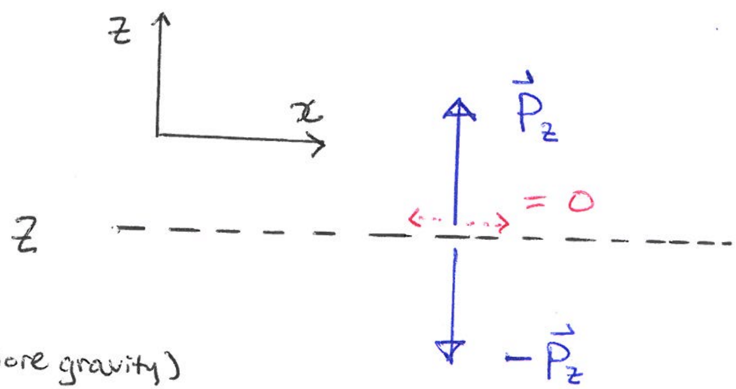
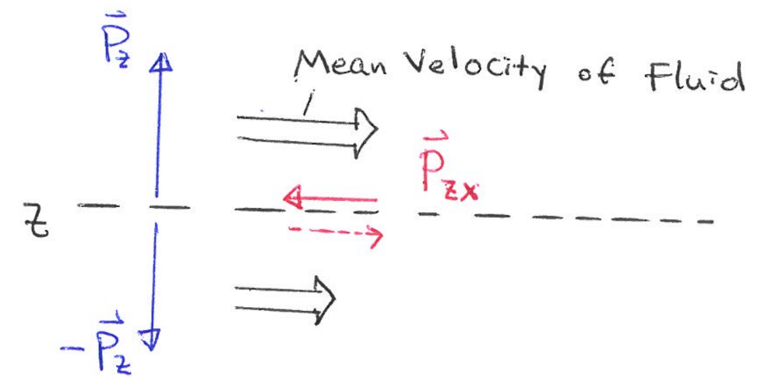
Fundamentals of Engineering Physics 2019

Week 14.

8.2. Viscosity and Transport of Momentum



(ignore gravity)



⊛ In equilibrium, the fluid below the plane exerts "mean force per unit area" (or mean stress) on the fluid above, and vice versa.

Mean stress normal to the plane is nothing but the pressure.

⊛ In this situation, there is no tangential component of stress. (// to the plane).

⊛ Now, mean velocity u_x is a function of z; $u_x = u_x(z)$.

Fig 8.3.

In this case, the fluid below the horizonal imaginary plane ---- exterts a tangential force on the fluid above the plane and vice versa:

This is the tangential component (P_{zx}) of the stress "tensor" $\overleftrightarrow{P}$.

" P_{zx} $\equiv$ the mean force, in the x direction per unit area of the plane, which the fluid below the plane exerts on the fluid above the plane "

*** The quantity P_{ij} (where i and j can denote x, y or z) is called the pressure tensor or stress tensor. It can be represented by a 3×3 matrix in Cartesian coordinate.

(divergence of vector → scalar
divergence of tensor → vector)

** The "pressure" is a diagonal component of the pressure tensor, i.e., $\bar{P}$ in the previous page $= P_{zz}$.

(mean force in normal direction per unit area).

(*) We have already seen that $P_{zx} = 0$ in equilibrium situation where $u_x(z)$ is independent of z. Therefore, if $u_x(z)$ varies smoothly in z and $\frac{\partial}{\partial z} u_x(z)$ is relatively small, we can get the following expression by Taylor expansion of $u_x(z)$ in z;

$$\boxed{P_{zx} = -\underline{\eta} \frac{\partial}{\partial z} u_x(z)} \qquad (14)$$

"Coefficient of viscosity"

(*) We can also conclude that : (from the Cartoon and the Newton's 2nd law).

$\underline{P_{zx}}$ = the mean increase, per unit time and per unit area of the plane, of the x component of momentum of the gas above the plane due to the net transport of momentum by molecules crossing this plane. (transport is in z-direction).

