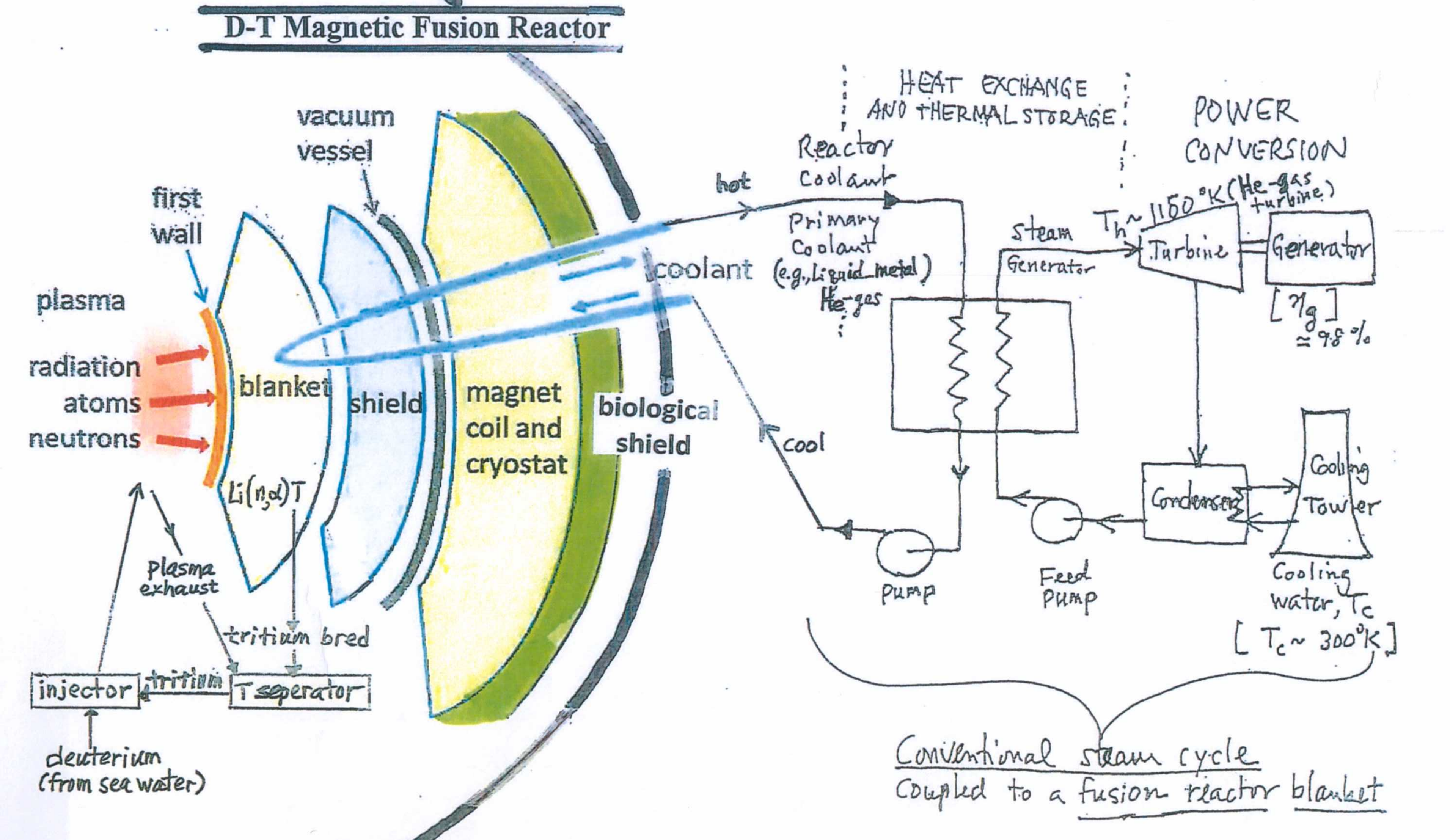
Fusion Reactor Technology 2
9 Sept 2019

Fusion Technology Issues



Conventional steam cycle
Coupled to a fusion reactor blanket

Engineering Problems / Technology Issues :

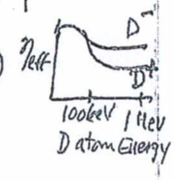
1. Magnets : e.g., In a toroidal device, $B = \dfrac{\mu_0 N I}{2\pi R}$ | $N = 20$ coils, $R = 2m$, $B = 3T$

$$\rightarrow I = \frac{2\pi \times 2 \times 3}{4\pi \times 10^{-7} \times 20} = 1.5\,MA$$

2. Plasma Heating :

   Ohmic heating : $P_{OH} = \vec{E}\cdot\vec{J} \underset{\vec{E}=\eta\vec{J}}{=} \eta J^2 \left(\frac{W}{m^3}\right)$, $\eta \sim T^{-3/2}$

   Neutral beam injection : $dW/dt \sim f(W_{beam})$



   wave heating : $\omega_{ce}, \omega_{ci}, \omega_{LH}, \omega_U$

   compressional heating : $pV^\gamma = const$, In tokamak, $a^2 B_t = const$, $B_t = B_0 R_0/R \rightarrow a^2/R = const.$

   Fusion-product heating : α-pcle slowing down

3. Materials :

   First wall : heat flux, neutron flux, high temperature, T-breeding material

   Blanket : structural integrity, tritium breeding capability, heat transfer requirement

   Shield : heat flux for magnets, biological shield against γ.

4. Vacuum : ultra high vacuum, $P \lesssim 1\,mPa$ $(1\,Pa \sim 10^{-5}\,atm \sim 7.5\,mTorr)$

   $P \sim 10^{-5}\,Pa$ $(7.5\times10^{-8}\,Torr)$

5. cryogenics : $T \sim 4°K$ (liq He), $T \sim 77°K$ (liq N)

6. Safety & Environment

7. Power Plant Designs

8. Fusion - Fission Hybrids

Fundamentals of Engineering Design

Good Engineer needs :

1. factual knowledge — basic physical sciences, computer, etc.
*2. skills
3. attitudes — role of education
4. capability of self-improvement

* SKILLS —

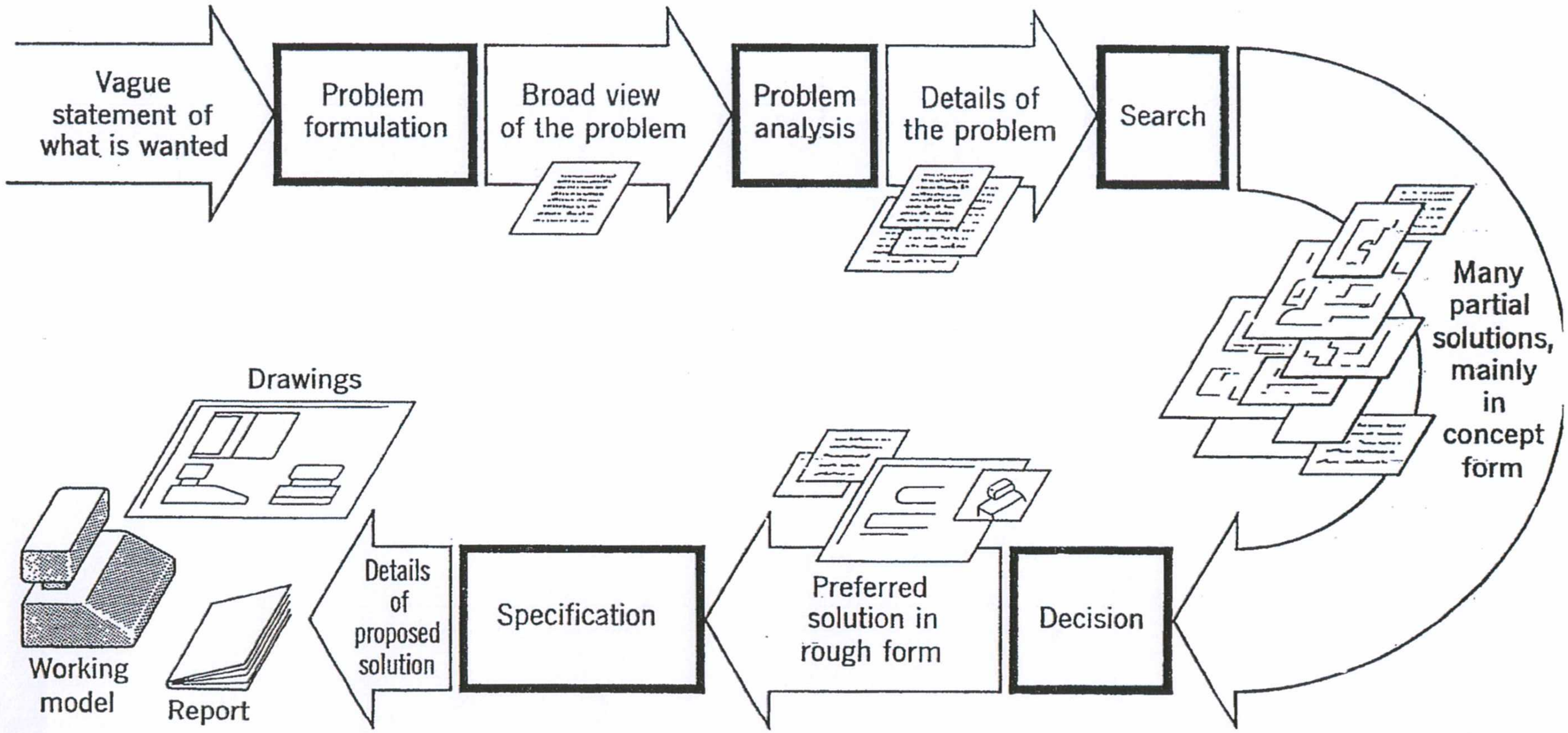
Problem ① solving ⟶ $\boxed{\text{Design}}$ § —
- ⓐ Define / Formulate problem
- ⓑ Problem Analysis
- ⓒ Search for Alternate Solutions
- ⓓ Decision phase (cf. Aesthetics)
- ⓔ Specifications of a solution
- [ⓕ Optimization]

② modeling ⟶ $\boxed{\text{Models}}$ —
- ⓐ Simulation
- ⓑ Numerical Methods
- ⓒ Modelling

③ optimization ⟶ $\boxed{\text{Optimization}}$ — ⓐ iteration (computer usage)

§ Alternate processes:
- ⓐ Recognizing the Need
- ⓑ Defining the Problem
- ⓒ Planning the Project
- ⓓ Gathering Information
- ⓔ Conceptualizing Alternate approaches
- ⓕ Evaluating the Alternatives
- ⓖ Selecting the Preferred Alternative
- ⓗ Communicating the Design
- ⓘ Implementing the Preferred Design

SKILLS : DESIGN PROCESS



§1.2* How can we make Fusion Reactors ? [*Dolan, MFT (2013)]

$kT_{sun} \simeq 1.3\ keV\ (15 \times 10^6\ {}^\circ K)$ under "gravity" ;

[∵ Room Temp, $300\,{}^\circ K \simeq \frac{1}{40}\ eV$

$1\ keV = 40 \times 300 \times 1000\,{}^\circ K = 12 \times 10^6\ {}^\circ K$]

[But $kT_{\text{Fusion reactor Plasma}} \simeq 10\ keV\ (116 \times 10^6\ {}^\circ K)$ under "magnetic pressure"

and $kT_{\text{reactor wall}} \simeq 1000\ {}^\circ K\ (\sim 0.1\ eV)$]

[1 atm = 14.7 psi

1 atm = 1.013×10^5 Pa

e.g., $\boxed{10\ atm = 1\ MPa}$]

Notation: $kT \equiv T$ (energy in keV)

"Kinetic temperature"

D-T (Deuterium-Tritium) Fusion Plant :

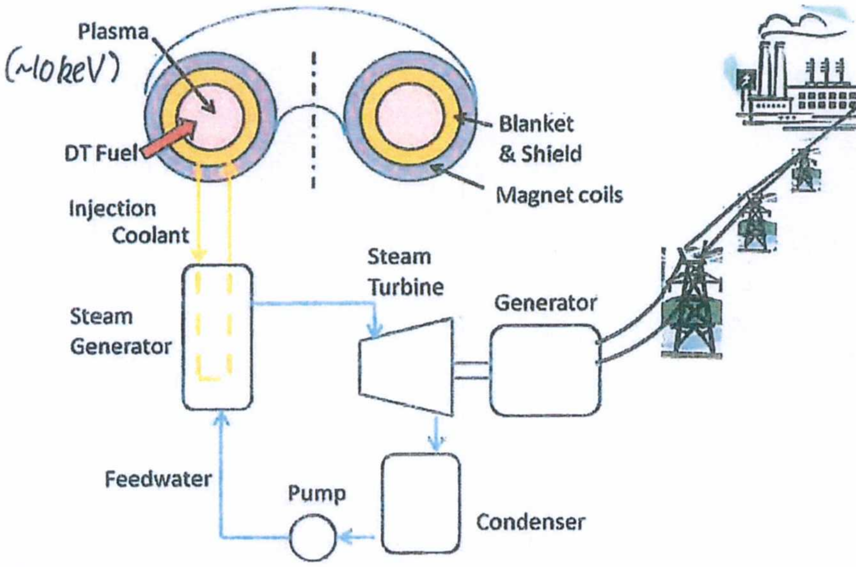
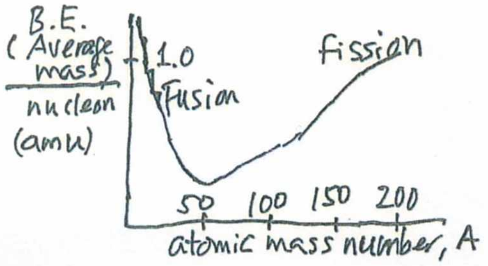


Fig. 1.4 Simplified cutaway view of a fusion reactor connected to a Rankine cycle (steam turbine cycle) to generate electricity. *DT* means "deuterium–tritium"

$${}^2_1D^+ + {}^3_1T^+ \rightarrow {}^1_0n + {}^4_2He^{++} + \underbrace{17.6\ MeV}_{E}$$



$\boxed{\begin{array}{l}\text{In Sea Water,}\\ D_2O : H_2O \Rightarrow\\ D : H = 0.0153\% : 1\\ \text{or}\quad 1 : 6500\end{array}}$

$m_{D^+} = 2.013553$ amu

$m_{T^+} = 3.015501$ "

$m_n = 1.008664$ "

$m_{{}^4He^{++}} = 4.001503$ "

1 amu = 931.481 MeV

$E = 0.018887$ amu $= 17.5928$ MeV

$$D^+ + T^+ \longrightarrow n + \alpha + 17.6\ \text{MeV}$$

$5 \times 1.67 \times 10^{-27}$ Kg
$\simeq 10^{-26}$ Kg

1 Kg of D^+-T^+ corresponds to

$$17.6\ \text{MeV} \times 10^{26}$$
$$= 17.6 \times 10^{6} \times 1.6 \times 10^{-19} \times 10^{26}\ \text{j}$$
$$= 2.86 \times 10^{14}\ \text{j}$$
$$= 2.86 \times 10^{14} \frac{\text{j}}{\text{sec}} \times \cancel{\text{sec}} \times \frac{\text{hr}}{3600\cancel{\text{sec}}}$$
$$= 7.94 \times 10^{10}\ \text{w}\cdot\text{h}$$
$$= 0.79 \times 10^{8}\ \text{KWh}$$
$$= 0.79 \times 10^{2}\ \text{GW}\cancel{\text{hr}} \cdot \frac{\text{day}}{24\cancel{\text{hr}}}$$
$$\simeq 3.29\ \text{GWd}$$

$$1\ \text{Kg of } D^+\text{-}T^+ \Rightarrow 10^{8}\ \text{KWh} \Rightarrow 1\ \text{GWd}$$

$$T_{\text{Fusion reactor plasma}} \simeq 10\ \text{keV} = 10 \times 10^{3} \times 40 \times 300\ {}^\circ\text{K}$$
$$= 1.2 \times 10^{8}\ {}^\circ\text{K}$$
$$= 120\ \text{M}\ {}^\circ\text{K}$$

$$10\ \text{keV} \simeq 100\ \text{M}\ {}^\circ\text{K}$$

[Hotter than the inner temperature of the Sun.]

§1.2.3 Fusion Reactions

Fusion power density, $P_f = \underset{\text{incoming projectile, e.g., } D^+}{n_1}\ \underset{\text{background target, e.g., } T^+}{n_2}\ \overset{\text{reaction rate}}{\langle \sigma v \rangle_{12}}\ \overset{\text{energy yield}}{W_{12}}$ $\left(\frac{watt}{m^3}\right)$

[p(pressure) $= nkT$]

For D-T Reactions

Recall Dolan, Vol. I, pp. 32-33, Equ. (2D12): [Fusion Research]

$$P_f\left(\frac{watt}{m^3}\right) = \frac{p^2 \langle \sigma v \rangle_{DT}\, W_{DT}}{4(kT_i)^2(1+T_e/T_i)^2}$$

$$\simeq \frac{2.9\times10^{-5}\,(p^2)}{(1+T_e/T_i)^2}\left(\frac{watt}{m^3}\right)$$

$(8 \le T_i \le 25\,keV)$, $W_{DT} = 17.6\,MeV$

$$= \frac{2.9\times10^{-5}\times(1\,MPa)^2}{(1+T_e/T_i)^2}$$

$p = 10\,atm$

$$= \frac{2.9\times10^{-5}\times(1\times10^6\,Pa)^2}{4}$$

$T_e = T_i$

$$= 7.2\times10^6\ W/m^3$$

$$\simeq 7\ \frac{MW}{m^3}$$

For Catalyzed-D

Dolan, Vol. I, pp. 32-33, Equ. (2D13): [Fusion Research]

$$P_f = \frac{p^2 \langle \sigma v \rangle_{DD}\, W_{DD}}{2(kT_i)^2(1+T_e/T_i)^2}\quad\left(\frac{W}{m^3}\right)$$

$W_{DD} = 21.6\,MeV = 3.46\times10^{-12}\,J$

$$\simeq \frac{6.9\times10^{-7}\,p^2}{(1+T_e/T_i)^2}\left(\frac{30\,keV}{T_i}\right)^{1/2}\left(\frac{W}{m^3}\right)$$

$\left(\frac{30\,keV}{T_i}\right)^{1/2} \approx 1$

$$\simeq \frac{6.9\times10^{-7}\times(10^{12}\,Pa^2)}{(2)^2}\cdot 1$$

$$= \left(\frac{6.9}{4}\right)\times10^5\ \left(\frac{W}{m^3}\right)$$

$$\simeq 1.7\times10^5\ W/m^3$$

$$\simeq 0.2\ \frac{MW}{m^3}$$

Table 1.3 Nuclear reactions of interest

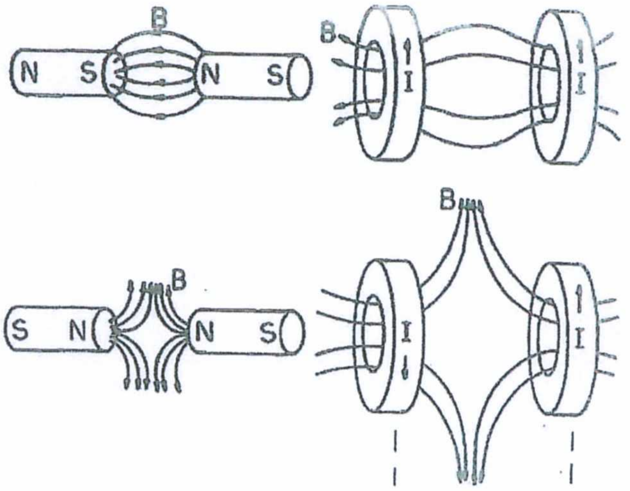
Name	Abbreviation	Reaction (energy, MeV)	Total (MeV)	Energy (10^{-12} J)
DT	$T(d,n)^4He$	$D + T \rightarrow {}^4He(3.54) + n(14.05)$	17.59	2.818
DDn	$D(d,n)^3He$	$D + D \rightarrow {}^3He(0.82) + n(2.45)$	3.27	0.524
DDp	$D(d,p)T$	$D + D \rightarrow T(1.01) + p(3.02)$	4.03	0.646
TT	$T(t,2n)\,{}^4He$	$T + T \rightarrow n + n + {}^4He$	11.3	1.81
$D\text{-}{}^3He$	${}^3He(d,p)\,{}^4He$	$D + {}^3He \rightarrow {}^4He(3.66) + p(14.6)$	18.3	2.93
$p\text{-}{}^6Li$	${}^6Li(p,\alpha)\,{}^3He$	${}^6Li + p \rightarrow {}^4He + {}^3He$	4.02	0.644
$p\text{-}{}^{11}B$	${}^{11}B(p,2\alpha)^4He$	${}^{11}B + p \rightarrow 3({}^4He)$	8.68	1.39
Reactions for breeding tritium (Natural lithium = 7.42 % ^{6}Li and 92.58 % ^{7}Li)				
$n\text{-}{}^6Li$	${}^6Li(n,\alpha)T$	${}^6Li + n(\text{thermal}) \rightarrow {}^4He(2.05) + T(2.73)$	4.78	0.766
$n\text{-}{}^7Li$	${}^7Li(n,n'+\alpha)T$	${}^7Li + n(\text{fast}) \rightarrow T + {}^4He + n$	−2.47 (endothermic)	−0.396

