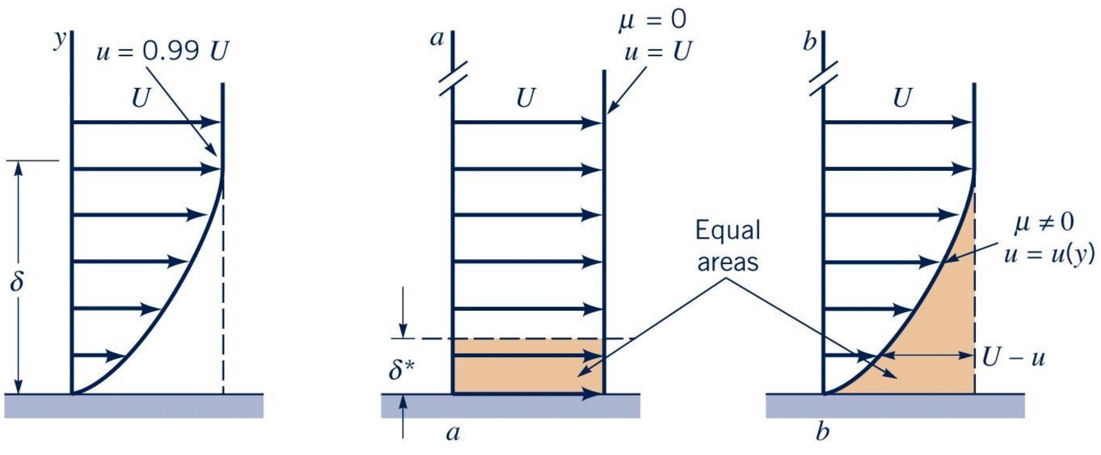
Lecture 25

Turbulent Boundary-Layer Flows (3)



Lecture 25 Turbulent Boundary-Layer Flows (3)

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25.2 Laws for Rough Walls

Objectives

- Study wall turbulence
- Derive equations of velocity distribution and friction coefficient for both smooth and rough walls

25.1 Power Law Fomulas for Smooth Walls

25.1.1 Power-law formulas: Smooth walls

- Logarithmic equations for velocity profile and shear-stress coeff .

~ universal

~ applicable over almost entire range of Reynolds numbers

- Power-law equations

~ applicable over only limited range of Reynolds numbers

~ simpler

~ explicit relations for $\bar{u}/U$ and c_f

~ explicit relations for δ in terms of *Re* and distance x

25.1 Power Law Fomulas for Smooth Walls

- Assumptions of power-law formulas

The power laws stem from two facts that hold for turbulent boundary layers with negligible pressure gradients when $\mathrm{Re}_\delta = \frac{U\delta}{\nu} < 5\times10^5$

i) Except very near the wall, mean velocity is closely proportional to a root of the distance *y* from the wall.

$$\bar{u} \propto y^{\frac{1}{n}} \qquad \text{(A)}$$

ii) Shear stress coeff. c_f is inversely proportional to a root of Re_δ

$$c_f \propto \frac{1}{\mathrm{Re}_\delta^m} \;,\quad \mathrm{Re}_\delta = \frac{U\delta}{\nu}$$

$$c_f = \frac{A}{\left(\frac{U\delta}{\nu}\right)^m} \qquad (25.1)$$

where *n, A* = constants; *m* = fraction

25.1 Power Law Fomulas for Smooth Walls

[Cf] Eq. (25.1) is similar to equation for laminar boundary layer, $c_f = \dfrac{3.32}{\mathrm{Re}_\delta}$