"momentum flux"

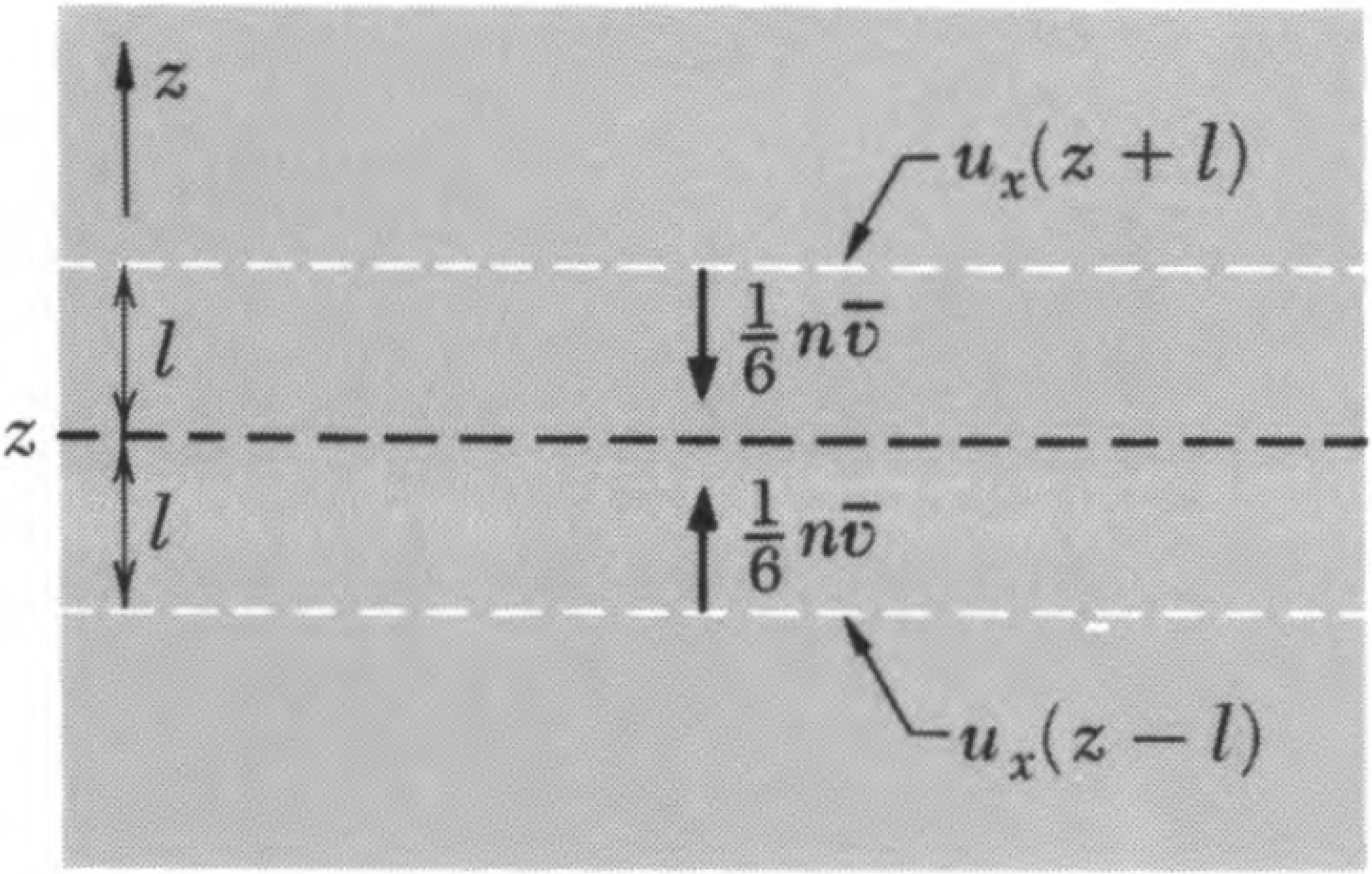


***Fig.* 8.5** Momentum transport by molecules crossing a plane.

(*) In Fig 8.5.,

* [The mean x component of momentum transported per unit time per unit area across the plane in the upward direction] $= \frac{1}{6} n \bar{v} \left[m\, u_x(z-l) \right]$ (16).