Numbers in parentheses are approximate energies of reaction products, MeV. The exact energies vary with angle and incident particle energies. The symbols p, d, t, n, and α represent protons, deuterons, tritons, neutrons, and alpha particles (^{4}He), respectively

§1.2.4 Magnetic Confinement

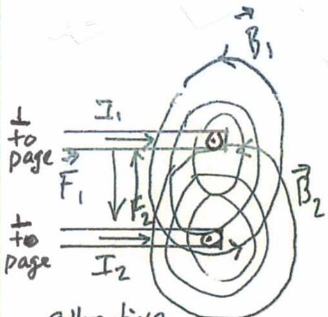
i) Magnetic Mirrors

Fig. 1.6 Simple magnetic "mirror" fields **B** (*top*) and "spindle cusp" fields (*bottom*) produced by bar magnets (*left side*) or by a pair of circular magnet coils carrying currents I (*right side*). Plasma could be confined in the central regions



parallel currents attract

opposite currents repel

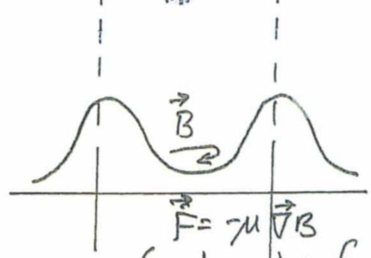


attractive

$\vec{F}_1 = q\vec{v}_1 \times \vec{B}_2$

$\vec{F}_2 = q\vec{v}_2 \times \vec{B}_1$

// currents → attractive force



$\frac{B^2}{2\mu_0}$ = magnetic pressure

$\vec{F} = -\mu \nabla B$ (mirroring force)

ii) Toroidal device, e.g., tokamak

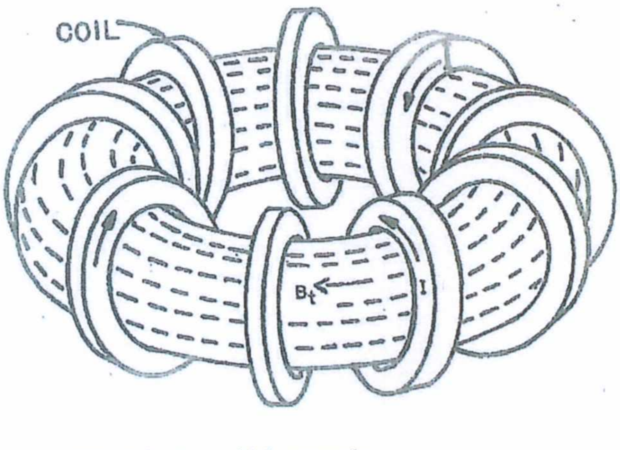
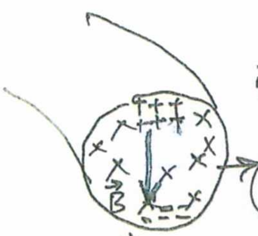


Fig. 1.8 A toroidal magnetic field. The magnet coil currents, I, create a toroidal magnetic field, B_t (*dashed lines*)

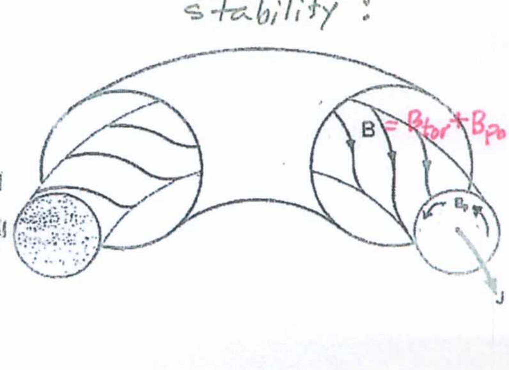
drift velocity, $\vec{v}_d = \frac{q\vec{E} \times \vec{B}}{qB^2}$



$\vec{v}_d$ ($\vec{E} \times \vec{B}$ drift)!!

Bad!

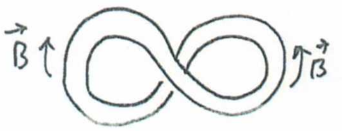
Twist $\vec{B}$ fields for stability:



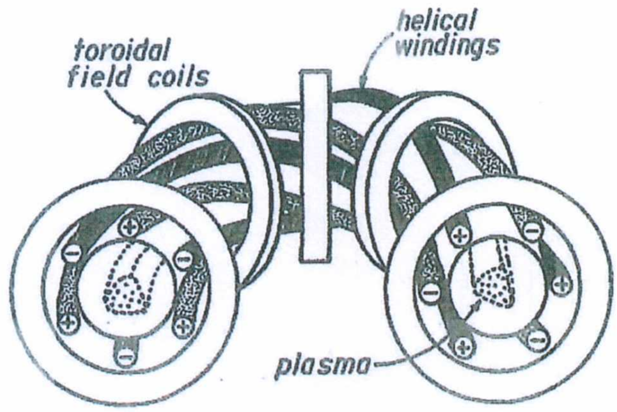
$\vec{B} = \vec{B}_{toroidal} + \vec{B}_{poloidal} + \vec{B}_{vertical}$

iii) Stellarator

twisted tokamak



No $\vec{E}\times\vec{B}$ drift losses (Good)



want plasma equilibrium

§ 1.2.5 Energy Gain ratio, Q

$$Q \equiv \frac{\text{fusion energy/pulse}}{\text{input energy / pulse}} \text{ or } \frac{\text{fusion power}}{\text{input power}}.$$

$$\text{plasma } \beta \equiv \frac{\text{plasma pressure}}{\text{magnetic pressure}} = \left.\frac{P}{\left(\frac{B^2}{2\mu_0}\right)}\right|_{B=1T} = \frac{P}{4\times10^5 \text{ Pa}} \sim 5 \sim 10\%$$

For $B = 1$ Tesla (T),

$$\frac{B^2}{2\mu_0} = \frac{1\ T^2\ (wb/m^2)^2}{2\times(4\pi\times10^{-7})\ H/m} = \frac{10^7}{8\pi} = 4\times10^5 \text{ Pa} = 4 \text{ atm}$$

$$Q = \frac{\text{fusion power out}}{\text{heating power in}} = \begin{cases} 0 & \text{if no fusion reactions take place} \\ \infty & \text{if no heating power is required to maintain the plasma (= \underline{ignition} mode)} \end{cases}$$

$$\therefore \quad 0 < Q < \infty$$

Consider D-T plasma:

$$Q\Big|_{DT} = \frac{\frac{1}{4} n^2 \langle\sigma v\rangle E_f \cdot Vol}{P_{heating}}$$

= fusion power $\Longleftrightarrow 5 \times P_\alpha$
(17.59 MeV) (3.54 MeV)

$$= \frac{5 P_\alpha}{P_{heating\,(H)}} \simeq \frac{5 P_\alpha}{P_L - P_\alpha},$$

P_L: $P_K + P_B$; P_α: (alpha heating)

(since $P_{heating\,(H)} + P_\alpha = P_{\substack{Cond\\Loss}\,(P_K)} + \overset{small}{P_{\substack{rad'n\\Loss}\,(P_B)}}$)

$P_\alpha = \frac{1}{4} n^2 \langle\sigma v\rangle E_\alpha \cdot Vol$, $P_K = \frac{3 n T}{\tau_E} \cdot Vol$

$$\therefore P_H = \left(\frac{3nT}{\tau_E} - \frac{1}{4} n^2 \langle\sigma v\rangle E_\alpha\right) \cdot Vol$$

where $P_\alpha = \frac{1}{4} n^2 \langle\sigma v\rangle E_\alpha$ (power density)

note: $p = 2nT$ (plasma pressure; electron + ion)

$$= \frac{1}{4}\left(\frac{p^2}{4T^2}\right)\langle\sigma v\rangle E_\alpha$$

$$= \frac{E_\alpha}{16} \frac{\langle\sigma v\rangle}{T^2} p^2 \equiv K_\alpha \frac{\langle\sigma v\rangle}{T^2} p^2$$

$P_L = P_{\substack{conduction\\Loss\,(P_K)}} + P_{\substack{Brem\text{-}radiation\\loss\,(P_B)}}$

with $P_K = \frac{3}{2}\frac{p}{\tau_E} \equiv K_K \frac{p}{\tau_E}$

and $P_B = C_B Z_{eff} n^2 T^{1/2} = \frac{1}{4} C_B Z_{eff} \frac{p^2}{T^{3/2}}$

$$\equiv K_B \frac{p^2}{T^{3/2}}$$

$$\therefore P_L = K_K \frac{p}{\tau_E} + K_B \frac{p}{T^{3/2}}$$

$$\therefore \boxed{Q} = \frac{5 K_\alpha \frac{\langle\sigma v\rangle}{T^2} p^2}{\left(K_K \frac{p}{\tau_E} + \underset{small}{K_B \frac{p^2}{T^{3/2}}}\right) - K_\alpha \frac{\langle\sigma v\rangle}{T^2} p^2}$$

$$= \frac{5 p \tau_E}{\left(\frac{K_K}{K_\alpha}\right)\frac{T^2}{\langle\sigma v\rangle} - p\tau_E} \simeq \boxed{\frac{5 n T \tau_E}{5\times 10^{21} - n T \tau_E}} \quad \text{Eq. (1.3)}$$

For condition for ignition, i.e., for $P_H = 0$,

$$\frac{3nT}{\tau_E} = \frac{1}{4} n^2 \langle\sigma v\rangle E_\alpha$$

$$\Rightarrow n\tau_E > \frac{12T}{\langle\sigma v\rangle E_\alpha},$$

where $\langle\sigma v\rangle\Big|_{\substack{DT\\10\text{-}20\,keV}} \simeq 1.1\times 10^{-24}\, T^2\, m^3 s^{-1}$

Lawson Criterion: $\Rightarrow n\tau_E > \frac{12T}{1.1\times10^{-24}\, T^2 (3.5\,MeV)}$

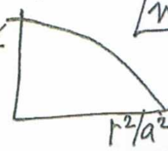
$$\Rightarrow nT\tau_E > 3.11\times 10^{21}\ (m^{-3}\cdot keV\cdot s)$$

Ignition Condition on flat profile

Ignition Condition for parabolic profile

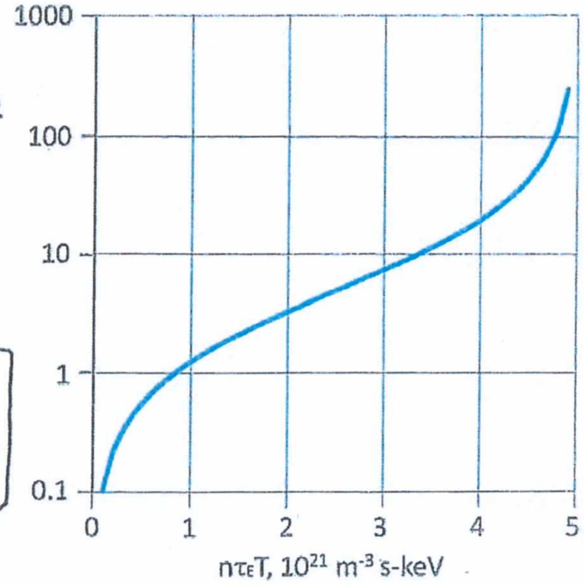
$$\boxed{nT\tau_E > 5\times 10^{21}\ (m^{-3}\cdot keV\cdot s)}$$

For n, T



For $T = 10\,keV$,
MFE, $n = 10^{20}\,m^{-3}$, $\tau_E = 3s$
IFE, $n = 10^{29}\,m^{-3}$, $\tau_E = 3ns$

Fig. 1.11 Energy gain ratio versus the "triple product" nTτ



$$Q \simeq \frac{5(nT\tau_E)}{5\times10^{21} - nT\tau_E} \quad \cdots \text{ Eq. (1.3)}$$

[Note: when $nT\tau_E = 5\times10^{21}\ m^{-3}\cdot keV\cdot s$, $Q = \infty$, ignition]

§ 1.2.6 Fusion Power Density

Reaction Rates, $R = n_1 n_2 \langle\sigma v\rangle$, $(\#/(m^3\cdot s))$

$\downarrow$

Fusion power density, $P_{DT} = R_{DT}\cdot W_{DT} = \frac{1}{4}n^2\langle\sigma v\rangle W_{DT}$ $\left(\frac{W}{m^3}\right)$

$P_{DD} = \left(\frac{1}{2!}\right)n^2\langle\sigma v\rangle\cdot W_{DD}$ $\left(\frac{W}{m^3}\right)$

Cat-DD reaction : $6D \rightarrow 2H + 2n + 2\alpha + 43.2\,MeV$ ✓

i.e., $2(D+D) \rightarrow n + {}^3He + \Delta E_1$
$ \rightarrow p + T + \Delta E_2$

Put ${}^3He$ & T back to D,

$D + T \rightarrow n + \alpha + \Delta E_3$
$+\ D + {}^3He \rightarrow p + \alpha + \Delta E_4$

$6D \rightarrow 2p + 2n + 2\alpha + 43.2\,MeV$

$\left(\frac{7.2\,MeV}{deuteron}\right)$

$3D \rightarrow \underline{21.6\,MeV}$

Example 1.1 Energy from water —

1 ℓ of $M_{DenT} \rightarrow W_D$ via Cat-D reaction? $W_D = N_D \cdot \frac{7.2\,MeV}{deut.}\cdot 1.6\times10^{-13}\frac{J}{MeV}$ (N_D: 1.02×10^{22} atoms)

$= 1.18\times10^{10}\,J = 11.8\,GJ$

1 ℓ of gasoline $\rightarrow W_{gasoline} = 1\ell \times \frac{1000\,cm^3}{\ell}\times 0.705\frac{g}{cm^3}\times\left(\frac{47\,MJ}{10^3 g}\right)$

$= 33.1\,MJ$

$\therefore \frac{11.8\times10^3\,MJ}{33.1\,MJ} = \underline{356.5}$ more energy produced in sea water!

Example 1.2 Fusion Power Density & Pressure

$$P \to P_{DT} = \frac{1}{4} n^2 \langle \sigma v \rangle W_{DT} \Big|_{n=10^{20}\,m^{-3},\ T=20\,keV}$$

power density

$\langle \sigma v \rangle \simeq 5.1 \times 10^{-22} [\ln T - 2.1]$ for $10 < T < 50\,keV$

p = Pressure $= n_i kT + n_e kT \Rightarrow 2nT$ (for $n_i = n_e$)

$(T \equiv kT) \to kT^{\circ} \to T^{\circ}$ (Kelvin)

$T^{\circ} = T(\text{joules}) / 1.38 \times 10^{-23}$

$T(eV) = T(\text{joules})/e$
$= T(\text{joules})/1.602 \times 10^{-19}$

$T(eV) = T^{\circ} \times 1.38 \times 10^{-23} / 1.602 \times 10^{-19}$
$= T^{\circ} \times 8.614 \times 10^{-5}$

$T^{\circ}(^{\circ}K) = T(eV) \times 1.16 \times 10^{4}$

$\to \boxed{1\,eV = 1.16 \times 10^{4}\ ^{\circ}K}$

(e.g. $\frac{1}{40} eV = 300\ ^{\circ}K$)

If impurity Z is present,

$p = n_i T_i + n_e T_e + \sum n_Z T_Z$

fuel ions, electrons, impurity ions (e.g., He, C, O, N, Fe…)

For few impurities, $n_i \simeq n_e$,

$\to p = 2nT$ or $n = \frac{p}{2T}$

$$P_f\Big|_{DT} = \frac{1}{4} n^2 \langle \sigma v \rangle W_{DT}$$
$$= \frac{1}{4}\left(\frac{p}{2T}\right)^2 \langle \sigma v \rangle W_{DT} \to \text{Choose max } P_{DT} \text{ for } \frac{\langle \sigma v \rangle}{T^2}$$

$W_{DT} = 2.82 \times 10^{-12}\,J$

$$P_{DT} \simeq \frac{1}{4}\frac{p^2}{4}\left(\frac{\langle \sigma v \rangle}{T^2}\right) W_{DT}$$
$$= \frac{1}{16} \times \frac{6.0118 \times 10^{-22}}{(1.6 \times 10^{-16})^2} \times 2.82 \times 10^{-12}\,J \Big|_{p=1\,Pa}$$
$$= 8.12 \times 10^{-6}\,Pa \quad \cdots\cdots (1.11)$$

$$P_{(d+D)} = \frac{1}{2!} n^2 \langle \sigma v \rangle_{DD} W_{DD}$$
$$= \frac{1}{2}\left(\frac{p}{2T}\right)^2 \langle \sigma v \rangle_{DD} W_{DD}$$
$$= \frac{p^2}{8}\left[\frac{\langle \sigma v \rangle_{DD}}{T^2}\right] \times 3.46 \times 10^{-12}\,J,$$

where
$\langle \sigma v \rangle_{DD} \equiv \langle \sigma v \rangle_{DD_n} + \langle \sigma v \rangle_{DD_p}$

$$\therefore P_{(d+D)} = \frac{p^2}{8}\left[\frac{\langle \sigma v \rangle_{DD}}{T^2}\right] \times 3.46 \times 10^{-12}\,J$$

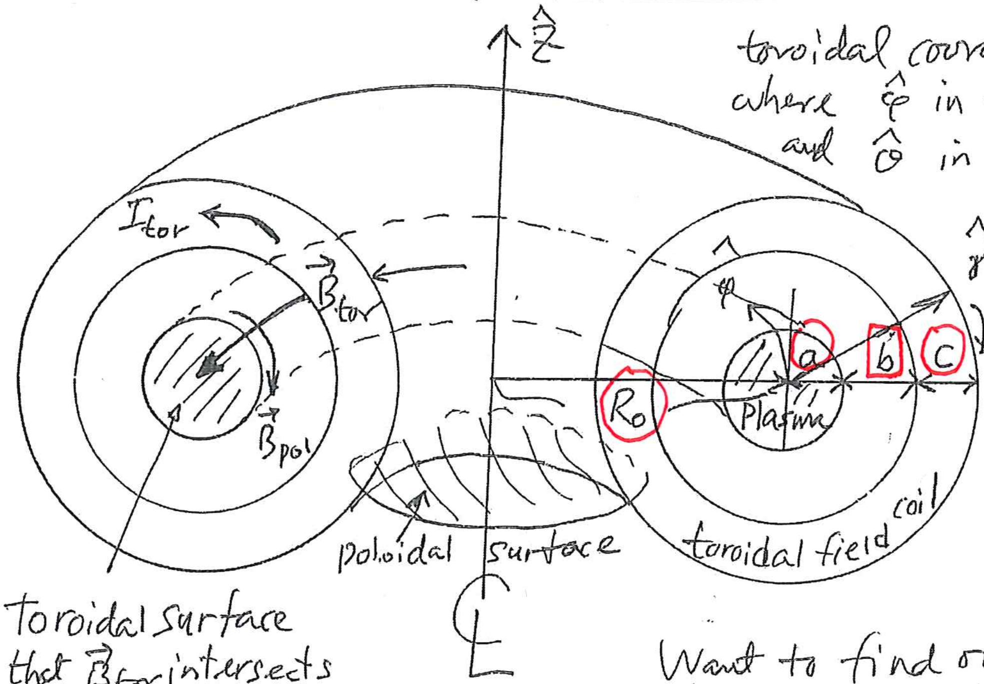
(@20keV) $\langle \sigma v \rangle_{DD} = 5.16 \times 10^{-24}\,m^3/s$, $T^2 = (20\,keV)^2$

$$= \frac{p^2}{8} \times \frac{1.29 \times 10^{-26}}{(1.6 \times 10^{-16})^2} \times 3.46 \times 10^{-12}\,J$$
$$= 2.17 \times 10^{-7}\,Pa$$

$\boxed{\text{Note: pressure for DT > DD by 40.}}$

Tokamak Reactor Power Balance & Reactor Design:

toroidal coord, (r, θ, φ)
where $\hat{\varphi}$ in toroidal direction
and $\hat{\theta}$ in poloidal direction



a = plasma radius

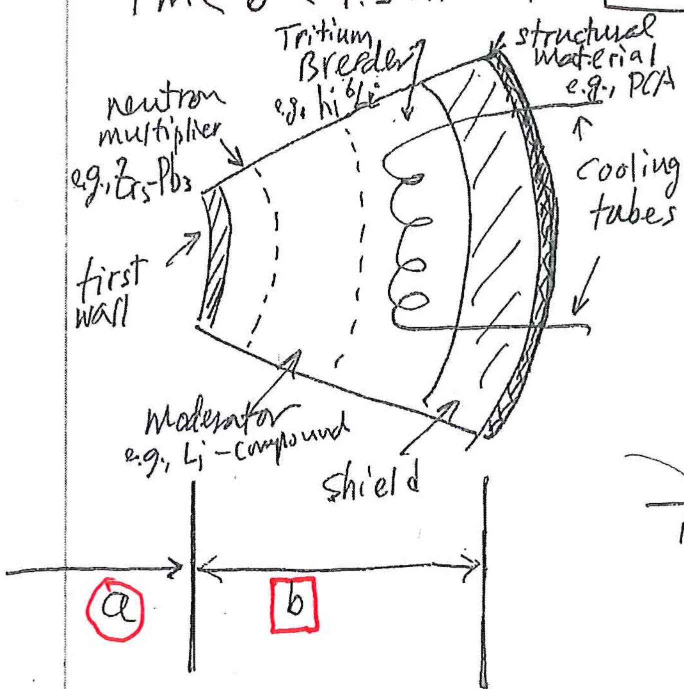
b = blanket + shield

c = toroidal field coil

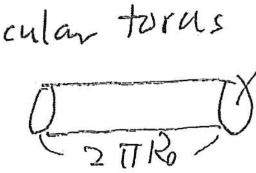
Want to find out the optimum values of a, b, and c.