- Derivation of power equation

Combine Eqs. (24.16) and (25.1)

$$c_f = \frac{A}{\left(\dfrac{U\delta}{\nu}\right)^m}$$

(24.16): $u_* = U\sqrt{\dfrac{c_f}{2}}$

$$\therefore \frac{U}{u_*} = \sqrt{\frac{2}{c_f}} = \frac{\sqrt{2}}{\sqrt{A}}\left(\frac{U\delta}{\nu}\right)^{\frac{m}{2}}$$

$$\frac{U^{1-\frac{m}{2}}}{u_*} = \frac{\sqrt{2}}{\sqrt{A}}\left(\frac{\delta}{\nu}\right)^{\frac{m}{2}}$$

$$\left(\frac{U}{u^*}\right)^{1-\frac{m}{2}} = \sqrt{\frac{2}{A}}\left(\frac{u_*\delta}{\nu}\right)^{\frac{m}{2}}$$

$$\therefore \frac{U}{u_*} = B\left(\frac{u_*\delta}{\nu}\right)^{\frac{m}{2-m}} \qquad (25.2)$$

25.1 Power Law Fomulas for Smooth Walls

Assume $\bar{u}$ depends on y by the same relation, Eq. (25.2), replacing δ with y

$$\frac{\bar{u}}{u_*} = B\left(\frac{u_* y}{\nu}\right)^{\frac{m}{2-m}} \qquad (25.3)$$

Divide (25.3) by (25.2)

$$\frac{\bar{u}}{U} = \left(\frac{y}{\delta}\right)^{\frac{m}{2-m}} \qquad (25.4)$$

This result indicates that all profiles are similar and can be represented by a single dimensionless curve like laminar boundary layer profile.

However, turbulent profiles are not truly similar, so Eq. (25.4) will apply for different Reynolds number ranges only if the constant m is varied.

25.1 Power Law Fomulas for Smooth Walls

Boundary-layer measurements shows that

For 3,000 < Re_δ < 70,000; $m=\frac{1}{4}$, $A=0.0466$, $B=8.74$

$$\frac{\bar{u}}{U}=\left(\frac{y}{\delta}\right)^{\frac{1}{7}} \quad (25.5)$$

$$\frac{U}{u_*}=8.74\left(\frac{u_*\delta}{\nu}\right)^{\frac{1}{7}} \quad (25.6)$$

$$\frac{\bar{u}}{u_*}=8.74\left(\frac{u_* y}{\nu}\right)^{\frac{1}{7}} \quad (25.7)$$

$$c_f=\frac{0.0466}{Re_\delta^{\frac{1}{4}}} \quad (25.8)$$

25.1 Power Law Fomulas for Smooth Walls

[Re] The Blasius solution for laminar boundary layer flows

For steady laminar flow over a flat plate with zero pressure gradient, Prandtl's (1904) 2-D boundary-layer equations become as follows:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (25.9)$$

Blasius (1908) obtained the solution to above PDE by assuming similar profiles along the plate at every *x*

$$\frac{u}{U}=F\left(\frac{y}{\delta}\right) \qquad (25.10)$$

$$\delta \sim \frac{x}{\mathrm{Re}_x^{\;1/2}} \qquad (25.11)$$

Blasius obtained the solution in the form of power series after he introduced a stream function for $\frac{y}{\delta}$