* [" x " " downward direction] $= \frac{1}{6} n \bar{v} \left[m\, u_x(z+l) \right]$ (17)

(16) minus (17)

$$\Rightarrow \boxed{P_{zx} = \frac{1}{6} n \bar{v} m \left[u_x(z-l) - u_x(z+l) \right]} \quad (18)$$

(*) Assuming mean free path $l \ll \left| \frac{1}{u_x} \frac{\partial}{\partial z} u_x \right|^{-1}$,

we can approximate $u_x(z \pm l) = u_x(z) \pm l \frac{\partial u_x}{\partial z}, \cdots$

to get

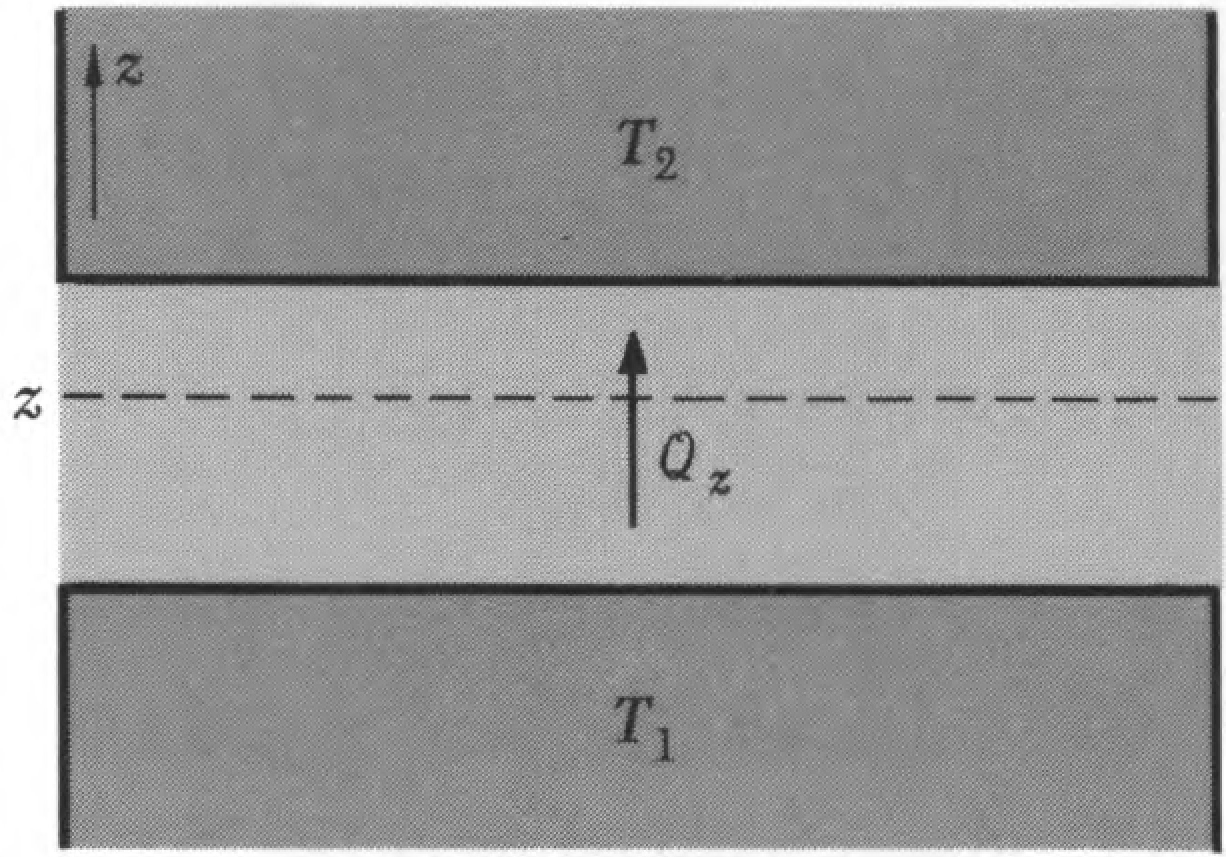
$$* \quad \boxed{P_{zx} \equiv -\eta \frac{\partial u_x}{\partial z}} \tag{19}$$