With the current material properties for 14.1 MeV neutrons interacting w/ the blanket-and-shield up to 4.0 MW/m^2 wall loading,

$1\,m < b < 1.5\,m \longrightarrow \boxed{b = 1.2\,m}$

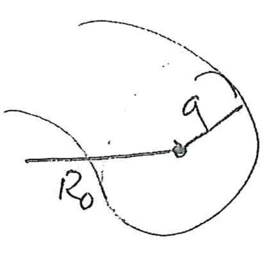


Approximate a circular torus by a cylinder



$V = (2\pi R_0)\cdot\pi a^2$

$S = 2\pi R_0 \cdot 2\pi a$



Gross electric power, P_E,

$P_E = 1.2\,\eta_e\,P_f \cdot V$ (watt)

P_f: fusion power density $\left(\frac{W}{m^3}\right)$

η_e: electric conversion efficiency

1.2: Blanket with ^6Li adds 20% to the total thermal power

$n + {}^6Li \rightarrow T + \alpha + 4.78\,MeV$

$\underbrace{D + T \rightarrow 17.59\,MeV}_{22.37\,MeV}$

where $P_f\Big|_{DT} = \dfrac{P_W}{0.8} \times \dfrac{S}{V}$, and P_W = neutron flux power @ wall $= 0.8\, P_f (V/S)$ $\left(\frac{W}{m^2}\right)$

↑ neutrons carry 80% of DT fusion power, $D+T \rightarrow n(14.1\,MeV) + \alpha(3.5\,MeV)$

↑ fusion power density

$$\therefore P_E = 1.2\, \eta_e \left(\frac{P_W}{0.8}\right) \frac{S}{\cancel{V}} \cdot \cancel{V}$$

$$= 1.2\, \eta_e \left(\frac{10}{8}\right) P_W \cdot S$$

$$= \frac{3}{2} S\, P_W\, \eta_e \quad \ldots\ldots (1.18)$$

Volume of blanket + shield + coil, V_{bc},

$$V_{bc} = 2\pi R_0 \cdot \pi\left[(a+b+c)^2 - a^2\right]$$

To minimize the cost of electricity (COE), i.e., cost/watt,

$$\frac{V_{bc}}{P_E} = \frac{\cancel{2\pi R_0} \cdot \cancel{\pi}\left[(a+b+c)^2 - a^2\right]}{\left(\frac{3}{2}\right) S\, P_W\, \eta_e} = \frac{\left[(a+b+c)^2 - a^2\right]}{3a\, P_W\, \eta_e}$$

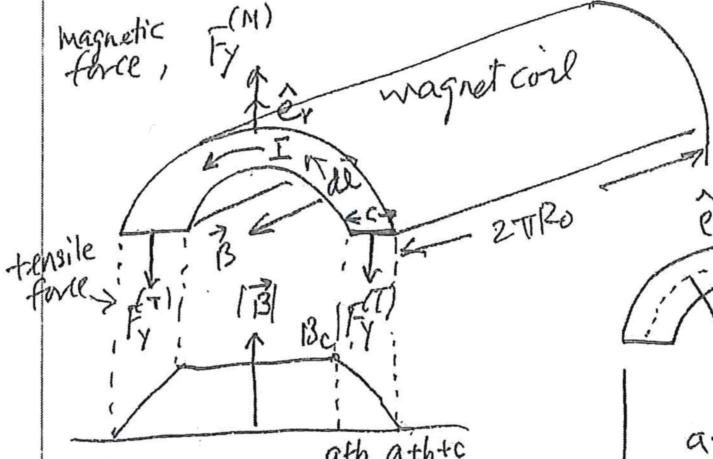
P_E: gross electric power

$S = \cancel{2\pi R_0} \cdot \cancel{2\pi} a$

η_e: thermal conversion efficiency, 0.4

$\left(\text{e.g., } \dfrac{1}{3\eta_e} = \dfrac{1}{1.2} = 0.83\right)$

To set minimum coil thickness, c:



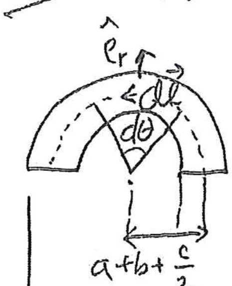
$$F^{(M)} = q\,\vec{v} \times \vec{B}$$

$$dF^{(M)}\hat{e} = B\, I\, dl\, \hat{e}_r, \text{ where } I = \frac{2\pi R_0 \cdot B_c}{\mu_0},\ dl = \left(a+b+\frac{c}{2}\right)d\theta$$

$$= \frac{B_c}{2} I\, dl\, \hat{e}_r$$

$\frac{B_c}{2}$: average magnetic field

$$= \frac{\pi R_0 B_c^2}{\mu_0}\left(a+b+\frac{c}{2}\right)d\theta\, \hat{e}_r.$$



upward component,

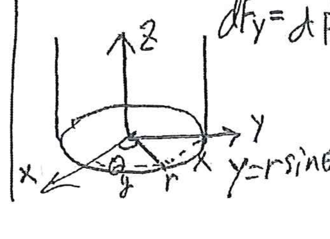
$$dF_y^{(M)} = dF^{(M)}\, \underbrace{\hat{e}_r \cdot \hat{e}_y}_{\sin\theta} = \frac{\pi R_0 B_c^2}{\mu_0}\left(a+b+\frac{c}{2}\right) \cdot \sin\theta\, d\theta$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\underbrace{\iint (\nabla \times \vec{B}) \cdot d\vec{S}}_{\int \vec{B} \cdot d\vec{l}} = \mu_0 I$$

$$B_c = \frac{\mu_0 I}{2\pi R_0}$$

$$\therefore I = \frac{2\pi R_0 B_c}{\mu_0}$$

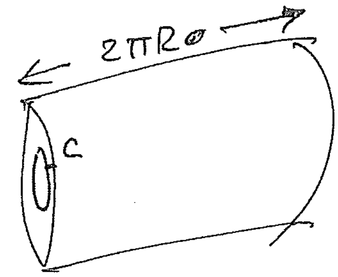


$$F_y^{(M)} = \int_{\theta=0}^{\pi} dF_y^{(M)} = \int_{\theta=0}^{\pi} \frac{\pi R_0 \cdot B_c^2}{\mu_0}\left(a+b+\frac{c}{2}\right) \underbrace{\sin\theta\, d\theta}_{\int_0^{\pi}\sin\theta d\theta = 2}$$

$$= \frac{2\pi R_0 B_c^2}{\mu_0}\left(a+b+\frac{c}{2}\right)$$

Now, tensile force,

$$F_y^{(T)} = \underset{\substack{\uparrow \\ \text{max stress}}}{\sigma_{max}} \left(2\pi R_0\, c\right)$$



Force balance, $2F_y^{(T)} = F_y^{(M)}$

$$\Rightarrow 2\sigma_{max}(\cancel{2\pi R_0}\cdot c) = \frac{\cancel{2\pi R_0}\cdot B_c^2}{\mu_0}\left(a+b+\frac{c}{2}\right)$$

Now, solve for c :

$$\frac{B_c^2}{\mu_0}(a+b) = 2\sigma_{max}\cdot c - \frac{B_c^2}{\mu_0}\left(\frac{c}{2}\right) = 2c\left[\sigma_{max} - \frac{B_c^2}{4\mu_0}\right]$$

$$\therefore c = \frac{\frac{B_c^2}{\mu_0}(a+b)}{2\left[\sigma_{max} - \frac{B_c^2}{4\mu_0}\right]} = \frac{B_c^2\,(a+b)}{2\sigma_{max}\mu_0\left[1 - \frac{B_c^2}{4\sigma_{max}\mu_0}\right]}$$

Set $\xi \equiv \dfrac{B_c^2}{4\sigma_{max}\cdot\mu_0}$

then

$$c = \frac{2\xi(a+b)}{1-\xi} \quad \ldots (1.21)$$

Use $B_c = 13T$ (maximum allowable magnetic field)

$\sigma_{max} = 300\,MPa$, [Note: 1 MPa ≃ 10 atm = 147 psi]

then $$\xi = \frac{B_c^2}{4\sigma_{max}\cdot\mu_0} = \frac{(13T)^2}{4\times 300\times10^6 Pa \times 4\pi\times10^{-7}} = 0.112$$

$$c = \frac{2\xi(a+b)}{1-\xi} = 0.252\,(a+b)$$

Recall optimum cost, i.e., Vol of blanket + shield + coil / gross electric power,

$$\frac{V_{bc}}{P_E} = \frac{[(a+b+c)^2 - a^2]}{3 a P_w \eta_e}, \quad \text{note: } c = \frac{2\xi(a+b)}{1-\xi}$$

$$= \frac{\left(a+b+\frac{2\xi(a+b)}{1-\xi}\right)^2 - a^2}{3 a P_w \eta_e}$$

$$= \frac{\left[a\left(1+\frac{2\xi}{1-\xi}\right)+b\left(\frac{2\xi}{1-\xi}+1\right)\right]^2 - a^2}{3 a P_w \eta_e}$$

$$\frac{\partial V_{bc}}{\partial P_E} = \frac{2\left[a\left(1+\frac{2\xi}{1-\xi}\right)+b\left(1+\frac{2\xi}{1-\xi}\right)\right]\left(1+\frac{2\xi}{1-\xi}\right) - 2a}{3 \eta_e a P_w} +$$

$$+ \frac{\left\{\left[a\left(1+\frac{2\xi}{1-\xi}\right)+b\left(1+\frac{2\xi}{1-\xi}\right)\right]^2 - a^2\right\}}{(-)\,3 \eta_e a^2 P_w}$$

$$= \frac{2a[\qquad]\left(1+\frac{2\xi}{1-\xi}\right) - \cancel{2}a^2 - [\qquad]^2 + \cancel{a^2}}{3 \eta_e a^2 P_w}$$

$$= 0$$

numerator:

$$\Rightarrow [\qquad]\left\{\underbrace{2a\left(1+\frac{2\xi}{1-\xi}\right) - \left[a\left(1+\frac{2\xi}{1-\xi}\right)}+b\left(1+\frac{2\xi}{1-\xi}\right)\right]\right\} - a^2 = 0$$

$$= \left[a\left(1+\frac{2\xi}{1-\xi}\right)+b\left(1+\frac{2\xi}{1-\xi}\right)\right]\left\{a\left(1+\frac{2\xi}{1-\xi}\right) - b\left(1+\frac{2\xi}{1-\xi}\right)\right\} - a^2 = 0$$

$$= \left[a\left(1+\frac{2\xi}{1-\xi}\right)\right]^2 - \left[b\left(1+\frac{2\xi}{1-\xi}\right)\right]^2 - a^2 = 0$$

$$\Rightarrow a^2\left[\left(1+\frac{2\xi}{1-\xi}\right)^2 - 1\right] - b^2\left(1+\frac{2\xi}{1-\xi}\right)^2 = 0$$

$$\Rightarrow a^2\left[\frac{4\xi}{1-\xi} + \frac{4\xi^2}{(1-\xi)^2}\right] - b^2\left(1+\frac{2\xi}{1-\xi}\right)^2 = 0$$

$$a^2\left(\frac{4\xi}{1-\xi}\right)\Big[\underbrace{1+\frac{\xi}{1-\xi}}_{\frac{1}{1-\xi}}\Big] - b^2\left(1+\frac{2\xi}{1-\xi}\right)^2 = 0$$

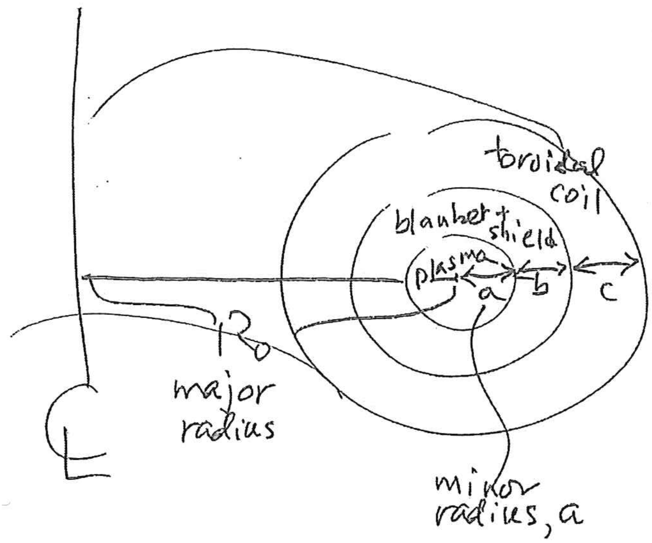
$$a^2\left(\frac{4\xi}{(1-\xi)^2}\right) - b^2\left(1+\frac{2\xi}{1-\xi}\right)^2 = 0$$

$$a = \frac{b\left(1+\frac{2\xi}{1-\xi}\right)}{\frac{2\sqrt{\xi}}{1-\xi}} = \frac{b(1-\xi)\left(1+\frac{2\xi}{1-\xi}\right)}{2\sqrt{\xi}} = \frac{b\left(1-\xi+\frac{\cancel{(1-\xi)}2\xi}{\cancel{(1-\xi)}}\right)}{2\sqrt{\xi}}$$

$$\therefore \boxed{a = \frac{b(1+\xi)}{2\sqrt{\xi}}} = \frac{(1.2m)(1+0.112)}{2\sqrt{0.112}} = 1.9936 \approx \boxed{2.0m}$$

now, $C = \dfrac{2\xi(a+b)}{1-\xi}$

$$= \frac{2\times 0.112\,(2.0\,m + 1.2\,m)}{1-0.112} = 0.80\,m$$



$\therefore\ a = 2.0\,m$

$b = 1.2\,m$

$c = 0.8\,m$

and

$$\frac{V_{bc}}{P_E} = \frac{(a+b+c)^2 - a^2}{3\eta_e a P_w}$$

$$= \frac{(2.0+1.2+0.8)^2 - 2^2\ m^2}{3\times 0.4\times(2.0m)\times 4\ MW/m^2}$$

$$= \boxed{1.25\ m^3/MW}$$

Two ways of getting the value R_0.

① Recall $P_W = (0.8)\,P_f\left(\frac{V}{S}\right),\left(\frac{W}{m^2}\right)$

where $S = (2\pi R_0)(2\pi a)$

$= 4\pi R_0^2\,(2.0)$

$= 78.95\,R_0 = 80R_0$

Eq. (1.18) $\rightarrow$ $P_E = \frac{3}{2} S P_W \cdot \eta_e \Big|_{\eta_e = 0.4}$

Gross electric power, assumed 1000 MW

$= \left(\frac{3}{2}\right)(80R_0)\left(4\frac{MW}{M^2}\right)\cdot 0.4$

$= 192\times 10^6\ R_0\ \text{(Watt)}$

$\therefore R_0 = \dfrac{1\times 10^9\,W}{192\times 10^6\,W} = \boxed{5.20\,M}$

Recall assumptions:
- $P_W = 4.0\ MW/m^2$ (neutron wall loading)
- $P_E = 1000\ MW$
- $B_{max} = 13$ Tesla (T)
- $\sigma_{max} = 300\ MPa = 3000\ atm$

② Alternat way of evaluating R_0
neutron wall loading, P_W (watt/m²),

$P_W \cdot (2\pi R_0 \cdot 2\pi a) = P_f \cdot \text{volume}$

$= \frac{1}{4} n^2 \langle \sigma v \rangle E_n \cdot (2\pi R_0 \cdot \pi a^2)$

Note: electric power out, P_E,

$P_E \Big|_{50/50\ D/T} = \frac{1}{4}\eta_t (E_\alpha + E_n + E_i)\cdot n^2 \langle \sigma v \rangle \cdot (2\pi R_0 \cdot \pi a^2)$

$$\therefore P_W = \frac{P_E \cdot E_n}{\eta_t (E_\alpha + E_n + E_i)}$$

Note: w/o Li-reaction $R_0 = 5.1\,m$

$$P_W \cdot 4\pi^2 R_0\, a = \frac{P_E \cdot E_n}{\eta_t (E_\alpha + E_n + E_i)}$$

$$R_0 = \frac{P_E \cdot E_n}{4\pi^2 a P_W \eta_t (E_\alpha + E_n + E_i)}$$

$$= \frac{E_n}{4\pi^2 \eta_t (E_\alpha + E_n + E_i)} \left(\frac{P_E}{a P_W}\right)$$

$$\underbrace{\frac{1}{4\pi^2 \cdot 0.4}}_{0.04} \cdot \frac{14.1}{3.5 + 14.1 + 4.78\ (\text{from } {}^6Li)} \left(\frac{P_E}{a P_W}\right) = \frac{0.04\times 1\times 10^3\,MW}{(2.0)\,4\,MW}$$

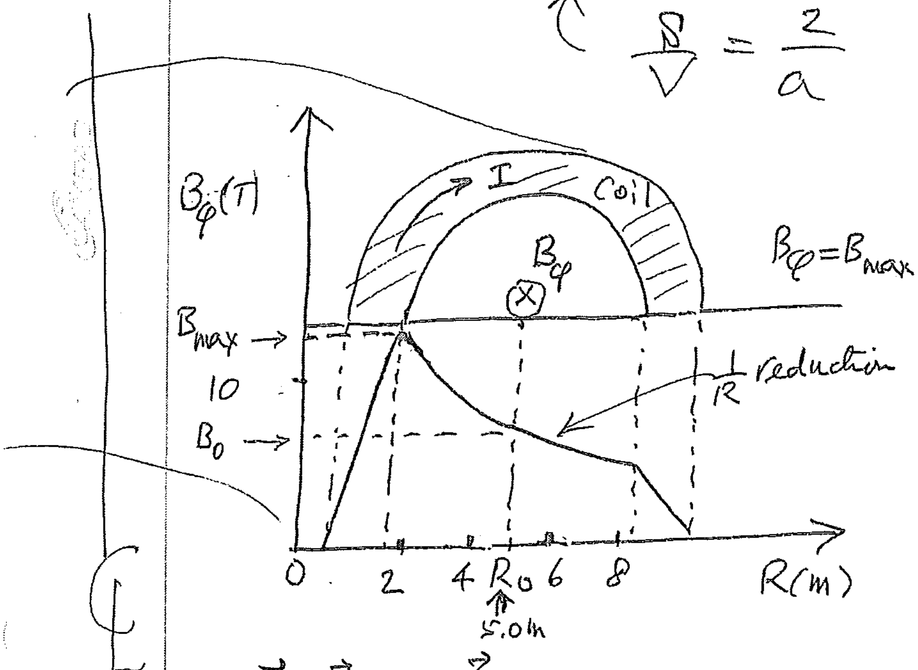
$= \boxed{5.0\,m}$

$$\text{Aspect ratio } (A) = \frac{\text{major radius } (R_0)}{\text{minor radius } (a)} = \frac{5.0}{2.0} \simeq \boxed{2.5}$$

$$\text{Surface area} = S = (2\pi R_0)(2\pi a) = 4\pi^2 (5.0)(2.0) = 394\,m^2 \simeq \boxed{400\,m^2}$$

$$\text{volume} = V = (2\pi R_0)(\pi a^2) = 2\pi^2 (5.0)(4.0) = 394\,m^3 \simeq \boxed{400\,m^3}$$

$$\frac{S}{V} = \frac{2}{a}$$



$$B \propto \frac{1}{R}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\rightarrow \int \underbrace{\vec{B} \cdot d\vec{l}}_{B \cdot 2\pi R} = \mu_0 \int d\vec{S} \cdot \vec{J} = \mu_0 I$$

$$\Rightarrow B = \frac{\mu_0 N I}{2\pi R_0} \quad \left[B \propto \frac{1}{R}\right]$$

assume $B_{max} = 13T$ but the coil is at $R_0 - a - b$

$$\underbrace{B_0 R_0}_{\substack{\text{constant} \\ \text{as it is} \\ \left(\frac{\mu_0 N I}{2\pi R_0}\right)}} = B_{max}(R_0 - a - b)$$

$$\therefore B_0 = \frac{B_{max}(R_0 - a - b)}{R_0}$$

$$= \frac{13T(5.0 - 2.0 - 1.2)}{5.0m}$$

$$= 4.68T$$

$$\simeq \boxed{4.7T}$$

$$\text{Eq.}(1.16) \rightarrow P_f = \frac{P_W}{0.8}\left(\frac{S}{V}\right), \text{ where } \frac{S}{V} = \frac{2}{a}$$

fusion power density

$$= \frac{P_W}{0.4a} = \frac{4\ \text{MW/m}^2}{(0.4 \times 2.0)} = \boxed{5\ \text{MW/m}^3}$$

Recall, Eq.(1.11)

$$P_f\big|_{DT} \simeq 8.12 \times 10^{-6}\, p^2$$

pressure

under "optimum $\frac{\langle \sigma v \rangle}{T^2}$ value" around $T = 15$ keV

$$\text{pressure, } p = \left(\frac{P_f|_{DT}}{8.12\times10^{-6}}\right)^{1/2} = \sqrt{\frac{5\times10^6\ \text{W/m}^3}{8.12\times10^{-6}}} = \boxed{7.8\times10^5\ \text{Pa}}$$

$$\beta \equiv \frac{\text{plasma pressure}}{\text{magnetic pressure}} = \frac{p}{\left(\frac{B^2}{2\mu_0}\right)} = \frac{7.8\times10^5\ \text{Pa}}{(4.7)^2/(2\times4\pi\times10^{-7})} = 0.088 = \boxed{8.8\%}$$