Blasius, H. (1883-1970): Prandtl's student

25.1 Power Law Fomulas for Smooth Walls

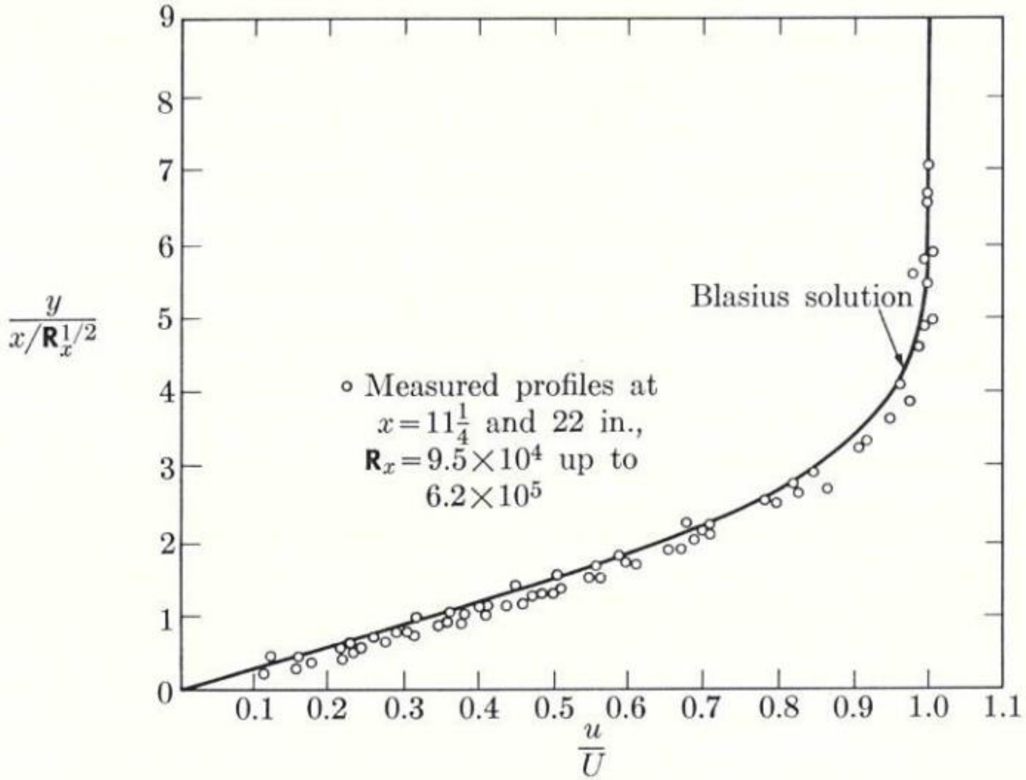


FIG. 10–4. Velocity profiles in a laminar boundary layer on a flat plate [2].

25.1 Power Law Fomulas for Smooth Walls

- Relation for δ

Adopt integral-momentum eq. for steady motion with $\frac{\partial p}{\partial x} = 0$

~ integration of Prandtl's 2-D boundary-layer equations

$$\frac{\tau_0}{\rho} = U^2 \frac{\partial \theta}{\partial x} = c_f \frac{U^2}{2} \leftarrow \tau_0 = \frac{\rho}{2} c_f U^2 \qquad (25.12)$$

where θ = momentum thickness

$$\theta = \int_0^h \frac{\bar{u}}{U}\left(1 - \frac{\bar{u}}{U}\right) dy \qquad (A)$$

Substitute Eq. (25.5) into (A) and integrate

25.1 Power Law Fomulas for Smooth Walls

$$\theta = \int_0^h \left(\frac{y}{\delta}\right)^{\frac{1}{7}} \left\{1 - \left(\frac{y}{\delta}\right)^{\frac{1}{7}}\right\} dy$$

$$= \frac{7}{72}\delta \approx 0.1\delta \qquad (25.13)$$

Substitute Eqs. (25.8) and (25.13) into (25.12) and integrate w.r.t. *x*

$$\delta = \frac{0.318x}{\left(\frac{Ux}{\nu}\right)^{\frac{1}{5}}} = \frac{0.318x}{\mathrm{Re}_x^{\frac{1}{5}}} \quad , \ \mathrm{Re}_x < 10^7 \qquad (25.14)$$

$$c_f = \frac{0.059}{\mathrm{Re}_x^{\frac{1}{5}}} \quad , \ \mathrm{Re}_x < 10^7 \qquad (25.15)$$

Integrate (25.15) over whole length, *l*, to get average coefficient

$$C_f = \frac{0.074}{\mathrm{Re}_l^{\frac{1}{5}}} \quad , \ \mathrm{Re}_l < 10^7 \qquad (25.16)$$