where

$$\boxed{\eta = \frac{1}{3} n \bar{v} m l} \tag{20}$$

(*) Now, we expressed the viscosity coefficient "η" in terms of the microscopic parameters characterizing the molecules of the gas.

Considering the simplicity of the model, coefficient "$\frac{1}{3}$" should not be taken too seriously. However, the scaling in Eq. (20) is essentially valid.



$T_2 > T_1, \quad Q_z < 0$

Fig. 8.6 A substance in thermal contact with two bodies at respective absolute temperatures T_1 and T_2. If $T_2 > T_1$, heat flows in the $-z$ direction from the region of higher to that of lower temperature; thus Q_z must be negative.

8.3. Thermal Conductivity and Transport of Energy

(*) Consider non-uniform temperature $T = T(z)$ and a plane $z = \text{const.}$

$Q_z \equiv$ the heat crossing unit area of the plane per unit time in the $+z$ direction.

i.e, the Heat Flux Density in z-direction.

- If T is uniform, $Q_z = 0$.
- If T is slowly varying in z, a Taylor expansion will give

$$\boxed{Q_z = -K \frac{\partial T(z)}{\partial z}} \qquad (29)$$

Coefficient of "Thermal Conductivity"

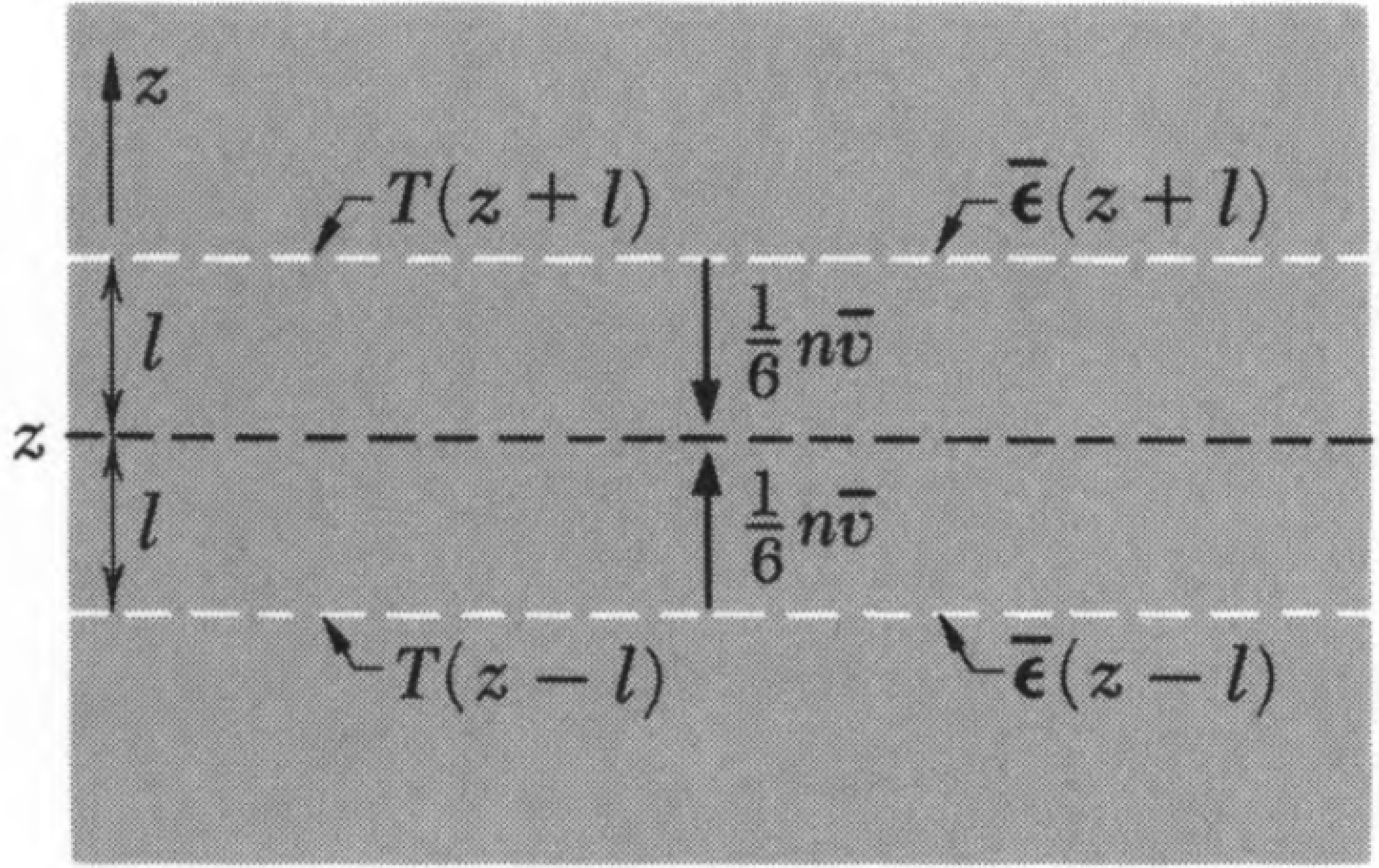


Fig. 8.7 Energy transport by molecules crossing a plane.

(*) From Fig. 8.7