So, (1) plasma pressure, $p = 7.8\times10^5\ \text{Pa} \simeq 7.7$ atm

(2) plasma temperature, $T = 15$ keV (optimum condition for $\langle \sigma v \rangle / T^2$)

Note: Example problem 1.3

fusion power density, P_f,

$$P_f\big|_{DT} = \frac{1}{4} n^2 \langle \sigma v \rangle W_{DT}$$

W_{DT}: 17.6 MeV $= 17.6\times10^6 \times 1.6\times10^{-19}$ joule

$P_f|_{DT}$: 5 MW/m³ estimated above

$\langle \sigma v \rangle \big|_{T=15\text{keV}} = 2.65\times10^{-22}\ \text{m}^3/\text{s}$ (note MFT, P.17 Table)

$$\text{(3) density, } n = \sqrt{\frac{4P_f|_{DT}}{\langle \sigma v \rangle W_{DT}}} = \sqrt{\frac{4\times5\times10^6\ \text{W/m}^3}{2.65\times10^{-22}\ \frac{\text{m}^3}{\text{s}} \times 17.6\times1.6\times10^{-13}\ \text{joule}}}$$

$$= \boxed{1.63\times10^{20}\ \text{m}^{-3}}$$

Recall $n T \tau_E \big|_{DT} \simeq 5\times10^{21}\ \text{m}^{-3}\cdot\text{keV}\cdot\text{s}$ for high Q-value (ignition)

$$\text{(4) confinement time, } \tau = \frac{n T \tau_E}{(nT)} = \frac{5.0\times10^{21}\ \text{m}^{-3}\cdot\text{keV}\cdot\text{s}}{1.63\times10^{20}\ \text{m}^{-3}\times15\text{keV}} = \boxed{2.04\ \text{sec}}$$

keV: 1.6×10^{-16} joule

nT: pressure

§1-2.8 Effect of Impurities (= "deadly")

$$n_e = \underset{(\text{ion})}{n_H} + \sum_Z n_Z \langle Z \rangle, \quad (m^{-3})$$

(n_Z = impurity density of charge Z)

$\langle Z \rangle$ = effective Z, $= \frac{\sum_k n_k \langle Z^2 \rangle_k}{n_e}$ (= Z_{eff})

(above) $\div\, n_e \Rightarrow 1 = \underbrace{\frac{n_H}{n_e}}_{\text{fraction of ion } f_H} + \sum_Z \underbrace{\frac{n_Z}{n_e}}_{f_Z \text{ (impurity fraction)}} \langle Z \rangle \Rightarrow \boxed{f_H = 1 - \sum_k f_k \langle Z \rangle}$

<u>Example 1.4</u> with 4% oxygen ; $f_H = 1 - 0.04(8) = 0.68$

Since $P_f \simeq n_H^2 \rightarrow$ So, fusion power density, $P_f \sim (0.68)^2 = 46\%$

$\therefore$ fusion power density P_f is reduced to <u>46%</u>

---

§1.2.9 Ignition

power balance,

$$\boxed{\frac{3}{2}\frac{\partial p}{\partial t} = P_{ext} + P_\alpha - \frac{3}{2}\frac{p}{\tau_E} - P_{rad}}$$

pressure

Brem-radiation power loss, $P_{rad} = \left(\frac{p}{2T}\right)^2 \sum_k f_k Q_k$

$f_k = n_k/n_e$

Q_k: radiation parameter

$n^2 = \left(\frac{p}{2T}\right)^2$

p = pressure

$\frac{\partial p}{\partial t}$: pressure / time

P_{ext}: ohmic + wave + neutral beam heating power density

P_α: α-pcle power density

$$P_\alpha = 0.2 P_{DT} = 0.2\left[\frac{1}{4}\left(\frac{p}{2T}\right)^2 \langle \sigma v \rangle W_{DT}\right] \cdots \text{Eq.}(1.10)$$

$$\frac{3}{2}\frac{\partial p}{\partial t} = P_{ext} + 0.2\left(\frac{p}{4T}\right)^2 \langle \sigma v \rangle W_{DT} - \frac{3}{2}\frac{p}{\tau_E} - \left(\frac{p}{2T}\right)^2 \sum_k f_k Q_k$$

If $P_\alpha \geq P_{Loss}$, ignition condition is reached,

i.e., $$0.2\frac{p^2}{(4T)^2}\cdot \langle \sigma v \rangle W_{DT} \geq \frac{3}{2}\frac{p}{\tau_E} + \underbrace{\frac{p^2}{(2T)^2}\sum_k f_k Q_k}_{\text{"radiation loss"}}$$

$$\Rightarrow \frac{0.2}{(4T)^2} p \tau_E \langle \sigma v \rangle W_{DT} \geq \frac{3}{2} + \frac{p \tau_E}{(2T)^2}\sum_k f_k Q_k$$

$$\Rightarrow p \tau_E \left[\frac{0.2}{(4T)^2}\langle \sigma v \rangle W_{DT} - \frac{1}{(2T)^2}\sum_k f_k Q_k\right] \geq \frac{3}{2}$$

(next page)

$$\therefore\ p\tau_E \geq \frac{\frac{3}{2}}{\frac{0.2}{(4T)^2}\langle\sigma v\rangle W_{DT} - \frac{1}{(2T)^2}\sum_k f_k Q_k}$$

$$= \frac{(4T)^2\left(\frac{3}{2}\right)}{0.2\langle\sigma v\rangle W_{DT} - 4\sum_k f_k Q_k}$$

$$p\tau_E \geq \frac{24T^2}{0.2\langle\sigma v\rangle W_{DT} - 4\sum_k f_k Q_k} \quad \cdots (1.39)$$

$p\tau_E = nT\tau_E$

Condition required to achieve ignition

Example 1.5 Ignition Condition w/ impurities

Consider DT plasma with $T = 15\,\text{keV}$ w/ 1% oxygen (i.e., $f_{oxy} = \frac{n_{oxy}}{n_e} = 1\%$)

assume $Q_{oxy} = 1.7\times10^{-34}\ \text{W}\cdot\text{m}^3$, and $Q_H = 2\times10^{-36}\ \text{W}\cdot\text{m}^3$.

Calculate energy confinement time, τ_E.

$$p\tau_E \geq \frac{24T^2}{(0.2)\langle\sigma v\rangle W_{DT} - 4\sum_k f_k Q_k}, \quad \begin{cases} \langle\sigma v\rangle\Big|_{15\text{keV, DT}} = 2.65\times10^{-22}\ \frac{\text{m}^3}{\text{s}} \ \text{(Table 1.5)}, \\ \text{and } f_H = n_H/n_e = 0.92 \end{cases}$$

$$= \frac{24(15\,\text{keV})^2}{(0.2)(2.65\times10^{-22}\frac{\text{m}^3}{\text{s}})\,17.6\,\text{MeV} - 4[\underbrace{f_H Q_H}_{0.92\times2\times10^{-36}} + \underbrace{f_{oxy}Q_{oxy}}_{0.01\times1.7\times10^{-34}}]}$$

$$= \frac{24\times(15\times1.6\times10^{-16}\,\text{J})^2}{(0.2)(2.65\times10^{-22})(17.6\times10^{-13}\,\text{J}) - 4[0.92\times2\times10^{-36} + 0.01\times1.7\times10^{-34}]}$$

$$= \frac{1.3824\times10^{-28}}{7.912\times10^{-35}} = 1.74\times10^6\ \text{Pa}\cdot\text{s} \quad [1\,\text{atm} \simeq 14.7\,\text{psi}]$$

$$= 1.74\ \text{MPa}\cdot\text{s} \quad [1\,\text{atm} \simeq 10^5\,\text{Pa}]$$

$$= \underline{\underline{17.4\ \text{atm}\cdot\text{s}}} \ \text{ to achieve ignition } (Q=\infty)$$

But one could operate reactor w/ $Q = 30$, however.

e.g., For ITER, $Q \sim 10$, $\tau \sim 400$ sec ("High Q" profile 2005)

Fusion Reactor Technology 2
11 Sept 2019

Notes Added to σ and ⟨σv⟩ values:

Dolan used this in his MFT book $\boxed{\text{Table 1.4}}$!

But should have used this instead generated by Li et al. (Attached paper)

Table 3. D+T fusion cross-section.

Energy in laboratory (keV)	Experimental data (ENDF/B VII.0) (b)	Resonant tunnelling, equation (1) (b)	3-Parameter fit equation (5) (b)	5-Parameter fit equation (7) (NRL Formulary) (b)
0.2	7.43E-39	7.15E-39	7.61E-39	2.83E-40
0.3	4.06E-31	3.90E-31	4.15E-31	2.92E-32
0.4	1.59E-26	1.53E-26	1.63E-26	1.67E-27
0.7	2.51E-19	2.42E-19	2.58E-19	4.80E-20
1	1.03E-15	9.90E-16	1.05E-15	2.66E-16
4	1.16E-06	1.13E-06	1.20E-06	6.57E-07
7	1.54E-04	1.51E-04	1.62E-04	1.07E-04
10	1.73E-03	1.71E-03	1.83E-03	1.32E-03
14	1.14E-02	1.13E-02	1.21E-02	9.47E-03
20	5.97E-02	6.00E-02	6.42E-02	5.34E-02
26	1.68E-01	1.71E-01	1.82E-01	1.57E-01
35	4.70E-01	4.81E-01	5.08E-01	4.53E-01
50	1.37E+00	1.40E+00	1.44E+00	1.32E+00
65	2.66E+00	2.69E+00	2.71E+00	2.53E+00
80	3.96E+00	3.97E+00	3.94E+00	3.74E+00
95	4.80E+00	4.79E+00	4.74E+00	4.59E+00
104	5.00E+00	4.98E+00	4.96E+00	4.86E+00
110	5.01E+00	5.00E+00	4.99E+00	4.93E+00
120	4.88E+00	4.88E+00	4.90E+00	4.90E+00
135	4.47E+00	4.48E+00	4.53E+00	4.62E+00
150	3.97E+00	4.00E+00	4.04E+00	4.19E+00
180	3.07E+00	3.10E+00	3.10E+00	3.29E+00
220	2.23E+00	2.25E+00	2.17E+00	2.36E+00
280	1.50E+00	1.49E+00	1.35E+00	1.53E+00

Figure 3 is for D + ^{3}He fusion, and figure 4 is for D + D fusion. The data points are from ENDF/B VII.0 of the National

tunnelling model (column 4) fits the data much better. Now we can see the importance of the $(-1/\theta^2)$ term in the denominator

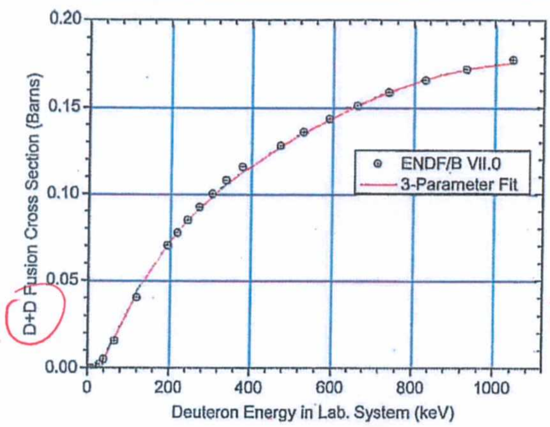


Figure 4. Comparison between D+D fusion data and the 3-parameter fit (solid line).

Table 1. Parameter list for 3-parameter fit formula.

	D+T	D+^{3}He	D+D (p+T and n+^{3}He)
C_1	−0.5405	−1.1334	−60.2641
C_2	0.005546	0.003039	0.05066
C_3	−0.3909	−0.6702	−54.9932

Table 2. NRL Plasma Formulary 5-parameter list.

	D+T Fusion	D+^{3}He Fusion	D+D Fusion p+T	D+D Fusion n+^{3}He
A_1	45.95	89.27	46.097	47.88
A_2	50200	25900	372	482
A_3	1.368×10^{-2}	3.98×10^{-3}	4.36×10^{-4}	3.08×10^{-4}
A_4	1.076	1.297	1.220	1.177
A_5	409	647	0	0

is almost a constant in the range of interest. $\ln(2ka)$ and $\Gamma(-I/ka_c)$ are slow functions of energy, which affect only the real part of w. As a result, the complex function, $w(E)$, may be simplified to

$$w = C_1 + C_2 E_{\mathrm{lab}} + iC_3. \quad (4)$$

The fusion cross-section may be written as

$$\sigma(E_{\mathrm{lab}}) = \frac{\pi}{(2\mu/\hbar^2)E_{\mathrm{lab}}(M_b/M_a + M_b)}\frac{1}{\theta^2} \times \frac{(-4C_3)}{(C_1 + C_2E_{\mathrm{lab}})^2 + \left(C_3 - (1/\theta^2)\right)^2}. \quad (5)$$

We may use experimental data to find these 3 fitting parameters (C_1, C_2, and C_3). In equation (5) and in the definitions of k and k_1, E has been replaced by E_{lab}, the energy of incident projectile in the laboratory system.

$$E_{\mathrm{lab}} = \frac{M_a + M_b}{M_b}E \quad (6)$$

M_a and M_b are the mass of the incident projectile and the target, respectively.

Figures 2–4 are the results of the calculation using this 3-parameter fitting formula (5). Figure 2 is for D + T fusion.

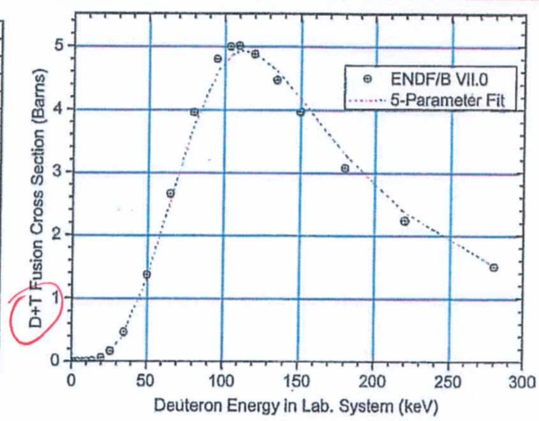


Figure 5. Comparison between D+T fusion data and the 5-parameter fit (dotted line).

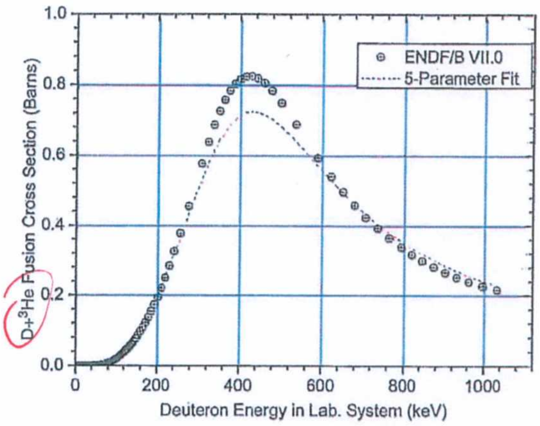


Figure 6. Comparison between D+^{3}He fusion data and the 5-parameter fit (dotted line).

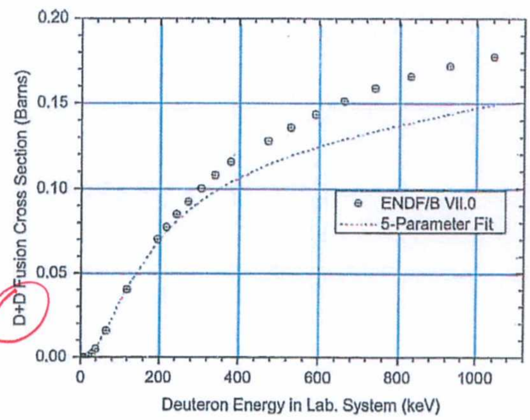


Figure 7. Comparison between D+D Fusion data and the 5-parameter fit (dotted line).

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A new simple formula for fusion cross-sections of light nuclei

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Abstract
The recent ENDF/B VII.0 data are compared with the fitting formula (NRL handbook—Plasma Formulary). The differences between experimental data and the fitting formula for three major fusion cross-sections reveal the need for a replacement to the old 5-parameter fitting formula. The new formula in this paper has only 3 parameters, but its fit with the experimental data is greatly improved because the energy dependence of the incident-channel (deuteron) width is taken explicitly into account through a penetrability function of the Mott form.

PACS numbers: 24.10.−i, 24.30.−v, 25.45.−z

(Some figures in this article are in colour only in the electronic version)

1. Introduction

A formula was published early in 1972 to describe the major fusion cross-sections using only 5 parameters. It was included in a famous handbook, NRL Plasma Formulary, published by the Naval Research Laboratory [1]. It correctly expressed the dependence of the cross-sections on energy, E: the geometric factor ($1/E$ dependence), the Gamow penetration factor ($\mathrm{Exp}[c/\sqrt{E}]$ dependence) and the resonance factor (Breit–Wigner type dependence). Hence it has been widely used by plasma physicists, although Bosch and Hale pointed out in 1992 [2] that this 5-parameter fitting formula did not give the correct results in the low energy region. Bosch and Hale proposed a 9-parameter fitting formula to show the correct dependence based on the R-matrix theory and thousands of experimental data. However, this old 5-parameter formula has still been cited in the later editions of NRL Plasma Formulary [3][1]even 15 years after the publication of the 9-parameter fitting formula. Many plasma scientists are still using this 5-parameter formula.

Moreover, this 5-parameter fitting formula was based on the Breit–Wigner theory, which might not be valid for light nuclei fusion reactions because it was proposed mainly for heavy or intermediate nuclei when fission reactor studies were dominant. The key assumption of Breit–Wigner theory was the compound nucleus model, i.e. the nuclear reaction might be divided into two *independent* steps—the formation of the compound nucleus and the decay of the compound nucleus. This implies that the collisions between the injected nucleon and the nucleons inside the target are strong enough to erase the particle 'memories'. The injected nucleon would become part of the compound nucleus and 'forget' its history. Hence, the decay of the compound nucleus would be totally independent of its formation. When several channels of decay are available, the compound nucleus would tend to decay into the fastest decay channel (the channel with the shortest lifetime). Since the neutron emission channel is the result of the strong nuclear interaction, usually it is the channel with the shortest lifetime. One may expect to see neutron emission after resonant tunnelling if the compound nucleus model remains valid. However, the compound nucleus model is correct only if many collisions occur inside the compound nucleus. This assumption might not be valid in the case of nuclei fusion where no strong collisions occur. The wave function of injected nucleon might still retain a memory of its phase before it is absorbed by the target. With this premise we can develop a better model to describe the fusion reaction between two light nuclei.

2. 3-Parameter model

In order to keep the memory of the phase of the wave function, a spherical square-potential well is assumed to represent the nuclear interaction between the incident projectile and the target nucleus (figure 1). In addition, an imaginary part of the nuclear potential is introduced to represent the effect of

[1] Indeed there was another mistake in the citation of NRL Plasma Formulary. The reference [28] there 'Rept.BNWL-1685 (Brookhaven National Laboratory, 1972)' should be corrected as 'Rept.BNWL-1685 (Pacific Northwest Laboratory, 1972), p 76'. This mistake was pointed out early in 2002 [8].