25.1 Power Law Fomulas for Smooth Walls

[Re] Derivation of (25.14) and (25.15)

$$U^2 \frac{\partial \theta}{\partial x} = c_f \frac{U^2}{2}$$

$$\frac{\partial \theta}{\partial x} = \frac{c_f}{2} \qquad \text{(B)}$$

Substitute (25.13) and (25.8) into (B)

$$\frac{\partial}{\partial x}\left(\frac{7}{72}\delta\right) = \frac{1}{2}\left(0.0466 \,/\, (\mathrm{Re}_\delta)^{\frac{1}{4}}\right)$$

$$\frac{7}{72}\frac{\partial \delta}{\partial x} = \frac{0.0233}{\mathrm{Re}_\delta^{\ \frac{1}{4}}} = \frac{0.0233}{\left(\dfrac{U\delta}{\nu}\right)^{\frac{1}{4}}}$$

$$\frac{\partial \delta}{\partial x} = \frac{0.2397}{\left(\dfrac{U\delta}{\nu}\right)^{\frac{1}{4}}}$$

25.1 Power Law Fomulas for Smooth Walls

Integrate once w.r.t. *x*

$$\delta = \frac{0.2397}{\left(\frac{U\delta}{\nu}\right)^{\frac{1}{4}}} x + C$$

B.C.: $\delta \cong 0$ at $x = 0$

$$0 = \frac{0.2397}{\left(\frac{U\delta}{\nu}\right)^{\frac{1}{4}}} 0 + C \quad \rightarrow \quad C = 0$$

$$\delta = \frac{0.2397}{\left(\frac{U\delta}{\nu}\right)^{\frac{1}{4}}} x$$

$$\delta^{\frac{5}{4}} = \frac{0.2397}{\left(\frac{Ux}{\nu}\right)^{\frac{1}{4}}} x^{\frac{5}{4}}$$

25.1 Power Law Fomulas for Smooth Walls

$$\therefore \delta = \left\{ \frac{0.2397}{\left(\frac{Ux}{\nu}\right)^{\frac{1}{4}}} x^{\frac{5}{4}} \right\}^{\frac{4}{5}} = \frac{(0.2397)^{\frac{4}{5}}}{\left(\frac{Ux}{\nu}\right)^{\frac{1}{5}}} x$$

$$\therefore \delta = \frac{0.318}{\left(\frac{Ux}{\nu}\right)^{\frac{1}{5}}} x = \frac{0.318}{\mathrm{Re}_x^{\frac{1}{5}}} x$$

→ Eq. (C)(25.14)

(25.17): $c_f = \frac{0.0466}{\mathrm{Re}_\delta^{\frac{1}{4}}} = \frac{0.0466}{\left(\frac{U\delta}{\nu}\right)^{\frac{1}{4}}}$ (C)

25.1 Power Law Fomulas for Smooth Walls

Substitute (25.14) into (C)

$$\therefore c_f = \frac{0.0466}{\left(\frac{U}{\nu}\right)^{\frac{1}{4}}\left[\frac{0.318}{\left(\frac{Ux}{\nu}\right)^{\frac{1}{5}}}\right]^{\frac{1}{4}}} = \frac{0.062}{\left(\frac{U}{\nu}\right)^{\frac{1}{4}}\left[\frac{x^{\frac{4}{5}}}{\left(\frac{U}{\nu}\right)^{\frac{1}{5}}}\right]^{\frac{1}{4}}}$$

$$= \frac{0.062}{\left(\frac{U}{\nu}\right)^{\frac{1}{4}}\frac{x^{\frac{1}{5}}}{\left(\frac{U}{\nu}\right)^{\frac{1}{20}}}} = \frac{0.062}{\left(\frac{U}{\nu}\right)^{\frac{1}{5}}x^{\frac{1}{5}}} = \frac{0.062}{\left(\frac{Ux}{\nu}\right)^{\frac{1}{5}}} = \frac{0.062}{\mathrm{Re}_x^{\frac{1}{5}}} \quad \rightarrow (25.15)$$