$$\left[\begin{array}{l}\text{The Mean Energy transported per} \\ \text{unit time per unit area across the} \\ \text{plane from below}\end{array}\right] = \frac{1}{6} n \bar{v} \, \bar{\epsilon}(z-l). \quad (30)$$

$$\left[\begin{array}{c}\text{"} \\ \\ \text{from above}\end{array}\right] = \frac{1}{6} n \bar{v} \, \bar{\epsilon}(z+l). \quad (31)$$

mean energy of a molecule.

(30) minus (31) $\Longrightarrow$ and Taylor expansion for small l, we obtain:

$$Q_z = -\frac{1}{3} n \bar{v} l \frac{\partial \bar{\epsilon}}{\partial T} \frac{\partial T(z)}{\partial z} \quad (32)$$

Chain rule used for $\frac{\partial \bar{\epsilon}}{\partial z}$.

* In summary,

$$\boxed{Q_z = -K \frac{\partial T}{\partial z}} \qquad (34)$$

where thermal conductivity $K = \frac{1}{3} n \bar{v} C_v l / N_a$ (35)

with heat capacity (at const volume) $\left(\frac{C_v}{N_a}\right) \equiv \frac{\partial \bar{\epsilon}}{\partial T}$, (33).

N_a : Avogadro's number.

(35) provides an expression for the thermal conductivity "K" of the gas in terms of fundamental molecular quantities.

8.4. Self-Diffusion and Transport of Molecules

* Consider non-uniform distribution of molecule number density $n_1(z)$ and a plane $z = $ constant.

* Flux density of molecules ; $J_z \equiv$ the mean number of labeled molecules crossing unit area of the plane per unit time in the $+z$ direction.