9.5 Nuclear Reactions for Fusion Plasmas

All data from the ENDF/B-VII nuclear data libraries. [4]

Abbreviations: n=neutron, p=^{1}H, d=^{2}H, t=^{3}H, h=^{3}He, α =^{4}He

	Reactants		Products (kinetic energy in MeV)	Branching ratio	Q-value (MeV)
1.	d + t	$\longrightarrow$	α(3.52) + n(14.07)	1.00	17.59
2.	d + d	$\longrightarrow$	t(1.01) + p(3.02)	0.50	4.03
		$\longrightarrow$	h(0.82) + n(2.45)	0.50	3.27
3.	d + h	$\longrightarrow$	α(3.67) + p(14.68)	1.00	18.35
4.	t + t	$\longrightarrow$	α + 2n	1.00	11.33
5.	h + t	$\longrightarrow$	α + p + n	0.51	12.10
		$\longrightarrow$	α(4.77) + d(9.54)	0.43	14.32
		$\longrightarrow$	^{5}He(1.87) + p(9.34)	0.06	11.21
6.	p + ^{6}Li	$\longrightarrow$	α(1.72) + h(2.30)	1.00	4.02
7.	p + ^{7}Li	$\longrightarrow$	2 α	0.20	17.35
		$\longrightarrow$	^{7}Be + n	0.80	-1.64
8.	d + ^{6}Li	$\longrightarrow$	2 α	1.00	22.37
9.	p + ^{11}B	$\longrightarrow$	3 α	1.00	8.62

9.6 Nuclear Reactions for Fusion Energy

All data from the ENDF/B-VII nuclear data libraries. [4]

Abbreviations: n=neutron, t=^{3}H, B=tritium breeding, M=neutron multiplication

	Reaction	Q-value [MeV]	Purpose	σ(0.025 eV) [barn]	σ(14.1 MeV) [barn]
1.	^{6}Li(n,t)^{4}He	4.78	B	978	0.03
2.	^{6}Li(n,2nα)^{6}Li	-3.96	M	-	0.08
3.	^{7}Li(n,2n)^{6}Li	-7.25	M	-	0.03
4.	^{7}Li(n,2nα)^{3}H	-8.72	B/M	-	0.02
5.	^{9}Be(n,2n)^{8}Be	-1.57	M	-	0.48
6.	^{204}Pb(n,2n)^{203}Pb	-8.39	M	-	2.22
7.	^{206}Pb(n,2n)^{205}Pb	-8.09	M	-	2.22
8.	^{207}Pb(n,2n)^{206}Pb	-6.74	M	-	2.29
9.	^{208}Pb(n,2n)^{207}Pb	-7.37	M	-	2.30

9.7 Fusion Cross Section Parametrization

The Bosch-Hale parametrization of the fusion reaction cross section [2]

$$\sigma(E_{\mathrm{keV}}) = \frac{S(E)}{E\exp(B_G/\sqrt{E})} \quad \text{[millibarn]}$$

where

$$S(E_{\mathrm{keV}}) = \frac{A_1 + E(A_2 + E(A_3 + E(A_4 + EA_5)))}{1 + E(B_1 + E(B_2 + E(B_3 + EB_4)))}$$

Bosch-Hale parametrization coefficients for several fusion reactions [2]

	^{2}H(d,n)^{3}He	^{2}H(d,p)^{3}H	^3H(d,n)^{4}He	^{3}He(d,p)^{4}He
B_G [keV]	31.3970	31.3970	34.3827	68.7508
A_1	5.3701×10^4	5.5576×10^4	6.927×10^4	5.7501×10^6
A_2	3.3027×10^2	2.1054×10^2	7.454×10^8	2.5226×10^3
A_3	-1.2706×10^{-1}	-3.2638×10^{-2}	2.050×10^6	4.5566×10^1
A_4	2.9327×10^{-5}	1.4987×10^{-6}	5.200×10^4	0.0
A_5	-2.5151×10^{-9}	1.8181×10^{-10}	0.0	0.0
B_1	0.0	0.0	6.380×10^1	-3.1995×10^{-3}
B_2	0.0	0.0	-9.950×10^{-1}	-8.5530×10^{-6}
B_3	0.0	0.0	6.981×10^{-5}	5.9014×10^{-8}
B_4	0.0	0.0	1.728×10^{-4}	0.0
Valid Range [keV]	0.5<E<4900	0.5<E<5000	0.5<E<550	0.3<E<900

Tabulated Bosch-Hale cross sections [millibarns] [2]

E (keV)	^{2}H(d,n)^{3}He	^{2}H(d,p)^{3}H	^3H(d,n)^{4}He	^{3}He(d,p)^{4}He
3	2.445×10^{-4}	2.513×10^{-4}	9.808×10^{-3}	1.119×10^{-11}
4	2.093×10^{-3}	2.146×10^{-3}	1.073×10^{-1}	1.718×10^{-9}
5	8.834×10^{-3}	9.038×10^{-3}	5.383×10^{-1}	5.199×10^{-8}
6	2.517×10^{-2}	2.569×10^{-2}	1.749×10^0	6.336×10^{-7}
7	5.616×10^{-2}	5.720×10^{-2}	4.335×10^0	4.373×10^{-6}
8	1.064×10^{-1}	1.081×10^{-1}	8.968×10^0	2.058×10^{-5}
9	1.794×10^{-1}	1.820×10^{-1}	1.632×10^1	7.374×10^{-5}
10	2.779×10^{-1}	2.812×10^{-1}	2.702×10^1	2.160×10^{-4}
12	5.563×10^{-1}	5.607×10^{-1}	6.065×10^2	1.206×10^{-3}
15	1.178×10^0	1.180×10^0	1.479×10^2	7.944×10^{-3}
20	2.691×10^0	2.670×10^0	4.077×10^2	6.568×10^{-2}

See Table V of the Bosch-Hale full paper

9.8 Fusion Reaction Rate Parametrization

The Bosch-Hale parametrization of the volumetric reaction rates [2]

$$\langle \sigma v \rangle = C_1 \cdot \theta \cdot \sqrt{\frac{\xi}{m_\mu c^2 T^3_{\text{i, keV}}}}\, e^{-3\xi} \qquad [\text{cm}^3\ \text{s}^{-1}] \text{ or } \times 10^{-6}\ [\text{m}^3\ \text{s}^{-1}]$$

where

$$\theta = T / \left(1 - \frac{T(C_2 + T(C_4 + TC_6))}{1 + T(C_3 + T(C_5 + TC_7))}\right) \qquad \xi = \left(\frac{B_G^2}{4\theta}\right)^{1/3}$$

Bosch-Hale parametrization coefficients for volumetric reaction rates [2]

	^{2}H(d,n)^{3}He	^{2}H(d,p)^{3}H	^3H(d,n)^{4}He	^{3}He(d,p)^{4}He
B_G [keV$^{1/2}$]	31.3970	31.3970	34.3827	68.7508
$m_\mu c^2$ [keV]	937 814	937 814	1 124 656	1 124 572
C_1	5.43360×10^{-12}	5.65718×10^{-12}	1.17302×10^{-9}	5.51036×10^{-10}
C_2	5.85778×10^{-3}	3.41267×10^{-3}	1.51361×10^{-2}	6.41918×10^{-3}
C_3	7.68222×10^{-3}	1.99167×10^{-3}	7.51886×10^{-2}	-2.02896×10^{-3}
C_4	0.0	0.0	4.60643×10^{-3}	-1.91080×10^{-5}
C_5	-2.96400×10^{-6}	1.05060×10^{-5}	1.35000×10^{-2}	1.35776×10^{-4}
C_6	0.0	0.0	-1.06750×10^{-4}	0.0
C_7	0.0	0.0	1.36600×10^{-5}	0.0
Valid range (keV)	$0.2<T_i<100$	$0.2<T_i<100$	$0.2<T_i<100$	$0.5<T_i<190$

Tabulated Bosch-Hale reaction rates [m^{3}s^{-1}] [2]

T (keV)	^{2}H(d,n)^{3}He	^{2}H(d,p)^{3}H	^3H(d,n)^{4}He	^{3}He(d,p)^{4}He
1.0	9.933×10^{-29}	1.017×10^{-28}	6.857×10^{-27}	3.057×10^{-32}
1.5	8.284×10^{-28}	8.431×10^{-28}	6.923×10^{-26}	1.317×10^{-30}
2.0	3.110×10^{-27}	3.150×10^{-27}	2.977×10^{-25}	1.399×10^{-29}
3.0	1.602×10^{-26}	1.608×10^{-26}	1.867×10^{-24}	2.676×10^{-28}
4.0	4.447×10^{-26}	4.428×10^{-26}	5.974×10^{-24}	1.710×10^{-27}
5.0	9.128×10^{-26}	9.024×10^{-26}	1.366×10^{-23}	6.377×10^{-27}
8.0	3.457×10^{-25}	3.354×10^{-25}	6.222×10^{-23}	7.504×10^{-26}
10.0	6.023×10^{-25}	5.781×10^{-25}	1.136×10^{-22}	2.126×10^{-25}
12.0	9.175×10^{-25}	8.723×10^{-25}	1.747×10^{-22}	4.715×10^{-25}
15.0	1.481×10^{-24}	1.390×10^{-24}	2.740×10^{-22}	1.175×10^{-24}
20.0	2.603×10^{-24}	2.399×10^{-24}	4.330×10^{-22}	3.482×10^{-24}

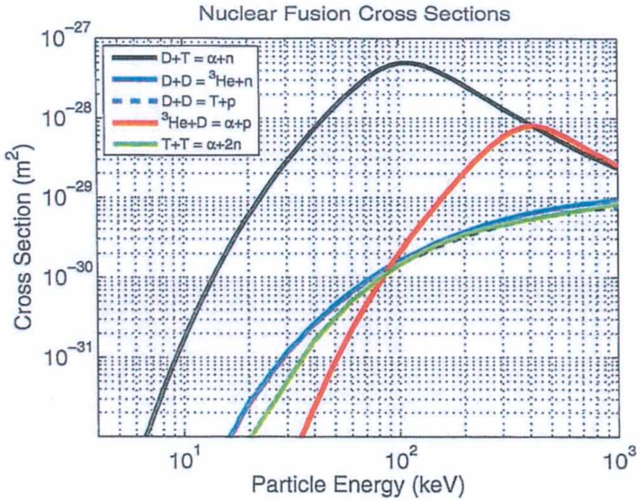
See Table VIII of the full paper

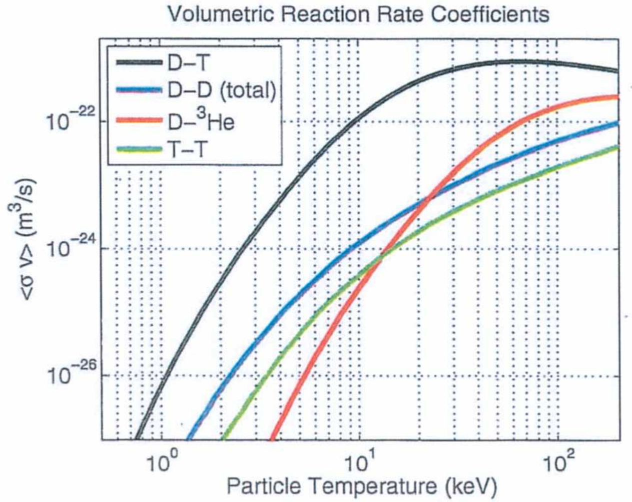
Approximate DT volumetric reaction rate ($10 \lesssim T$ [keV] $\lesssim 20$) [26:7]

$$\langle \sigma v \rangle_{\text{DT}} = 1.1 \times 10^{-24} T^2_{\text{keV}} \quad [\text{m}^3\ \text{s}^{-1}]$$

9.9 Cross Section and Reaction Rate Plots

Data from the ENDF/B-VII nuclear data libraries [4] is plotted directly below and used to calculate the volumetric reaction rate coefficients (thermal reactivity) [3].





IMPROVED FORMULAS FOR FUSION CROSS-SECTIONS AND THERMAL REACTIVITIES

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ABSTRACT. For interpreting fusion rate measurements in present fusion experiments and predicting the fusion performance of future devices or of d–t experiments in present devices, it is important to know the fusion cross-sections as precisely as possible. Usually, it is not measured data that are used, but parametrizations of the cross-section as a function of the ion energy and parametrizations of the Maxwellian reactivity as a function of the ion temperature. Since the publication of the parametrizations now in use, new measurements have been made and evaluations of the measured data have been improved by applying R-matrix theory. The paper shows that the old parametrizations no longer adequately represent the experimental data and presents new parametrizations based on R-matrix calculations for fusion cross-sections and Maxwellian reactivities for the reactions $D(d,n)^3He$, $D(d,p)T$, $T(d,n)^4He$ and $^3He(d,p)^4He$.

1. INTRODUCTION

Fusion cross-sections have been of great interest from the beginning of fusion research, mainly for two reasons: While the detection of fusion reaction products has always been a signature of success, beginning with qualitative measurements of neutron production in the earliest experiments, modern fusion experiments use the measured fusion rates as plasma diagnostics. If one wants to draw reliable conclusions from these measurements about the plasma parameters, detailed knowledge of the fusion cross-sections is more important than ever before. Since the uncertainties in the neutron rate measurements are of the order of 10%, the uncertainties in the cross-sections should be of similar size or slightly lower, i.e. of the order of 5–10%.

As the experimental devices become larger from one generation to the next, it becomes increasingly more important for the design to predict their behaviour reliably. Among other parameters, the fusion rate in these plasmas must be determined accurately. Also, for reactor design and blanket studies, it is necessary to know the energy dependence of the fusion cross-sections with the greatest possible precision, i.e. an uncertainty in the fusion cross-sections of the order of 5% is desirable.

Beginning in the 1940s, there have been many measurements of the fusion cross-sections, mainly for the four most important reactions, $D(d,n)^3He$, $D(d,p)T$, $T(d,n)^4He$ and $^3He(d,p)^4He$. Since these experiments cover only a limited range of energy and are not always in agreement, the data have been periodically collected and reviewed, usually resulting in some graphical or, more recently, analytical representation of the cross-sections. Today, such parametrizations of the data are essential, since the use of tables is not so convenient.

Two approximate representations of the fusion cross-sections have been widely used, one derived by Duane in 1972 [1], which is also listed in the NRL formulary [2], and the other derived by Peres in 1979 [3]. Duane's cross-sections pose particular difficulties if they are extrapolated to energies below about 20 keV because of the unphysical parametrization of the penetrabilities that was used. In general, however, both approximations as well as the corresponding parametrizations for the Maxwellian reactivity $\langle \sigma v \rangle$ calculated by Hively [4, 5] (on the basis of Duane's cross-section formula) and by Peres [3] are no longer adequate, for two reasons: First, in the last few years there have been new and more accurate measurements of the fusion cross-sections which in any case justify new parametrizations. Second, the development since the 1970s of multi-reaction, many-observable R-matrix analyses for systems containing the major fusion reactions has provided a much better method of evaluating the cross-section data.

In this paper, we make use of cross-section information based on more recent experiments and on the more reliable representations given by R-matrix theory [6] to

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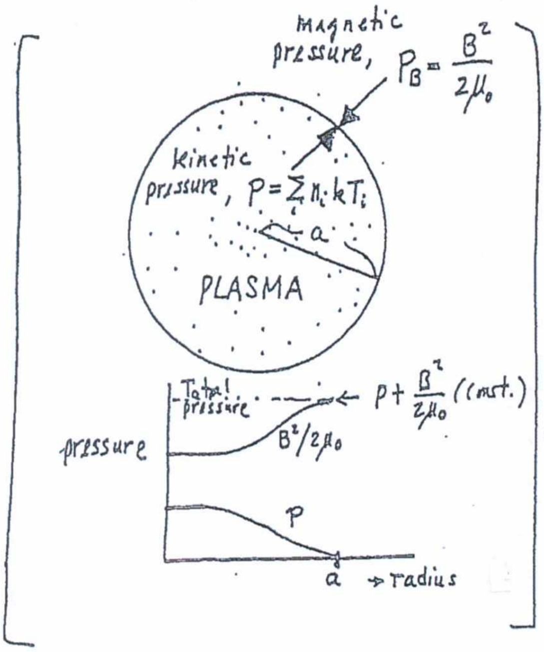
Reactor Design (Continued) —

1. magnetic confinement:



- Condition of equilibrium,

$$p = \frac{B^2}{2\mu_0} \quad \left(\text{or} = \frac{B^2}{8\pi} \right)$$

[assuming that plasma contains no internal mag. field due to diamagnetic property of the plasma.]

- Plasma beta,

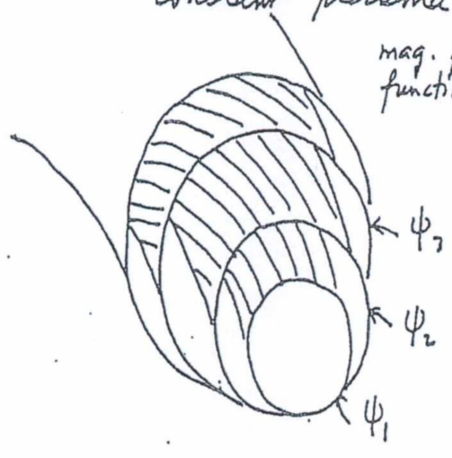
$$\beta \equiv \frac{p}{\left(\frac{B^2}{2\mu_0}\right)} \quad \left(\begin{array}{l} = 1 \text{ desirable, but} \\ < 1 \text{ in practice} \end{array} \right)$$

Eq. of motion: $\rho_m \frac{d\vec{u}}{dt} = \vec{J} \times \vec{B} - \vec{\nabla} p$, where $\rho_m \equiv n_i m_i + n_e m_e \simeq n m_i$,

and $\vec{J} = ne\vec{u}$ [Eq. 5D39] Dolan, Vol. I

At equilibrium: $\vec{J} \times \vec{B} = \vec{\nabla} p$ [assumption of scalar pressure is not valid for anisotropic plasmas]

$\left(\begin{array}{l} \vec{B} \cdot \vec{\nabla} p = 0 \\ \vec{J} \cdot \vec{\nabla} p = 0 \end{array} \right) \Rightarrow$ Both $\vec{B}$ and $\vec{J}$ are perpendicular to the pressure gradient, so $\vec{B}$ and $\vec{J}$ both lie on surfaces of constant pressure. Thus, a toroidal magnetic surface traced out "ergodically" by a field line is also a surface of constant plasma pressure:

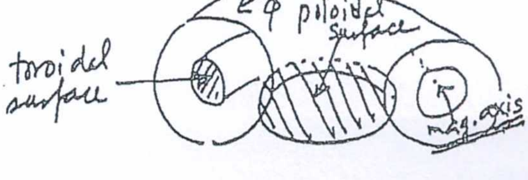


mag. flux function, $\psi \equiv \iint d\vec{S} \cdot \vec{B} \quad (2\pi R A_\phi)$ $\quad \vec{B}_p \equiv \vec{\nabla} \times \vec{A}_\phi$

$= \oint d\vec{l} \cdot \vec{A}$

$\psi_p \equiv \iint d\vec{S} \cdot \vec{B}_p$ [$d\vec{S}$ = poloidal surface]

$\psi_t \equiv \iint d\vec{S} \cdot \vec{B}_t$ [$d\vec{S}$ = toroidal surface]



$$\vec{\nabla} p = \vec{J} \times \vec{B} = \frac{(\vec{\nabla} \times \vec{B})}{\mu_0} \times \vec{B}$$

[Note: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$
$\simeq \mu_0 \vec{J}$ (for $\frac{\partial \vec{E}}{\partial t} \simeq 0$ case)]

$$= \frac{(\vec{B} \cdot \vec{\nabla})\vec{B}}{\mu_0} - \frac{\vec{\nabla} B^2}{2\mu_0}$$

[Note also: $\vec{\nabla}(\vec{A} \cdot \vec{B}) = (\vec{A} \cdot \vec{\nabla})\vec{B} + (\vec{B} \cdot \vec{\nabla})\vec{A} + \vec{A} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{A})$]

$$\Rightarrow \vec{\nabla}\left(p + \frac{B^2}{2\mu_0}\right) = \underbrace{\frac{(\vec{B} \cdot \vec{\nabla})\vec{B}}{\mu_0}}$$: **Pressure** balance eqn.

related to curvature of $\vec{B}$ lines;
and is zero if field lines are straight (as a uniform B_z in a cylinder)

[For field lines with little curvature,

$$\frac{(\vec{B} \cdot \vec{\nabla})\vec{B}}{\mu_0} \simeq 0,$$

then

$$\boxed{p + \frac{B^2}{2\mu_0} \simeq \text{constant}}$$
]

Plasma beta,

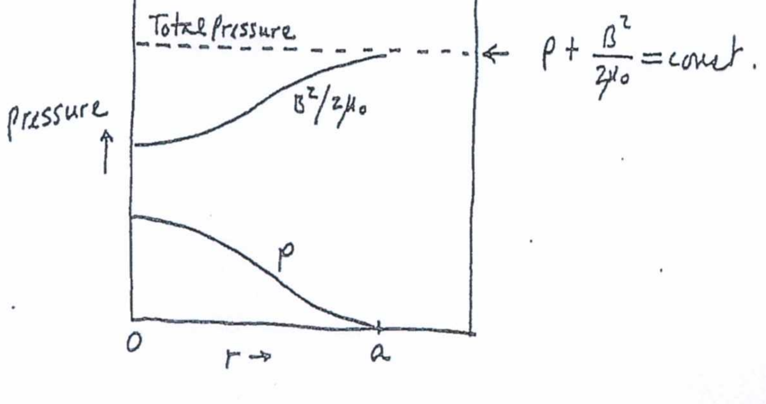
$$\boxed{\beta \equiv \frac{p}{\left(\frac{B^2}{2\mu_0}\right)}}$$

peak $\hat{\beta} \equiv \frac{p_{max}}{\left(\frac{B^2}{2\mu_0}\right)}$

average $\langle\beta\rangle \equiv \frac{\langle p \rangle}{(B^2/2\mu_0)}$ [$\langle\ \rangle \equiv$ volume average]

rms $\beta_* \equiv \frac{\sqrt{\langle p^2 \rangle}}{(B^2/2\mu_0)}$ $\longleftrightarrow$ [$\langle p^2 \rangle = \frac{\beta_*^2 B^4}{(2\mu_0)^2}$
$\langle P_f \rangle \propto \beta_*^2 B^4$ [Eq. 2D18 Dolan, Vol. I]]

For $\hat{\beta} = 0.3$, and $\langle\beta\rangle = 0.1$, we have



2. Plasma Equilibrium :

Axisymmetric Toroidal Equil. $\left[\frac{\partial}{\partial\phi}=0\right]$ (ϕ = toroidal direction)

→ Grad-Shafranov Equation : [See Textbook, p. 175 for derivation [Vol. I Dolan]]

⇒ Pressure balance equ. with $\vec{B}$ in terms of mag. flux function, ψ

$$\psi \equiv \iint d\vec{s}\cdot\vec{B} = \iint d\vec{s}\cdot(\vec{\nabla}\times\vec{A}) = \oint d\vec{l}\cdot\vec{A} \rightarrow \text{e.g. } \psi_p = 2\pi R A_\phi$$

Now $\vec{B}\equiv\vec{\nabla}\times\vec{A} \rightarrow \vec{B}_p = \vec{\nabla}\times\vec{A}_\phi$

Cylindrical coord. (R,ϕ,z):

$$\psi \equiv RA_\phi = \frac{\psi_p}{2\pi}$$

$$\vec{\nabla}\phi = \frac{\hat{\phi}}{R}$$

$$\vec{A}_\phi = \frac{\psi}{R}\hat{\phi}$$

using

$$\therefore \vec{B}_p = \vec{\nabla}\times\left(\frac{\psi}{R}\hat{\phi}\right)$$

$$= \vec{\nabla}\times(\psi\vec{\nabla}\phi)$$

$$= \vec{\nabla}\psi\times\vec{\nabla}\phi$$

$$= \vec{\nabla}\psi\times\frac{\hat{\phi}}{R} = \frac{\hat{R}}{R}(-)\frac{\partial\psi}{\partial z} + \frac{\hat{z}}{R}\frac{\partial\psi}{\partial R}$$

& $\vec{\nabla}p = (\vec{J}\times\vec{B})$

$$= (\vec{J}_{pol}+\vec{J}_{tor})\times(\vec{B}_{pol}+\vec{B}_{tor})$$

$$= (\vec{J}_p\times\vec{B}_t) + (\vec{J}_t\times\vec{B}_p)$$

$$= \left(\frac{\vec{\nabla}\times\vec{B}_t}{\mu_0}\times B_t\right) + \left(\frac{\vec{\nabla}\times\vec{B}_p}{\mu_0}\times\vec{B}_p\right)$$

$$= \frac{1}{\mu_0}\left[(\vec{\nabla}\times\vec{B}_t)\times\vec{B}_t + (\vec{\nabla}\times\vec{B}_p)\times B_p\right]$$

$$\left[\vec{B}_t = B_t\,\hat{\phi}\right]$$

Grad-Shafranov equ. is :

$$\Rightarrow \boxed{-\left[R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial}{\partial R}\right)+\frac{\partial^2}{\partial z^2}\right]\psi = \mu_0\left[R^2\cdot\frac{dP(\psi)}{d\psi} + \frac{I_p(\psi)}{2\pi}\cdot\frac{1}{2\pi}\frac{dI_p(\psi)}{d\psi}\right]} : \psi(R,z)$$

$\vec{J}_t\times\vec{B}_p$ force; press. grad.; $\vec{J}_p\times\vec{B}_t$ force

poloidal currents

Note: $\nabla^2\big|_{cyl} \equiv \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial z^2}$: Given ψ, can find (I, I'), P (pressure); but in general ψ is solved with assumed I & P.