Integrate (25.15) over l

$$C_f = \frac{1}{l}\int_0^l \frac{0.062}{\mathrm{Re}_x^{\frac{1}{5}}}dx = \frac{0.062}{l}\int_0^l \frac{1}{\left(\frac{Ux}{\nu}\right)^{\frac{1}{5}}}dx \, \frac{0.062}{l\left(\frac{U}{\nu}\right)^{\frac{1}{5}}}\int_0^l \frac{1}{x^{\frac{1}{5}}}dx = \frac{0.076}{\mathrm{Re}_l^{\frac{1}{5}}}$$

25.2 Laws for Rough Walls

25.2.1 Effects of roughness

❖ **Effects of roughness**

▪ Rough walls:

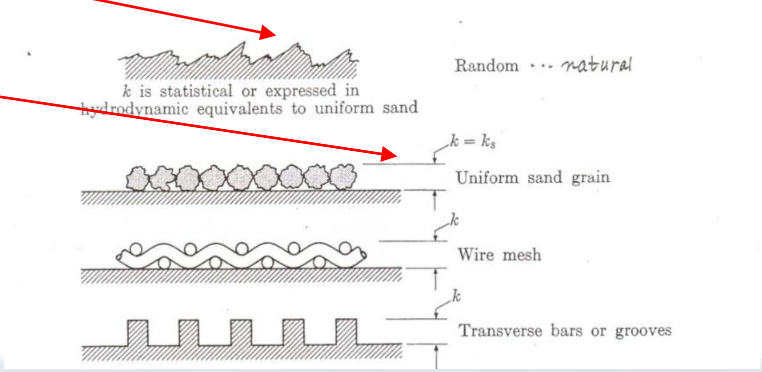
- Velocity distribution and resistance = *f* (Reynolds number, roughness)

[Cf] Smooth walls:

- velocity distribution and resistance = *f* (Reynolds number)

▪ For natural roughness, *k* is random, and statistical quantity

$\rightarrow \quad k = k_s$ = uniform sand grain



25.2 Laws for Rough Walls

- Measurement of roughness effects

  a) experiments with sand grains cemented to smooth surfaces - Nikuradse

  b) evaluate roughness value ≡ height k_s

  c) compare hydrodynamic behavior with other types and magnitude of roughness

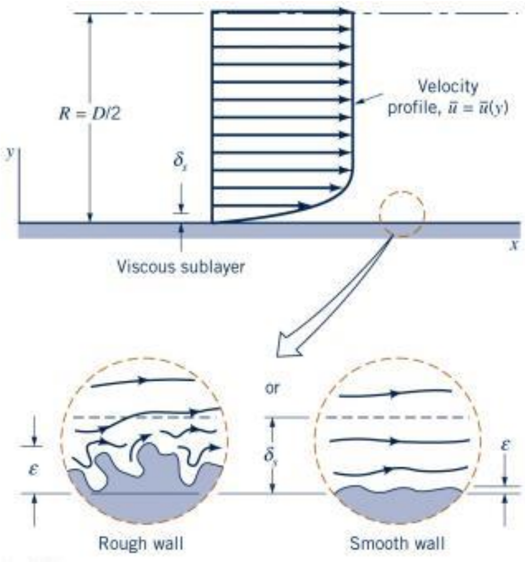
- Effects of roughness

i) $\frac{k_s}{\delta'} < 1$

~ roughness has negligible effect on the wall shear

→ <u>hydrodynamically smooth</u>

$\delta' = \frac{4\nu}{u_*}$ = laminar sublayer thickness



25.2 Laws for Rough Walls

ii) $\frac{k_s}{\delta'} > 1$

~ roughness effects appear

~ roughness disrupts the laminar sublayer

~ smooth-wall relations for velocity and C_f no longer hold

→ hydrodynamically rough

iii) $\frac{k_s}{\delta'}$ > 15 ~25

~ friction and velocity distribution depend only on roughness rather than Reynolds number