* Once again, $J_z = 0$ if n_1 is uniform. If n_1 is slowly varying in z, a similar consideration as before leads to :

$$\boxed{J_z = -D \frac{\partial n_1(z)}{\partial z}} \qquad (41)$$

Coefficient of self-diffusion.

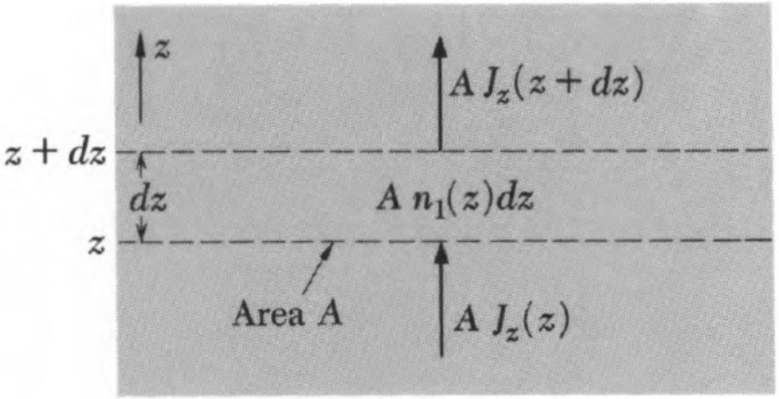


Fig. 8.8 Diagram illustrating the conservation of the number of labeled molecules during diffusion.

The diffusion equation

It is useful to point out that the quantity n_1 satisfies, by virtue of the relation (41), a simple differential equation. Consider a one-dimensional problem where $n_1(z,t)$ is the mean number of labeled molecules per unit volume located at time t near the position z. Focus attention on a slab of substance of thickness dz and of area A. Since the total number of labeled molecules is conserved, we can make the statement that {the increase per unit time in the number of labeled molecules contained within the slab} must be equal to {the number of labeled molecules entering the slab per unit time through its surface at z} minus {the number of labeled molecules leaving the slab per unit time through its surface at $(z + dz)$}. In symbols,

$$\frac{\partial}{\partial t}(n_1 A\, dz) = A\,J_z(z) - A\,J_z(z+dz).$$

Hence

$$\frac{\partial n_1}{\partial t}dz = J_z(z) - \left[J_z(z) + \frac{\partial J_z}{\partial z}dz\right]$$

or

$$\frac{\partial n_1}{\partial t} = -\frac{\partial J_z}{\partial z}. \tag{42}$$

This equation expresses merely the conservation of the number of labeled molecules. Using the relation (41), this becomes

$$\boxed{\frac{\partial n_1}{\partial t} = D\frac{\partial^2 n_1}{\partial z^2}.} \tag{43}$$

This is the desired partial differential equation, the *diffusion equation,* satisfied by $n_1(z,t)$.

