ; \quad \nu = 1.25 \times 10^{-5} ft^2 / \sec$$

Eq. (9.18)

$$\therefore u_* = \sqrt{\frac{\tau_0}{\rho}} = \sqrt{\frac{0.485}{1.938}} = 0.5 ft / \sec$$

$$c_f = 2\left(\frac{u_*}{U}\right)^2 = 2\left(\frac{0.5}{10}\right)^2 = 0.005$$

$$\frac{u_* k_s}{\nu} = \frac{0.5(0.001)}{1.25 \times 10^{-5}} = 40$$

Eq. (9.44): $$\frac{\Delta u}{u_*} = 5.6 \log\left(\frac{u_* k_s}{\nu}\right) - 3.3$$

$$\therefore \Delta u = 0.5\{5.6 \log 40 - 3.3\} = \underline{2.83 ft / \sec}$$

9.3 Mean-flow characteristics for turbulent boundary layer

(b) Velocity $\overline{u}$ on each plate at y =0.007 *ft*

i) For rough plate

Eq. (9.43): $\dfrac{\overline{u}}{u_*} = 5.6\log\dfrac{y}{k} + 8.2$

$$\therefore \overline{u} = 0.5(5.6\log\frac{0.007}{0.001} + 8.2) = \underline{6.47\,ft/\sec}$$

ii) For smooth plate,

Eq. (9.13): $\dfrac{\overline{u}}{u_*} = 5.6\log\dfrac{u_* y}{\nu} + 4.9$

$$\frac{u_* y}{\nu} = \frac{0.5(0.007)}{1.25\times 10^{-5}} = 280$$

$$\therefore \overline{u} = 0.5\{5.6\log(280) + 4.9\} = \underline{9.3\,ft/\sec}$$

Check $\Delta\overline{u} = 9.3 - 6.47 = 2.83$ → same result as (a)

9.3 Mean-flow characteristics for turbulent boundary layer

(c) Boundary layer thickness δ on the rough plate

Eq. (9.46):

$$\frac{1}{\sqrt{c_f}} = 3.96 \log \frac{\delta}{k_s} + 7.55$$

$$\therefore \log \frac{\delta}{k_s} = \frac{1}{3.96}\left\{\frac{1}{\sqrt{0.005}} - 7.55\right\} = 1.66$$

$$\therefore \frac{\delta}{k_s} = 46 \rightarrow \delta = 0.046ft = 0.52in = 1.4cm$$