→ fully rough flow condition

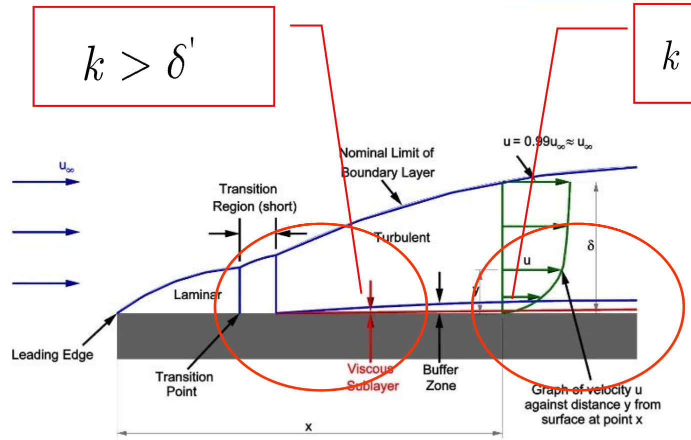
25.2 Laws for Rough Walls

- Critical roughness, k_{crit}

$$k_{crit} = \delta'$$

$$= \frac{4\nu}{u_*} = \frac{4\nu}{U\sqrt{c_f / 2}} \propto \mathrm{Re}_x \propto x$$

$$c_f \propto \frac{1}{\mathrm{Re}_x}$$



If x increases, then c_f decreases, and δ' increases.

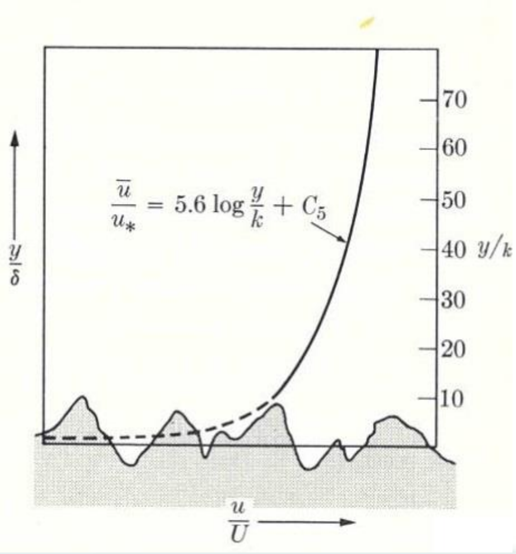
Therefore, for a surface of uniform roughness, it is possible to be hydrodynamically rough upstream, and hydrodynamically smooth downstream.

25.2 Laws for Rough Walls

(2) Rough-wall velocity profiles

Assume roughness height k accounts for magnitude, form, and distribution of the roughness. Then

$$\frac{\bar{u}}{u_*} = f\left(\frac{y}{k}\right) \qquad (25.18)$$



25.2 Laws for Rough Walls

Make *f* in Eq. (25.18) be a logarithmic function to overlap the velocity-defect law, Eq. (24.14), which is applicable for both rough and smooth boundaries.

(24.14): $$\frac{U - \bar{u}}{u_*} = 5.6 \log\left(\frac{y}{\delta}\right) + 2.5, \quad \frac{y}{\delta} < 0.15$$

i) For rough walls, in the wall region

$$\frac{\bar{u}_{rough}}{u_*} = -5.6 \log\left(\frac{k}{y}\right) + C_5, \quad \frac{u_* y}{\nu} > 50 \sim 100, \quad \frac{y}{\delta} < 0.15 \qquad (25.19)$$

where C_5 = constant = *f* (size, shape, distribution of the roughness)