Note Added –

For a magnetic mirror field which is symmetric around the axis (Fig. 8A1) $B_\phi = 0$. This equation tells us that any change in the axial field B_z in the z direction must be accompanied by a change in the radial field B_r. If we integrate over a small cylindrical volume element $2\pi r dr dz$, we find

$$\partial\psi_r/\partial r = - \partial\psi_z/\partial z \tag{8B12}$$

where ψ_r and ψ_z are the magnetic fluxes in the r and z directions. Thus, this equation represents conservation of magnetic flux. All the field lines which enter the volume must leave it.

A similar relation applies to the current density. In view of vector relation (F25), the divergence of Eq. (8B2) yields

$$\vec{\nabla}\cdot\vec{J} = 0 \quad . \tag{8B13}$$

In a toroidal coordinate system with nearly circular flux surfaces

$$\vec{\nabla}\cdot\vec{B} \simeq \frac{1}{rR}\frac{\partial}{\partial r}(rRB_r) + \frac{1}{rR}\frac{\partial}{\partial\theta}(RB_p) + \frac{1}{R}\frac{\partial}{\partial\phi}B_t = 0 \tag{8B14}$$

and likewise for $\vec{J}$. By virtue of Eq. (8B5) B_r and J_r are zero for circular magnetic surfaces, and $(\partial/\partial\phi)$ is zero for an axisymmetric torus, so this equation reduces to

$$\frac{\partial}{\partial\theta} RB_p = 0, \qquad RB_p = (\text{function of } r), \quad B_p = B_p(r)R_0/R$$

$$\frac{\partial}{\partial\theta} RJ_p = 0, \qquad RJ_p = (\text{function of } r), \quad J_p = J_p(r)R_0/R \quad . \tag{8B15}$$

8C. Axisymmetric Toroidal Equilibrium

derivation of Grad-Shafranov Equation

The pressure balance equation (8B10) was derived from the force balance equation $\vec{\nabla}p = \vec{J}\times\vec{B}$ and the Maxwell equation $\vec{J} = \vec{\nabla}\times\vec{B}/\mu_0$. This equation can be rederived, expressing $\vec{B}$ in terms of a magnetic flux function labelling the magnetic surfaces. The result is known as the Grad-Shafranov Equation.

Since $\vec{J}$ and $\vec{B}$ both lie along magnetic surfaces, they can be resolved into toroidal and poloidal components, with no component perpendicular to the surfaces:

$$\vec{J} = \vec{J}_p + \vec{J}_t$$

$$\vec{B} = \vec{B}_p + \vec{B}_t \tag{8C1}$$

For simplicity, we will consider only axisymmetric cases here, in which all derivatives in the toroidal direction ϕ are zero. Then from Eq. (5B14) the poloidal magnetic field may be written as

$\vec{B} = \vec{\nabla}\times\vec{A}$

$$\vec{B}_p = \vec{\nabla}\times\vec{A}_\phi \tag{8C2}$$

where $\vec{A}_\phi$ is the magnetic vector potential in the toroidal direction. We will use the cylindrical coordinate system of Fig. 8B1. Let

$$\psi \equiv RA_\phi \quad . \tag{8C3}$$

From the definition of poloidal flux ψ_p (Table 8B1), it can be shown that

$$\psi = \psi_p/2\pi \quad . \tag{8C4}$$

Let $\hat{\phi}$ be a unit vector in the toroidal direction. Using the fact that $\vec{\nabla}\phi = \hat{\phi}/R$ and the relation $\vec{A}_\phi = \psi\hat{\phi}/R$, we can write Eq. (8C2) in the form

$$\vec{B}_p = \vec{\nabla}\times(\psi\hat{\phi}/R) = \vec{\nabla}\times(\psi\vec{\nabla}\phi) = \vec{\nabla}\psi\times\vec{\nabla}\phi = \vec{\nabla}\psi\times\hat{\phi}/R = -(\hat{R}/R)(\partial\psi/\partial z) + (\hat{z}/R)(\partial\psi/\partial R) \tag{8C5}$$

where $\hat{R}$ and $\hat{z}$ are unit vectors, and vector relations (F23), (F24), and (F32) have been used (Appendix F).

Using Eqs. (8C1) in the force-balance Eq. (8B7), we find

$$\vec{\nabla}p = \underline{\vec{J}_p\times\vec{B}_p} + \vec{J}_p\times\vec{B}_t + \vec{J}_t\times\vec{B}_p + \underline{\vec{J}_t\times\vec{B}_t} \quad , \tag{8C6}$$

where the underlined terms are equal to zero. The Maxwell equation (8B2) may be written

$$\vec{J} = \nabla\times(\vec{B}_p+\vec{B}_t)/\mu_0 = \vec{\nabla}\times\vec{B}_p/\mu_0 + \vec{\nabla}\times\vec{B}_t/\mu_0 \tag{8C7}$$

and we can identify

$$\vec{J}_p = \vec{\nabla}\times\vec{B}_t/\mu_0 \qquad \vec{J}_t = \vec{\nabla}\times\vec{B}_p/\mu_0 \quad . \tag{8C8}$$

After multiplying Eq. (8C6) by μ_0 and using (8C8), we have

$$\mu_0\vec{\nabla}p = (\vec{\nabla}\times\vec{B}_t)\times\vec{B}_t + (\vec{\nabla}\times\vec{B}_p)\times\vec{B}_p \quad . \tag{8C9}$$

With $\vec{B}_t = B_t\hat{\phi}$ and $\vec{B}_p$ given by various expressions in Eq. (8C5), this becomes

$$\mu_0\vec{\nabla}p = [\vec{\nabla}\times B_t\hat{\phi}]\times B_t\hat{\phi} + [\vec{\nabla}\times(-\frac{1}{R}\frac{\partial\psi}{\partial z}\hat{R} + \frac{1}{R}\frac{\partial\psi}{\partial R}\hat{z})]\times(\vec{\nabla}\psi\times\hat{\phi}/R)$$

$$= [-\frac{\partial B_t}{\partial z}\hat{R} + \frac{1}{R}\frac{\partial}{\partial R}(RB_t)\hat{z}]\times B_t\hat{\phi} - [\frac{\partial}{\partial R}(\frac{1}{R}\frac{\partial\psi}{\partial R}) + \frac{1}{R}\frac{\partial^2\psi}{\partial z^2}]\hat{\phi}\times(\vec{\nabla}\psi\times\hat{\phi}/R) \quad , \tag{8C10}$$

See next page

where (F34) has been used to evaluate the components of the vector curls in cylindrical coordinates (with $r \rightarrow R$). Since the parameter ψ is proportional to the poloidal flux [Eq. (8C4)] we can use it to label the magnetic surfaces, and write

$$p = p(\psi) \quad . \tag{8C11}$$

Then

Appendix F. Vector Relations

Identities

$$\vec{\nabla}p(\psi) = \frac{dp}{d\psi}\vec{\nabla}\psi \tag{F13}$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B}) \equiv (\vec{A}\vec{B}\vec{C}) \tag{F14}$$

$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C}) \tag{F15}$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C} = -(\vec{B}\times\vec{C})\times\vec{A} \tag{F16}$$

$$(\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D}) = (\vec{A}\vec{B}\vec{D})\vec{C} - (\vec{A}\vec{B}\vec{C})\vec{D} = (\vec{A}\vec{C}\vec{D})\vec{B} - (\vec{B}\vec{C}\vec{D})\vec{A} \tag{F17}$$

$$\vec{\nabla}(\phi+\psi) = \vec{\nabla}\phi + \vec{\nabla}\psi \tag{F18}$$

$$\vec{\nabla}\cdot(\vec{A}+\vec{B}) = \vec{\nabla}\cdot\vec{A} + \vec{\nabla}\cdot\vec{B} \tag{F19}$$

$$\vec{\nabla}\times(\vec{A}+\vec{B}) = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{B} \tag{F20}$$

$$\vec{\nabla}(\phi\psi) = \phi\vec{\nabla}\psi + \psi\vec{\nabla}\phi \tag{F21}$$

$$\vec{\nabla}\cdot(\phi\vec{A}) = \phi\vec{\nabla}\cdot\vec{A} + \vec{\nabla}\phi\cdot\vec{A} \tag{F22}$$

$$\vec{\nabla}\times(\phi\vec{A}) = \phi\vec{\nabla}\times\vec{A} + \vec{\nabla}\phi\times\vec{A} \tag{F23}$$

$$\vec{\nabla}\times\vec{\nabla}\phi = 0 \tag{F24}$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0 \tag{F25}$$

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B}) \tag{F26}$$

$$\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A}) \tag{F27}$$

$$(\vec{A}\cdot\vec{\nabla})\vec{B} \equiv A_x\frac{\partial\vec{B}}{\partial x} + A_y\frac{\partial\vec{B}}{\partial y} + A_z\frac{\partial\vec{B}}{\partial z} \tag{F28}$$

$$\vec{\nabla}(\vec{A}\cdot\vec{B}) = (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A} + \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) \tag{F29}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} \tag{F30}$$

Cylindrical Geometry

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\phi}}{r}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z} \tag{F31}$$

$$\vec{\nabla}\psi = \hat{r}\frac{\partial\psi}{\partial r} + \frac{\hat{\phi}}{r}\frac{\partial\psi}{\partial\phi} + \hat{z}\frac{\partial\psi}{\partial z} \tag{F32}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \tag{F33}$$

$$\vec{\nabla}\times\vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\phi} + \left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) - \frac{1}{r}\frac{\partial A_r}{\partial\phi}\right)\hat{z} \tag{F34}$$

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} \tag{F35}$$

$$\nabla^2\vec{A} = \left[\nabla^2 A_r - \frac{1}{r^2}\left(A_r + 2\frac{\partial A_\phi}{\partial\phi}\right)\right]\hat{r} + \left[\nabla^2 A_\phi - \frac{1}{r^2}\left(A_\phi - 2\frac{\partial A_r}{\partial\phi}\right)\right]\hat{\phi} + \nabla^2 A_z\hat{z} \tag{F36}$$

$$\begin{aligned}(\vec{A}\cdot\vec{\nabla})\vec{B} = &\left[A_r\frac{\partial B_r}{\partial r} + \frac{A_\phi}{r}\frac{\partial B_r}{\partial\phi} + A_z\frac{\partial B_r}{\partial z} - \frac{1}{r}A_\phi B_\phi\right]\hat{r}\\ &\left[A_r\frac{\partial B_\phi}{\partial r} + \frac{A_\phi}{r}\frac{\partial B_\phi}{\partial\phi} + A_z\frac{\partial B_\phi}{\partial z} + \frac{1}{r}A_\phi B_r\right]\hat{\phi}\\ &\left[A_r\frac{\partial B_z}{\partial r} + \frac{A_\phi}{r}\frac{\partial B_z}{\partial\phi} + A_z\frac{\partial B_z}{\partial z}\right]\hat{z}\end{aligned} \tag{F37}$$

$$\vec{\nabla} p = p' \vec{\nabla}\psi \tag{8C12}$$

where $p' \equiv dp/d\psi$. By definition, let

$$\Delta^*\psi \equiv R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right) + \frac{\partial^2\psi}{\partial z^2} \quad . \tag{8C13}$$

Using Eq. (F16), Eq. (8C10) becomes

$$\mu_0 p'\vec{\nabla}\psi = -\frac{B_t}{R}\left[\frac{\partial}{\partial R}(RB_t)\hat{R} + R\frac{\partial B_t}{\partial z}\hat{z}\right] + \frac{1}{R^2}\Delta^*\psi\left[\vec{\nabla}\psi(\hat{\phi}\cdot\hat{\phi})\ ) - \underline{\hat{\phi}(\vec{\nabla}\psi\cdot\hat{\phi})}\right] \tag{8C14}$$

where the underlined term is zero, since $\vec{\nabla}\psi$ is perpendicular to the magnetic surfaces, hence orthogonal to $\hat{\phi}$. The product $\hat{\phi}\cdot\hat{\phi} = 1$. With the definition of the gradient in cylindrical geometry (F32), we can write Eq. (8C14) as

$$\mu_0 p'\vec{\nabla}\psi = -\frac{(RB_t)}{R^2}\vec{\nabla}(RB_t) - \frac{1}{R^2}\Delta^*\psi\ \vec{\nabla}\psi \quad . \tag{8C15}$$

Since this is a vector equation, $\vec{\nabla}(RB_t)$ must be parallel to $\vec{\nabla}\psi$, which means that RB_t is also a constant on each magnetic surface. Therefore, we can let

$$I(\psi) \equiv RB_t \quad . \tag{8C16}$$

From the definition of poloidal current density (Table 8B1), it can be shown that

$$I(\psi) = \mu_0 I_p/2\pi \quad , \qquad I_p \equiv \int_{\text{Poloidal Surface}} d\vec{S}\cdot\vec{J}_p \tag{8C17}$$

Now

$$\vec{\nabla}(RB_t) = I'\vec{\nabla}\psi \quad , \qquad I' \equiv \frac{dI}{d\psi} \tag{8C18}$$

and the vector $\vec{\nabla}\psi$ cancels out of Eq. (8C15), leaving

$$\boxed{R^2\mu_0 p' = -II' - \Delta^*\psi} \tag{8C19}$$

which is called the *Grad-Shafranov Equation*. The pressure gradient p' is balanced by the terms II' representing the $\vec{J}_p \times \vec{B}_t$ force and $\Delta^*\psi$ representing the $\vec{J}_t \times \vec{B}_p$ force. If the shape of the flux surfaces were known, then the poloidal current and pressure distributions $I(\psi)$ and $p(\psi)$ could be determined from the transport equations (Section 8F). Usually, some simple distributions $I(\psi)$ and $p(\psi)$ are assumed, and the Grad-Shafranov equation is solved for $\psi(R,z)$.

properties of the Grad-Shafranov equation

There are three simple limiting cases of Eq. (8C19) (Bateman, 1978):

(1) $II' \approx 0$. In this case $J_p \approx 0$ inside the plasma, and $B_t \propto 1/R$. The plasma pressure is balanced entirely by the $\vec{J}_t \times \vec{B}_p$ pinching force of the

self-magnetic field induced by the plasma toroidal current. For this case the poloidal beta $\beta_p \approx 1$.

(2) Regions where $\vec{J}$ is parallel to $\vec{B}$. In such regions, the force $\vec{J}\times\vec{B} \approx 0$, $p' \approx 0$, and the pressure is uniform. If such a force-free region covers most of the plasma, then confinement is poor, and $\beta_p \ll 1$.

(3) $|II'| \gg |\Delta^*\psi|$. In this case, the plasma pressure is balanced mainly by the $\vec{J}_p\times\vec{B}_t$ force, and $\beta_p > 1$.

Variations of B_t with R for these cases are shown in Fig. 8C1.

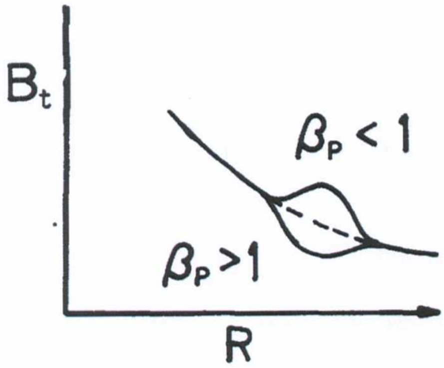


Fig. 8C1. Variations of toroidal magnetic field B_t with major radius R for three extreme cases of poloidal beta. Reprinted from MHD Instabilities by G. Bateman, Fig. 4.5, by permission of the MIT Press, Cambridge, Massachusetts. Copyright 1978 by the Massachusetts Institute of Technology.

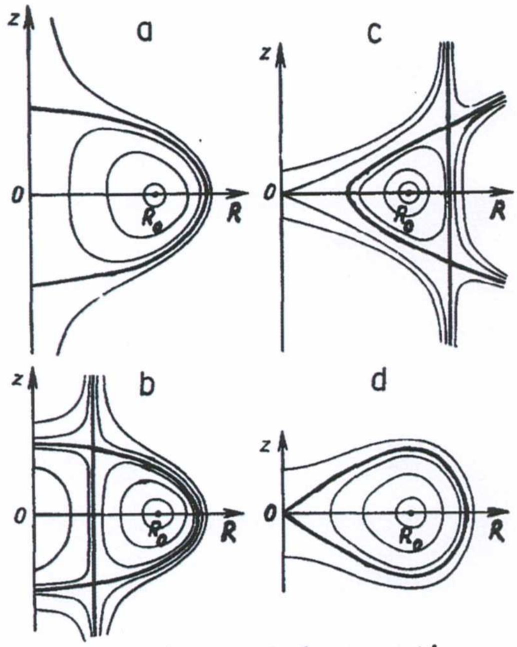


Fig. 8C2. Shapes of the magnetic surfaces ψ = constant for the simple equilibrium case of Eq. (8C21), for various values of b/a. (a) b/a = 0, (b) b/a = - 1/7, (c) b/a = -7, (d) b/a = 1. From L.S. Solov'ev, Hydromagnetic stability of closed plasma configurations, Reviews of Plasma Physics, Vol. 6, Consultants Bureau, New York, 1976, Fig. 2.

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The variation of the $\vec{J}_p\times\vec{B}_t$ force with R makes the plasma tend to move in the R direction. If a vertical magnetic field B_z is applied, the $\vec{J}_t\times\vec{B}_z$ force can be used to control this motion.

The Grad-Shafranov equation has been solved analytically only for a few simple cases. One such case is

$$p' = -a$$

$$II' = -bR_0^2 \tag{8C20}$$

where a and b are constants. The exact solution for this case is

$$\psi(R,z) = \frac{1}{2}(bR_0^2 + cR^2)z^2 + \frac{1}{8}(a - c)(R^2 - R_0^2)^2 \tag{8C21}$$

where R_0 is the radius of the magnetic axis and c is a constant. The shapes of the magnetic surfaces ψ = constant for this case are shown in Fig. 8C2 for various values of the ratio b/a.

It is usually not feasible to choose $\psi(R,z)$ and then try to find $p'(\psi)$ and $II'(\psi)$, because only a few flux functions $\psi(R,z)$ are suitable for MHD equilibria.

Since the flux function ψ is a natural coordinate of the plasma, it is often convenient to use a coordinate system based on (ϕ, ψ, θ) rather than the cylindrical coordinates (R, ϕ, z), where θ is a coordinate in the poloidal direction, similar to the angle θ of Fig. 7E1. It is possible to choose θ in such a way that the magnetic field lines appear as straight lines in the (ϕ,θ) plane. Such coordinates simplify visualization of MHD stability problems in toroidal devices.

8D. MHD Instabilities

Most of this Section follows the works of Schmidt (1979) and Bateman (1978).

<u>the ball analogy</u>

Consider the cases of a ball in a hole, on a level surface, and on a hill, Fig. 8D1. All three are equilibrium cases. If the ball on the hill is given a slight initial velocity, it will roll downhill and be accelerated as it goes, so the hilltop is an unstable equilibrium. If the ball in the hole is given a small initial velocity, it will oscillate about its initial position, with friction damping the oscillations until it comes to rest at its initial position. The tendency to return to its equilibrium condition is called *stability*.