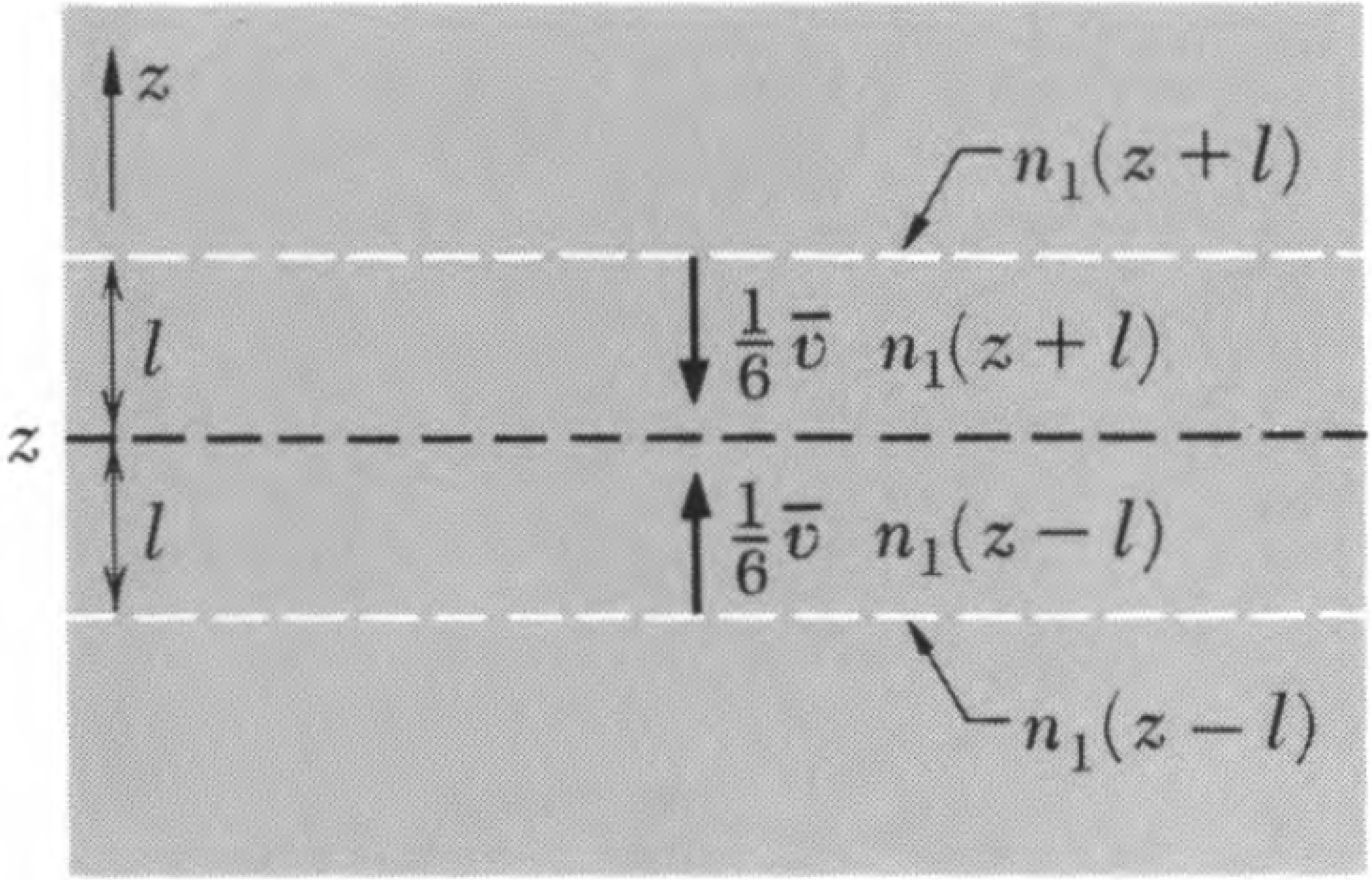


Fig. 8.9 Transport of labeled molecules across a plane.

From Fig 8.9, following the same procedures used for simple calculations of viscosity and thermal conductivity, we get

$$\boxed{J_z = -D \frac{\partial n_1}{\partial z}} \qquad (44)$$

where

$D = \frac{1}{3} \bar{v} l$: Coeff. of self-diffusion in terms of fundamental molecular quantities.

Once again, "$\frac{1}{3}$" factor should not be taken too seriously considering the simplicity of the model.

" Flux-Gradient Relation "

i) • $P_{zx} = -\eta \dfrac{\partial u_x}{\partial z}$

ii) • $Q_z = -K \dfrac{\partial T}{\partial z}$

iii) • $J_z = -D \dfrac{\partial n_1}{\partial z}$

LHS:	RHS:		RHS:
" z-directed Flux " of	" Effective transport coefficient of "	×	gradient (in z direction) of
i) x-directed momentum	" viscosity "		x-comp of velocity
ii) heat	" thermal conductivity "		temperature
iii) and particles	" self-diffusion "		and density