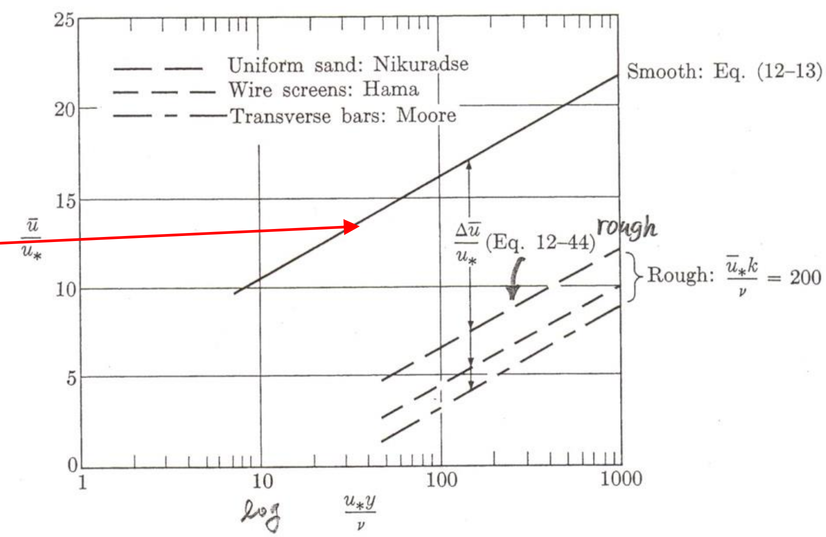
25.2 Laws for Rough Walls

ii) For smooth walls, in the wall region

(24.11):

$$\frac{\bar{u}_{smooth}}{u_*} = 5.6\log\left(\frac{u_* y}{\nu}\right) + C_2, \quad \frac{u_* y}{\nu} > 30 \sim 70, \quad \frac{y}{\delta} < 0.15$$

where C_2 = 4.9



Subtract Eq. (25.19) from Eq. (24.11)

$$\frac{\Delta\bar{u}}{u_*} = \frac{\bar{u}_{smooth} - \bar{u}_{rough}}{u_*} = 5.6\log\left(\frac{u_* k}{\nu}\right) + C_6 \qquad (25.20)$$

→ <u>Roughness reduces the local mean velocity</u> $\bar{u}$

in the wall region

where C_5 and C_6 → Table 9-4 (D&H)

25.2 Laws for Rough Walls

TABLE 12–4

Values of Constants in Rough-Wall Equations for the Wall Region

($y/\delta < 0.15$; $u_* k/\nu > 50$ to 100)

Roughness type	Source of data	C_5, Eq. (12–43)	C_6, Eq. (12–44)	C_8, Eq. (12–46)
Uniform sand grains	Nikuradse [11] (pipes)	8.2	−3.3	7.55
Wire screens	Hama [12] (plates)	6.1	−1.2	6.1
Transverse bars	Moore [7] (plates)	4.9	0	5.25

Eq. (12–43): $\bar{u}/u_* = -5.6 \log (k/y) + C_5$,

Eq. (12–44): $\Delta\bar{u}/u_* = 5.6 \log (u_* k/\nu) + C_6$,

Eq. (12–46): $1/\sqrt{c_f} = 3.96 \log (\delta/k) + C_8$.

(Constants in this table were evaluated graphically from Fig. 12–12.)

25.2 Laws for Rough Walls

(3) Surface-resistance formulas: rough walls

Combine Eqs. (25.19) and (24.14)

$$\frac{U-\bar{u}}{u_*} = -5.6\log\left(\frac{y}{\delta}\right) + 2.5 \ , \quad \frac{y}{\delta} < 0.15$$

$$+ \left| \ \frac{\bar{u}}{u_*} = -5.6\log\left(\frac{k}{y}\right) + C_5 \right.$$

$$\rightarrow \quad \frac{U}{u_*} = \ 5.6\log\left(\frac{\delta}{k}\right) + C_7$$