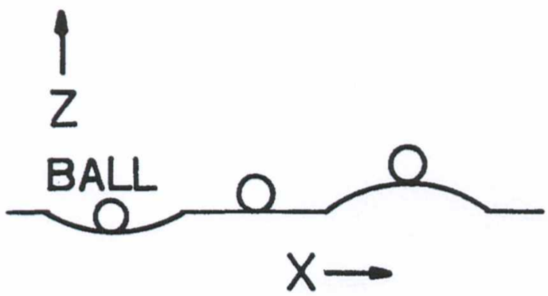


Fig. 8D1. The mechanical analogy of the stability of various equilibrium positions.

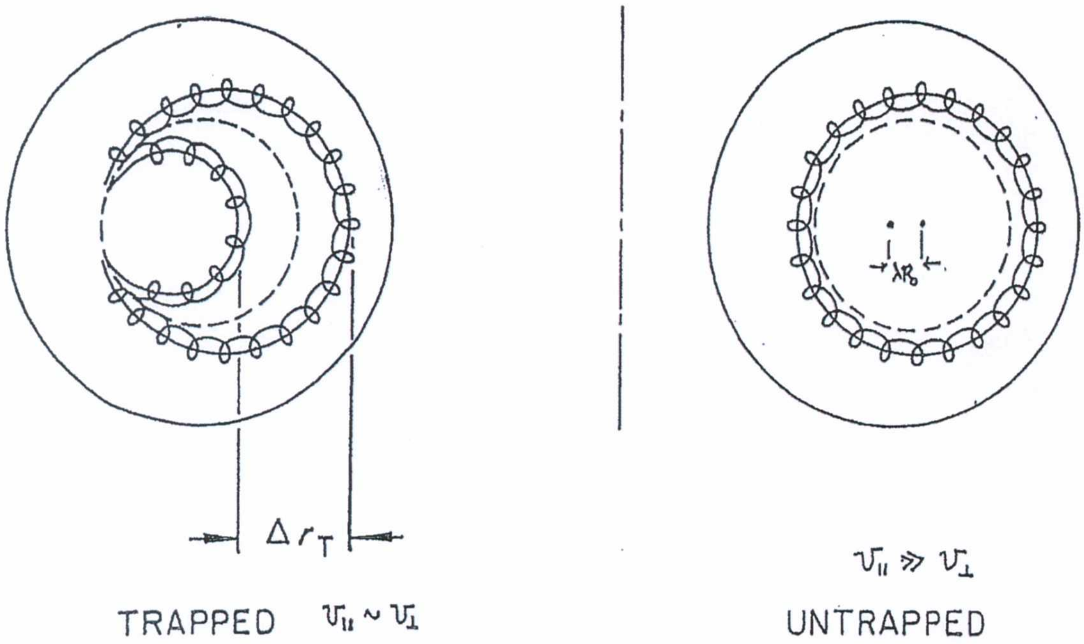
The stability depends upon the sign of the force experienced by the ball as it moves in the x direction. We use a Taylor series expansion for the force F around the initial position x_0, and write the equation of motion

$$m(d^2x/dt^2) = F(x) = F(x_0) + (x-x_0)(dF/dx) + (x-x_0)^2(d^2F/dx^2)/2! + \ldots \qquad (8D1)$$

If x_0 is an equilibrium position, then $F(x_0) = 0$. By definition, let

$$\xi \equiv (x - x_0) \quad . \qquad (8D2)$$

Tokamak Plasma Shift due to $\vec{v}_{drift}$:



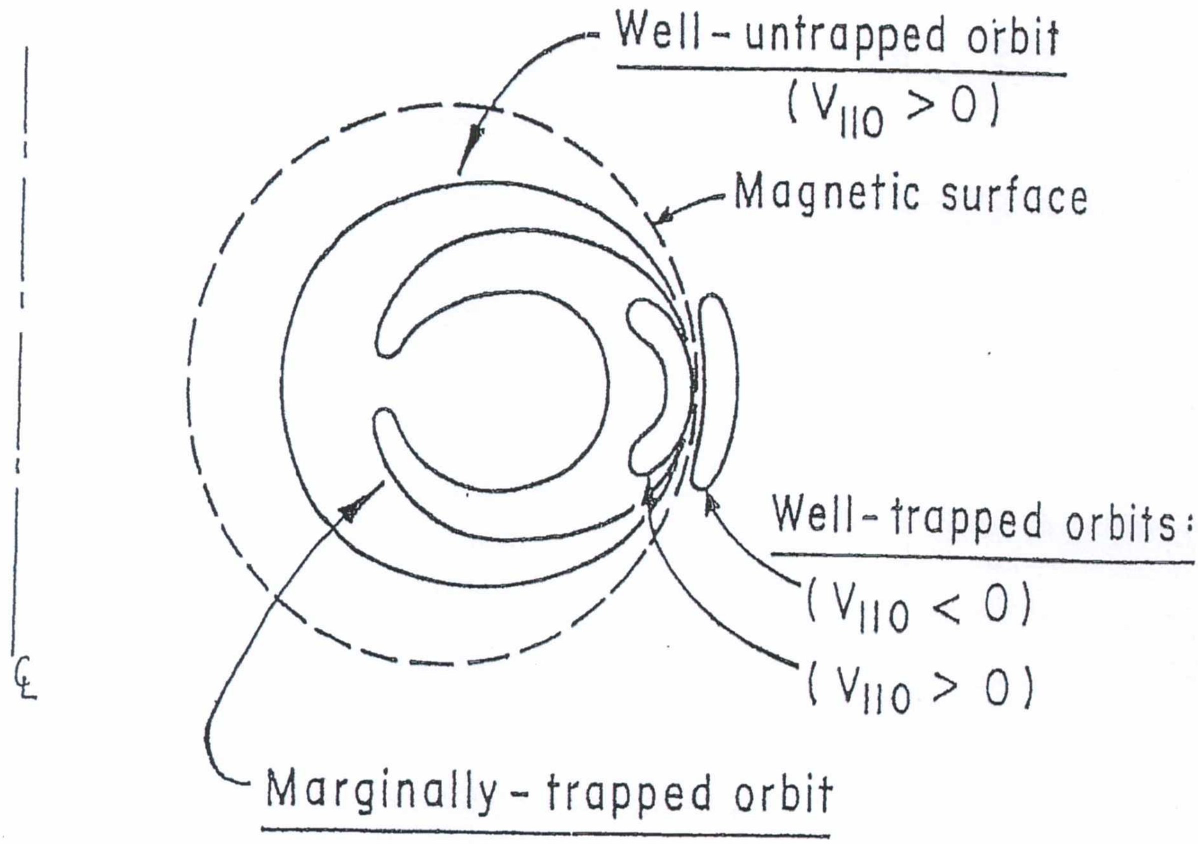


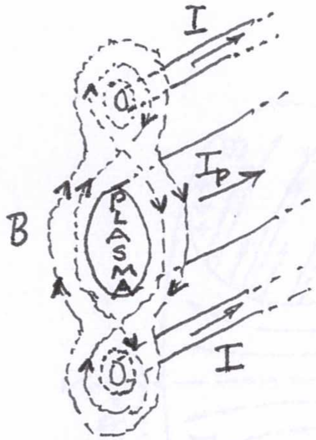
Fig. 3

"Ion" Bananas

Consider a cylinder with elongated cross section —

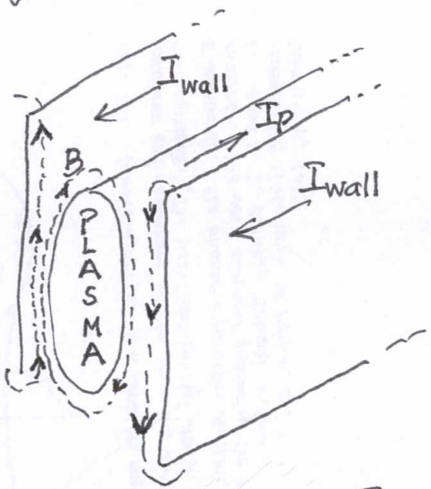
One must use externally applied magnetic fields to change the plasma shape as a straight circular cylinder can maintain its own shape entirely by currents within the plasma.

Two ways to elongate a current-carrying plasma:



$I \parallel I_p$

Divertor-type elongation by pulling on the ends



I_{wall} opposite to I_p

Increased strength of the magnetic field along the side of the plasma squeezes the plasma; an ellipse with uniform longitudinal current density

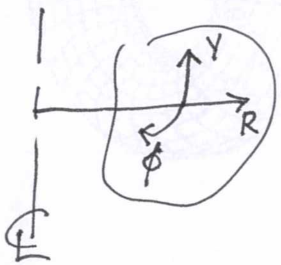
An externally applied quadrupole field will generate the shaping forces.

The shape of the plasma is produced by external currents via quadrupole field: magnetic flux function, ψ,

$$\psi_{\text{equilibrium}} = \psi_{\substack{\text{plasma}\\ \text{currents}}} + \psi_{\substack{\text{external}\\ \text{currents}}} \quad \cdots (4.5.9)$$

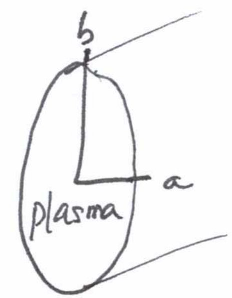
where $\psi \equiv RA_\phi = \psi_{pol}/2\pi$

poloidal flux function



Ref. Glenn Bateman, MHD Instabilities, MIT (1978). P. 71
ISBN: 0-262-02131-5

$$\psi_{equil} = \psi_{\substack{plasma\\ current}} + \psi_{\substack{external\\ currents}},$$



where $$\psi_{equil} = \psi_0 - \frac{\mu}{2}\frac{a^2b^2}{b^2+a^2} J_{z0}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) \cdots (4.5.2)$$

$$\psi_{\substack{plasma\\ current}} = \psi_0 - \frac{\mu}{2}\frac{J_{z0}}{b+a}(bx^2+ay^2) + \cdots \quad (4.5.7)$$

and $$\psi_{\substack{external\\ currents}} = \mu\frac{I_q}{\pi}\left[\left(\frac{x}{r_0}\right)^2 - \left(\frac{y}{r_0}\right)^2 + O\left(\left(\frac{x}{r_0}\right)^6, \left(\frac{y}{r_0}\right)^6\right)\right) \cdots (4.5.10)$$

r_0 is the location of the wires, with I_q, producing quadrupole field, from the center of the plasma

$$\rightarrow \psi_0 - \frac{\mu}{2}\frac{a^2b^2}{b^2+a^2} J_{z0}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) = \psi_0 - \frac{\mu}{2}\frac{J_{z0}}{b+a}(bx^2+ay^2) + \cdots$$
$$+ \mu\frac{I_q}{\pi}\left[\left(\frac{x}{r_0}\right)^2 - \left(\frac{y}{r_0}\right)^2 + \cdots\right]$$

Rearranging the terms,

$$\cancel{\mu}\frac{I_q}{\pi}\left[\left(\frac{x}{r_0}\right)^2 - \left(\frac{y}{r_0}\right)^2 + \cdots\right] = \cancel{\psi_0} - \frac{\mu}{2}\frac{J_{z0}}{b+a}(bx^2+ay^2) + \cdots$$
$$- \cancel{\psi_0} + \frac{\mu}{2}\frac{a^2b^2}{b^2+a^2} J_{z0}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)$$
$$= \frac{\cancel{\mu}}{2} J_{z0}\left[\frac{a^2b^2}{b^2+a^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) - \frac{1}{b+a}(bx^2+ay^2)\cdots\right]$$

For x-component of the lowest terms, x^2,

$$I_q x^2 = \pi \cdot r_0^2 \frac{J_{z0}}{2}\left[\frac{\cancel{a^2}b^2}{b^2+a^2}\frac{1}{\cancel{a^2}} - \frac{b}{b+a}\right]x^2$$

(continue)

x-component (cont'd)

$$I_q = (\pi r_0^2)\cdot\frac{J_{z_0}}{2}\cdot\left[\underbrace{\frac{b^2(b+a)-b(b^2+a^2)}{(b^2+a^2)(b+a)}}\right]$$

$$\frac{\cancel{b^3}+b^2a-\cancel{b^3}-ba^2}{(b^2+a^2)(b+a)} = \frac{ab(b-a)}{(b^2+a^2)(b+a)}\cdot\left[\frac{(b+a)}{(b+a)}\right]^{{}^{\shortparallel}\,1}$$

$$= \frac{ab(b^2-a^2)}{(b^2+a^2)(b+a)^2}$$

$$= ab\cdot\frac{\left[\left(\frac{b}{a}\right)^2-1\right]}{\left(\frac{b^2}{a^2}+1\right)\left(\frac{b}{a}+1\right)^2\cdot a^2}$$

$$I_q = \underbrace{\pi(ab)J_{z_0}}_{I_{plasma}}\cdot\frac{r_0^2}{a^2}\cdot\left[\frac{1}{2}\frac{\left[\left(\frac{b}{a}\right)^2-1\right]}{\left(\frac{b^2}{a^2}+1\right)}\frac{1}{\left(\frac{b}{a}+1\right)^2}\right]$$

$$= \frac{1}{2}\left[\frac{\left(\frac{b}{a}\right)^2-1}{\left(\frac{b}{a}\right)^2+1}\right]\cdot\left[\frac{1}{\left(\frac{b}{a}+1\right)^2}\right]\cdot\frac{r_0^2}{a^2}I_{plasma} \quad \ldots (4.5.11)$$

By symmetry, y-component yields the same result, by exchanging $a \leftrightarrow b$ with the minus sign, thus $-\left(1-\left(\frac{b}{a}\right)^2\right) \Rightarrow \left(\frac{b}{a}\right)^2-1$ in the numerator;

I_q = the same as for the x-component.

$\therefore$ In order to maintain the external current for plasma equilibrium, ($I_q > 0$) $\underline{\underline{b > a}}$ (Non-circular x-section!) e.g., $\underline{\underline{\text{ITER}}}$

When $I_q = 0$, then $\underline{\underline{b = a}} \Rightarrow$ circular x-section! (Tore Supra)

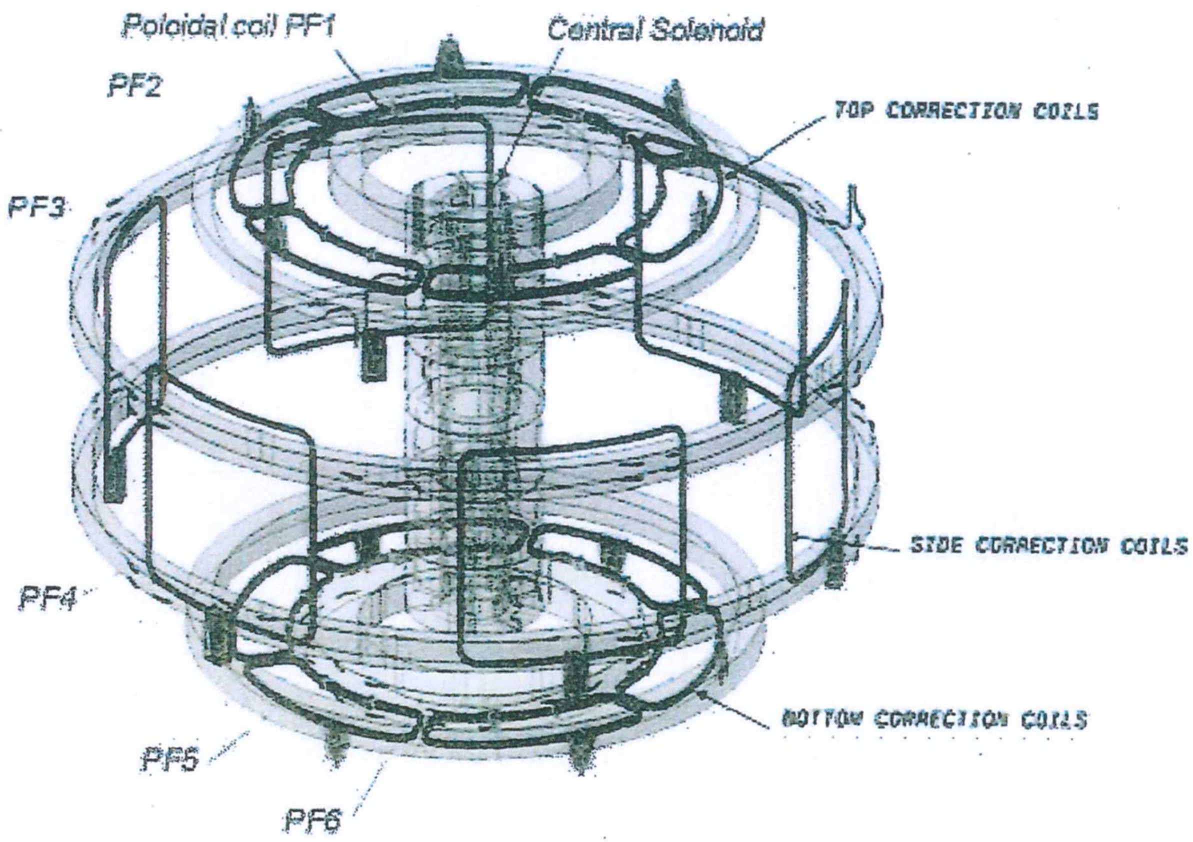


Fig. 1. Layout of correction coils system.

Table 4.8 ITER coil sets

System	Energy (GJ)	Peak field (T)	Cond. length (km)	Total weight (t)
Toroidal field (TE)	41	11.8	82.2	6,540
Central solenoid (CS)	6.4	13.0	35.6	974
Poloidal field (PF)	4	6.0	61.4	2,163
Correction coils(CC)	–	4.2	8.2	85

	Tore Supra (France)	Asdex-U (Germany)	Textor (Germany)	JET (European Union in the UK)	TFTR (USA) (machine closed)	DIIID (USA)	JT-60U (Japan)
Specificity	·· =	r	=	r1<	= 1	r	r
Major radius R (m)	2.36	1.65	1.75	2.96	2.48	1.67	3.45
Minor radius a (m)	0.8	0.5	0.5	1.25	0.85	0.67	1.2
Minorradius b (m)	0.8	0.8	0.5	2.10	0.85	1.36	1.68
Toroidal field (Teslas)	4.5	4	2	3.45	5.2	2.2	4.4
Plasma current (MA)	1.7	1.6	0.65	7	2.5	3.5	5

r : machine fitted with an axisymmetric divertor

1 : machine suitable for tritium

< : machine with remote handling of internal components

= : machine fitted with a limiter

·· : permanent toroidal magnetic field produced supra-conducting magnets

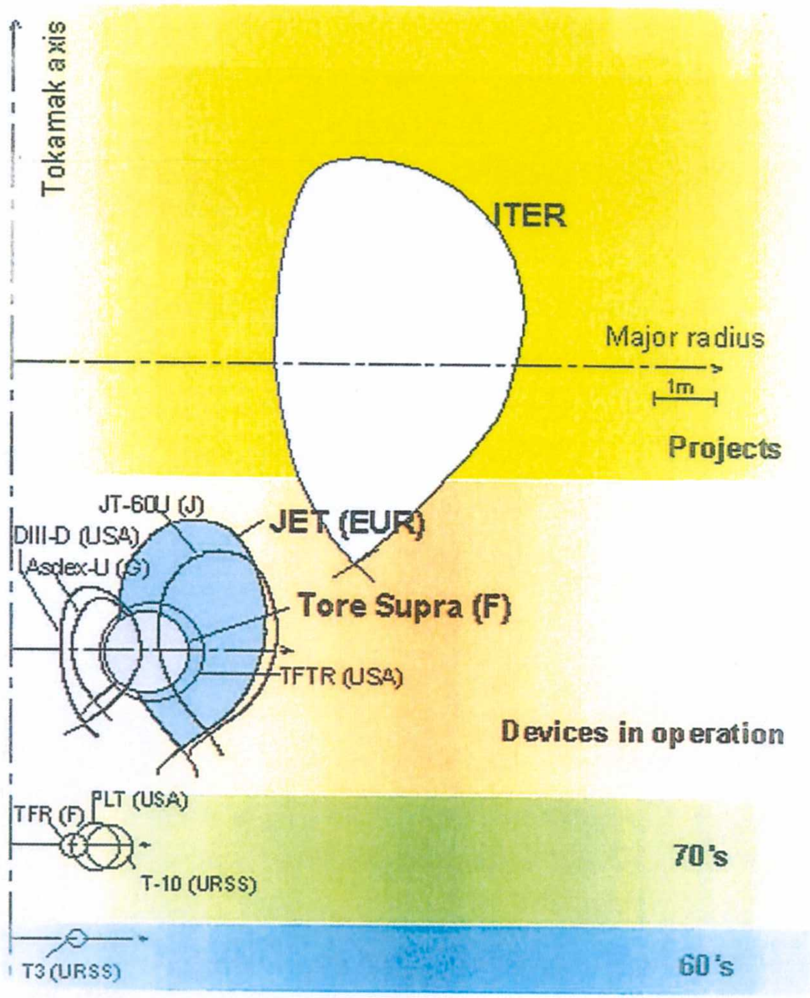


Table 1.7 Some large tokamaks

	D III-D	JT-60	JET	ITER
Location	San Diego, USA	Naka, Japan	Culham, UK	Cadarache, France
R_o, m	1.66	3.4	2.96	6.2
a, m	0.67	1	0.96	2.0
B, T	2.2	4.2	4	5.3
Current I, MA	3	5	6	15
Electron cyclotron heating (ECH) MW	6	4	–	20
Ion cyclotron heating (ICH) MW	5	10	12	20
Neutral beam injection (NBI) MW	20	40	24	39
Lower hybrid waves (LH) MW	–	8	7	0
Main achievements	$\beta > 12$ %	Long pulses ~ 28 s equivalent $Q > 1$. Being upgraded	P(DT) = 15 MW Be walls	Expect $Q \sim 10$

R_o and a are defined in Fig. 1.~~12~~ 1.13 (Additional data may be found at www.tokamak.info)

[Ref. Dolan, MFT, P. 18]

14.14 Plasma shape*

* Ref. J. Wesson, Tokamaks, Oxford (2004).