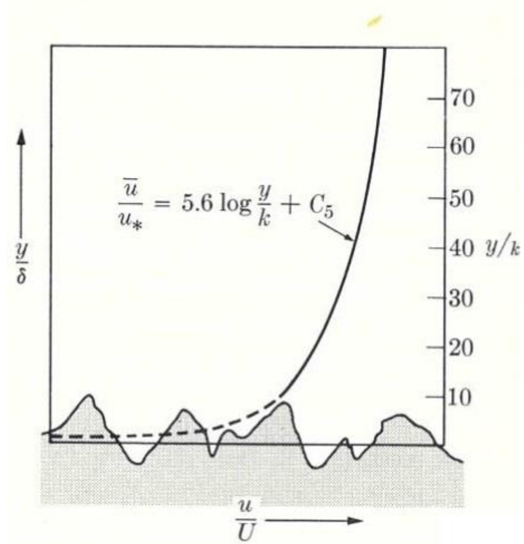
$$\therefore \frac{U}{u_*} = \sqrt{\frac{2}{c_f}} = 5.6\log\left(\frac{\delta}{k}\right) + C_7 \qquad (25.21)$$

$$\boxed{\frac{1}{\sqrt{c_f}} = 3.96\log\left(\frac{\delta}{k}\right) + C_8} \qquad (25.22)$$

25.2 Laws for Rough Walls

[Ex 25.1] Rough wall velocity distribution and local skin friction coefficient

- Comparison of the boundary layers on a smooth plate and a plate roughened by sand grains



▪ Given: $\tau_0 = 0.485\ lb/ft^2$ on both plates

U = 10 ft/sec past the rough plate

k_s = 0.001 ft

Water temp. = 58 °F on both plates

25.2 Laws for Rough Walls

(a) Velocity reduction Δu due to roughness

From Table 1-3 (D&H):

$$\rho = 1.938 \;\; slug / ft^3; \quad \nu = 1.25 \times 10^{-5} ft^2 / \sec$$

Eq. (24.16)

$$\therefore u_* = \sqrt{\frac{\tau_0}{\rho}} = \sqrt{\frac{0.485}{1.938}} = 0.5 ft / \sec$$

$$c_f = 2\left(\frac{u_*}{U}\right)^2 = 2\left(\frac{0.5}{10}\right)^2 = 0.005$$

$$\frac{u_* k_s}{\nu} = \frac{0.5(0.001)}{1.25 \times 10^{-5}} = 40$$

Eq. (25.20): $\dfrac{\Delta u}{u_*} = 5.6 \log\left(\dfrac{u_* k_s}{\nu}\right) - 3.3$

$$\therefore \Delta u = 0.5\{5.6 \log 40 - 3.3\} = \underline{2.83 ft / \sec}$$

25.2 Laws for Rough Walls

(b) Velocity $\overline{u}$ on each plate at y =0.007 ft

i) For rough plate

Eq. (25.19): $\dfrac{\overline{u}}{u_*} = 5.6\log\dfrac{y}{k} + 8.2$

$$\therefore\ \overline{u} = 0.5(5.6\log\frac{0.007}{0.001} + 8.2) = \underline{6.47\,ft\,/\,\sec}$$

ii) For smooth plate,

Eq. (24.11): $\dfrac{\overline{u}}{u_*} = 5.6\log\dfrac{u_* y}{\nu} + 4.9$

$$\frac{u_* y}{\nu} = \frac{0.5(0.007)}{1.25\times10^{-5}} = 280$$

$$\therefore\ \overline{u} = 0.5\{5.6\log(280) + 4.9\} = \underline{9.3\,ft\,/\,\sec}$$

Check $\Delta\overline{u} = 9.3 - 6.47 = 2.83 \longrightarrow$ same result as (a)

25.2 Laws for Rough Walls

(c) Boundary layer thickness δ on the rough plate

Eq. (25.22):

$$\frac{1}{\sqrt{c_f}} = 3.96 \log \frac{\delta}{k_s} + 7.55$$

$$\therefore \log \frac{\delta}{k_s} = \frac{1}{3.96}\left\{\frac{1}{\sqrt{0.005}} - 7.55\right\} = 1.66$$

$$\therefore \frac{\delta}{k_s} = 46 \rightarrow \delta = 0.046 ft = 0.52 in = 1.4 cm$$