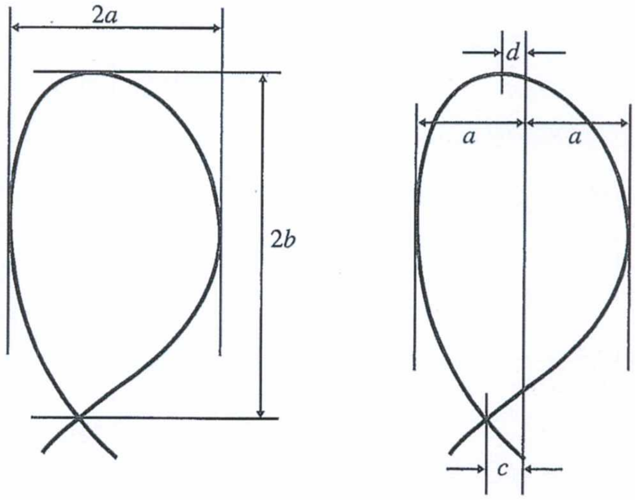


Fig. 14.14.1
Definitions of plasma dimensions

The elongation and triangularity of the plasma are defined using the dimensions shown in Fig. 14.14.1.

Elongation

$\kappa = b/a$. where a is the plasma minor radius, and b is the height of the plasma measured from the equatorial plane.

Triangularity

$$\delta = \frac{(c+d)/2}{a},$$ i.e, the horizontal distance between the plasma major radius R and the X point.

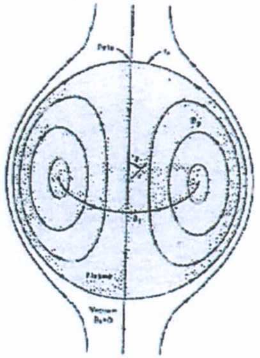
The vertical and horizontal limit of the plasma are defined either by tangents to the plasma edge or by the X-point of a separatrix.

The spectrum of toroidal configurations

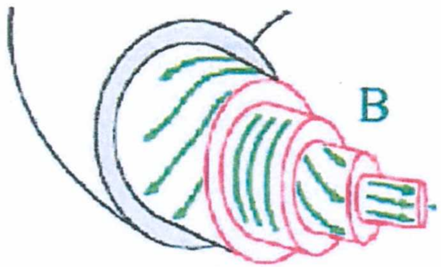
weak ——— *Magnetic field (external)* ———→ *strong*

self-organized

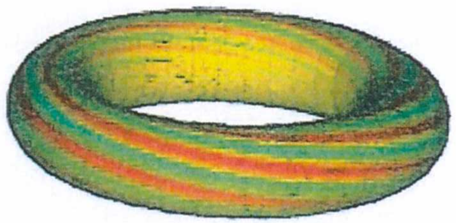
externally controlled



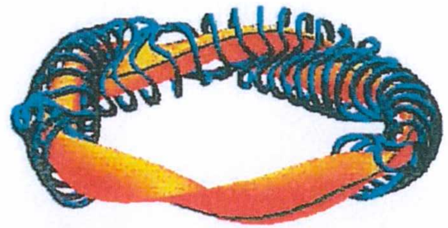
compact torus



reversed field pinch



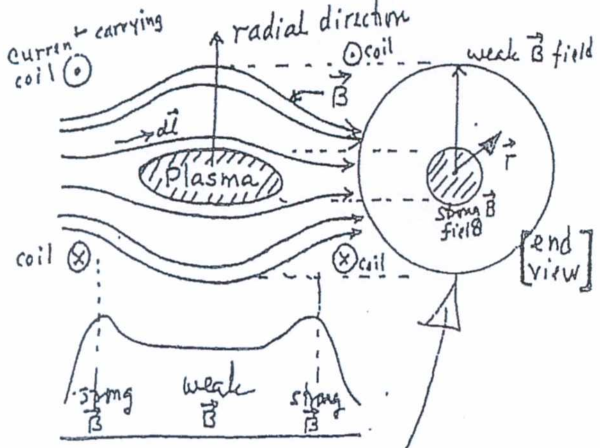
tokamak



stellarator

Plasma Stability of a Mirror Reactor :

A) Simple Mirror



Note : Magnetic field, $\vec{B}$ is decreasing with radius; i.e., "outwardly" "decreasing";

$\therefore \delta B$ decreases

$\Rightarrow \delta \int \frac{dl}{B}$ increases.

$\Rightarrow \Phi \delta \int \frac{dl}{B} > 0$

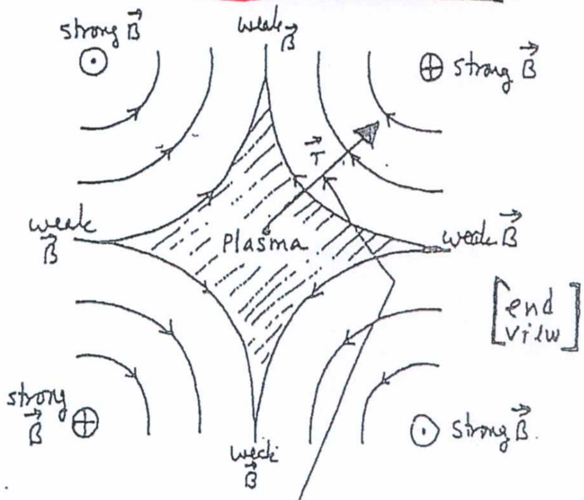
$\Rightarrow \delta V > 0$

Hence, $\delta p \, \delta V < 0$, since $\delta p < 0$ (in general), which leads to $\delta W < 0$ for instability!

$\therefore$ Simple mirror is unstable to interchange mode.

[Note: Convex field curvature is unstable!]

B) Mirror with Ioffe Bars (Cusped Magnetic Field)



Note : Magnetic field, $\vec{B}$ is increasing with radius; i.e., "outwardly" "increasing", which means that the $\vec{B}$ associated with the plasma is the minimum $\vec{B}$ field.

$\therefore \delta B$ increases.

$\Rightarrow \delta \int \frac{dl}{B}$ decreases.

$\Rightarrow \Phi \delta \int \frac{dl}{B} < 0$

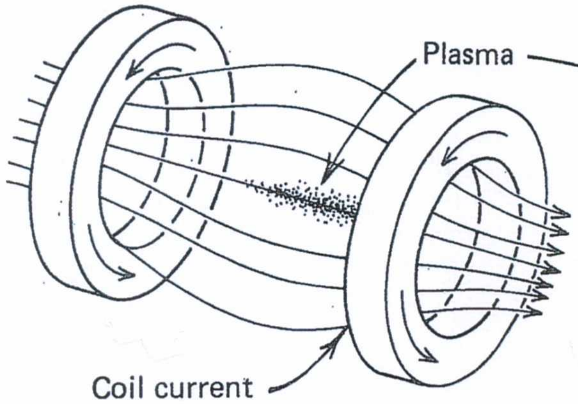
$\Rightarrow \delta V < 0$

Hence, $\delta p \, \delta V > 0$ which leads to $\delta W > 0$ for stability!

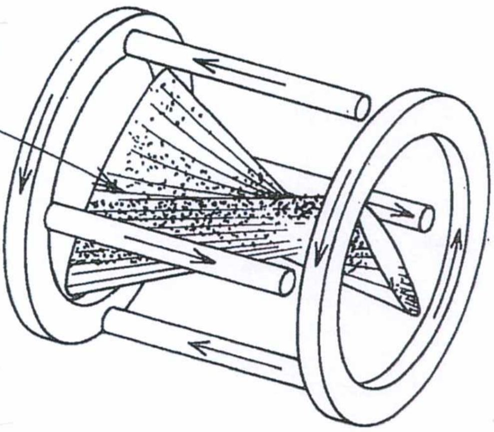
$\therefore$ Cusped mirror geometry is stable to interchange instability.

[Note: Concave field curvature is stable!]

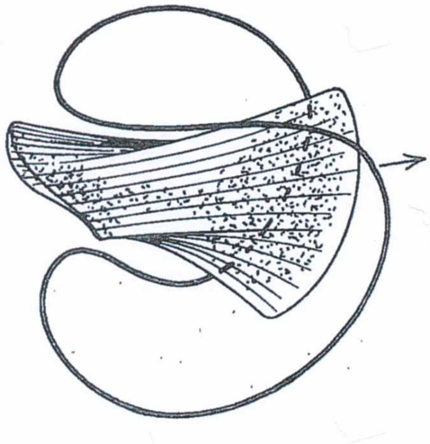
Minimum-B Mirror Configuration:



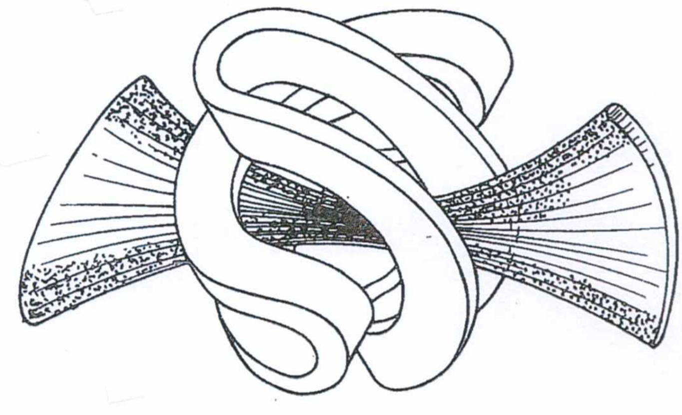
(a) Simple mirror



(b) Joffe bars



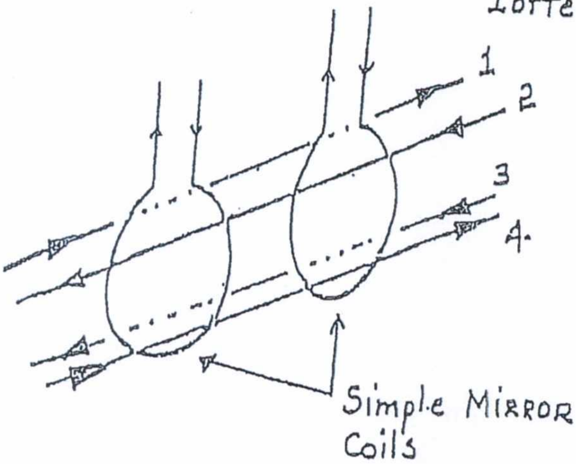
(c) Baseball coil

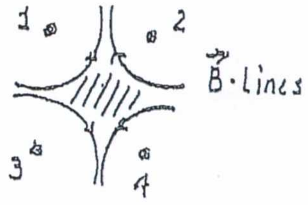


(d) Yin—yang coils

To OBTAIN MINIMUM $\vec{B}$-FIELDS

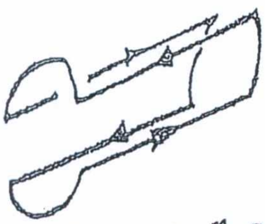
Ioffe Bars (to stabilize simple mirror)





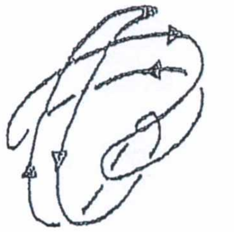
Cross Section

(A.)



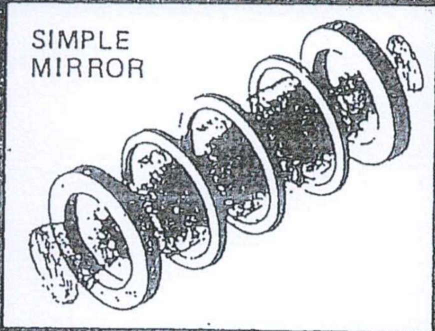
(B.) Evolution of Ioffe Bars to a continuous coil;

(C.) and to a baseball design;



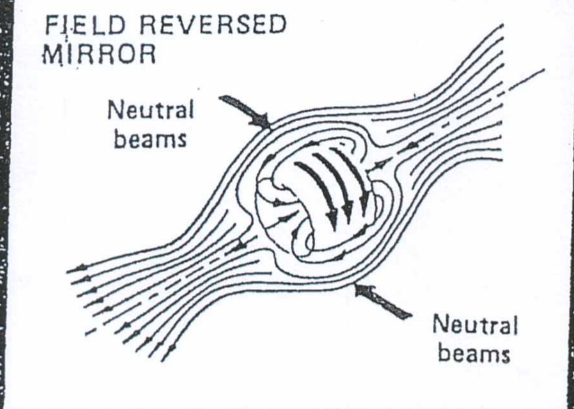
(D.) and finally to the interlocking baseball seams of Yin-Yang coil.

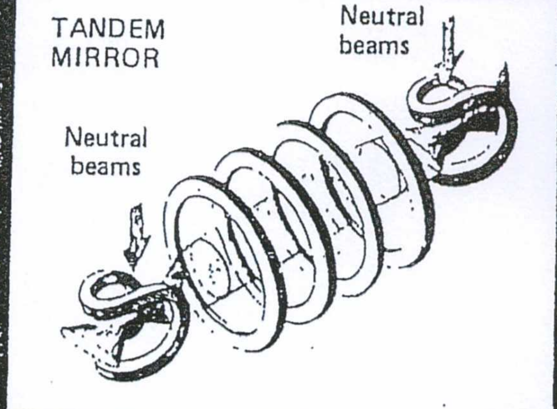
EVOLUTION OF MIRROR FUSION IDEAS



Plasma in simple mirror is vulnerable to <u>interchange</u> <u>instabilities</u>

<u>Stability</u> is achieved by creating <u>minimum - $\vec{B}$</u> field by <u>baseball coil</u> (or by Ioffe bars, or Yin-Yang. coils)





<u>Stability</u> can be also achieved by high-current <u>neutral beam injection</u> to produce <u>strong ion current</u> which reverses the axial magnetic field

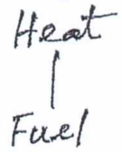
Fusion Power Plant Designs – Revisited

[Recall Lecture Note on "Reactor Power Balance"]

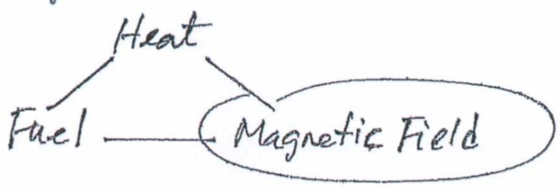
The well-known chemical triangle is:

Without each of these three elements, there is no fire.
Controlling these elements controls the burn → burns steady-state.

Fusion fire "triangle" becomes

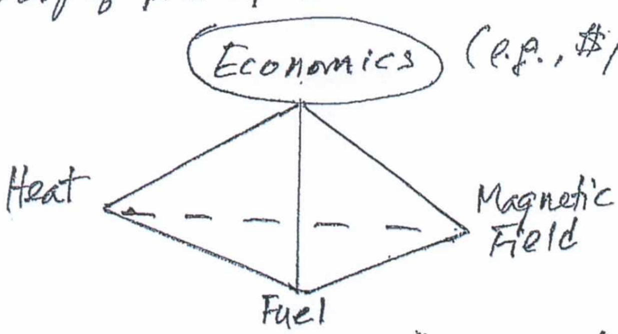


and may be hard to control w/o the third element,



Now controlling the magnetic field allows control of the burn
→ steady-state operation

However, proof of economic feasibility is as important as the proof of principle:



(e.g., \$/KWh, plant load factors...)

Electric utility operating "economics" and how it applies to fusion reactor design should also be considered.

Utilities / industries want fusion reactors

- Small, modular units → mass production; cost effectiveness;
- Clean → environmental advantage & siting flexibility; advanced-fuel operation
- simple → easy maintenance

(• steady-state)
(• economics)
(• public acceptance)

* Economic considerations : - *Simplicity
- Capital cost of plant / COE
- Construction time
- Lifetime (~60 yrs?)
- Fuel-cycle
- Reliability
- Availability (> 90%)
- Load following / Capacity factor
- Maintainability
- Risk
- Finance
- Staff
- Market
- Transmission
- Grid Stability
- Resources
- Natural hazards
- Waste
- Decommissioning
- International (markets?)

* Regulatory simplicity: - Regulations
  - Law suits
  - Safety
  - Emergency planning
  - Emissions (waste heat, etc.)
  - Worker exposure
  - Licensing

* Public acceptance: - Public understanding
  - Environment
  - Waste heat
  - Emissions
  - Radwastes
  - Public concerns
  - Perception
  - Nonproliferation.

※ Reliability –

Component "failure" rate, $\lambda = \dfrac{\text{Number of component failures}}{\begin{pmatrix}\text{Number of operating} \\ \text{Component in} \\ \text{the time interval} \\ \text{"N"}\end{pmatrix} \times \begin{pmatrix}\text{time interval} \\ \text{of interest,} \\ \text{e.g. test period} \\ \text{"t"}\end{pmatrix}}$

Reliability over time period T,

$$Rel = \exp(-\lambda T)$$

<u>Example</u>: A laser fusion experiment has 192 flash lamps used to augument a laser pulse.
Over 20,000 pulses, two lamps fail to operate.
(Note: "pulse" is used instead of a unit of time)
What is the failure rate of the flash lamps and the reliability over 50,000 pulses?

<u>Ans</u>. Failure rate, $\lambda = \dfrac{2 \text{ lamps failures}}{(192 \text{ lamps}) \times (20,000 \text{ pulses})}$

$= \underline{5.2 \times 10^{-7}}$ failures/pulse

Reliability, $Rel = \exp(-\lambda T)$

$= \exp(-5.2 \times 10^{-7} \times 50,000)$

$= 0.974 \quad = \underline{97.4\%}$

※ Availability – $A \equiv \dfrac{\text{uptime}}{\text{uptime} + \text{downtime}} = \dfrac{\text{Mean time bet}^{n} \text{ failures (MTBF)}}{\text{MTBF} + \text{Mean time to repair (MTTR)}}$

<u>Example</u>: The ITER experiment plans to perform 3,000 pulses per year, with 400-second pulses about 45 minutes.
What is the experiment availability?

<u>Ans</u>. $A = \dfrac{0.75 \text{ hr} \times 3,000 \text{ pulses/yr}}{24 \text{ hr} \times 365 \text{ days}} = 0.25 = 25\%$

ITER would operate for about 25% of a calendar year.

Maintainability –

Probability that a failed item or system in a plant can be repaired in a specified time using a specific set of resources

Mean Time To Repair (MTTR),

$$MTTR \equiv \sum_i \frac{T_i}{n},$$

where T_i = summation of repair times
and n = number of component malfunctions that were repaired in some time interval

Example: Machine shop technicians for a fusion experiment repaired 5 different electronic modules removed from the plant instrumentation and control system in a month. The shop times for module repair were 4 hrs, 0.5 hr, 9 hrs, 2.25 hrs, and 1.75 hrs. What is the MTTR ?

ans. $$MTTR = \sum_i \frac{T_i}{n}$$

$$= \frac{4\text{ hr} + 0.5\text{ hr} + 9\text{ hr} + 2.25\text{ hr} + 1.75\text{ hr}}{5\text{ repairs}}$$

$$= 3.5\text{ hr/repair}$$

Capacity factor – $\left(\text{plant Availability } A\right) \times \left(\begin{array}{l}\text{Fraction of the plant's design}\\ \text{electrical rating that}\\ \text{the plant is producing}\end{array}\right)$

Since 1960, over 50 fusion power plant design studies were conducted; 100 tokamaks built and 35 still operating.

STARFIRE Commercial Tokamak Fusion Power Plant Study (1980)
— D-T, steady-state, non-inductive plasma based on continuous rf lower-hybrid current drive and auxiliary ECRH-assisted, limited OH coil for plasma startup.

ARIES (Advanced Reactor Innovative Engineering Study) (1990)
— 1 GWe first-stability regime tokamak, D–T & D–^{3}He

ITER (International Thermonuclear Experimental Reactor) (2000)

— Date	Event
1985	ITER began as a Regan-Gorbachev initiative
1996	ITER Detail Design Report (DDR) completed.
2006-11-21	Seven participants (European Union, India, Japan, China, Russia, Republic of Korea, and the United States) formally agreed to fund the ITER fusion reactor
2008	Site preparation start, ITER itinerary start.
2009	Site preparation completion.
2010	Tokamak complex excavation starts.
2013	Tokamak complex construction starts.
2015	Predicted: Tokamak assembly starts.
2019 ?	Predicted: Tokamak assembly completion. Torus pumpdown starts.
2020	Predicted: Achievement of first plasma
2027	Predicted: Start of deuterium-tritium operation.

European Power Plant Conceptual Study (PPCS) (2004)
— 1 GWe, steady state tokamaks

Force-Free Helical Reactor (FFHR) Large Helical Device (LHD)
— 3 GW_{th} heliotron reactor, tritium breeder blanket concept

Wendelstein 7-X Stellarator construction completed (Oct. 2015)
— $\ell = 2$ stellarator, beyond W VII-AS (Advanced Stellarator).

Review: 50 Years of Fusion Research on MFE & IFE (2010).