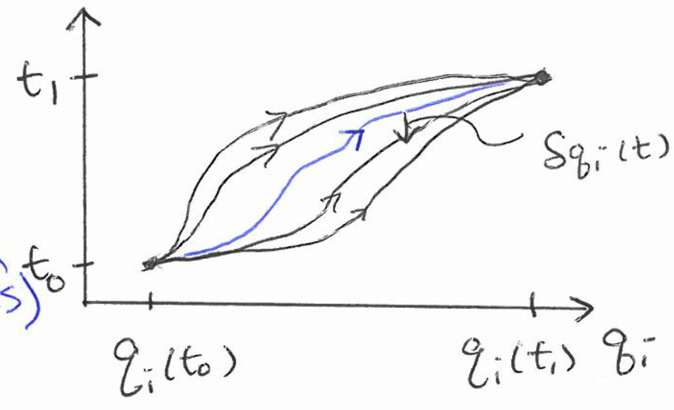
Fundamentals of Engineering Physics 2019

Week 4.

Principle of Least Action.

$$\mathcal{S} \equiv \int_{t_0}^{t_1} \mathcal{L}(q_i, \dot{q}_i)\, dt$$



⊛ Out of infinitely many (and uncountable) trajectories $q_i(t)$ which satisfy fixed values (position (coordinates)) $q_i(t_0)$ at initial time and $q_i(t_1)$ at final time, the nature will choose a path which minimizes "$\mathcal{S}$", i.e. the action.

$$\Rightarrow \quad \delta\mathcal{S} = \int_{t_0}^{t_1} \delta\mathcal{L}(q_i, \dot{q}_i)\, dt = 0$$

→ variational calculus;

$$\delta\mathcal{L}(q_i, \dot{q}_i) = \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\delta\dot{q}_i + \frac{\partial\mathcal{L}}{\partial q_i}\delta q_i \,;$$

now $\delta\dot{q}_i = \delta\dfrac{dq_i}{dt} = \dfrac{d}{dt}\delta q_i$.

Derivation of Euler-Lagrange Eqn.

⊛ → $\delta \mathcal{L}(q_i, \dot{q}_i) = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{d}{dt} \delta q_i + \frac{\partial \mathcal{L}}{\partial q_i} \delta q_i$

$$\therefore \delta S = \int_{t_0}^{t_1} \left\{ \underline{\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) \frac{d}{dt} \delta q_i} + \frac{\partial \mathcal{L}}{\partial q_i} \delta q_i \right\} dt$$

Integrate by parts → $\underbrace{\left. \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta q_i \right]_{t_0}^{t_1}}_{=0} - \int_{t_0}^{t_1} \left(\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) \right) \delta q_i$

0 because $q_i(t_0)$ is fixed → $\delta q_i(t_0) = 0$

$q_i(t_1)$ 〃 → $\delta q_i(t_1) = 0$

$$= \int_{t_0}^{t_1} \left\{ \frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) \right\} \delta q_i \, dt = 0$$

- Since this should be satisfied for arbitrary δq_i, we obtain the following Euler-Lagrange Equation.

$$* \quad \boxed{\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) = \frac{\partial \mathcal{L}}{\partial q_i}}$$

Lecture 7 : Symmetries and Conservation Laws

⊛ The relationship between symmetries and conservation laws is one of the big main themes of modern physics.

Eg.1. Consider a system consisting with two ptls in 1-d.

$$\mathcal{L} = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - V(x_1 - x_2).$$

→ The Lagrangian depends on both x_1 and x_2, neither is cyclic and neither momentum is conserved.

But $\frac{dP_1}{dt} = -\frac{\partial}{\partial x_1}V(x_1-x_2)$ and $\frac{dP_2}{dt} = -\frac{\partial}{\partial x_2}V(x_1-x_2)$ from Eqs. of motion.

If we add them, $\frac{d}{dt}(P_1+P_2) = -\frac{\partial}{\partial x_1}V(x_1-x_2) - \underbrace{\frac{\partial}{\partial x_2}V(x_1-x_2)}_{=-\frac{\partial}{\partial x_1}V(x_1-x_2)} = 0.$

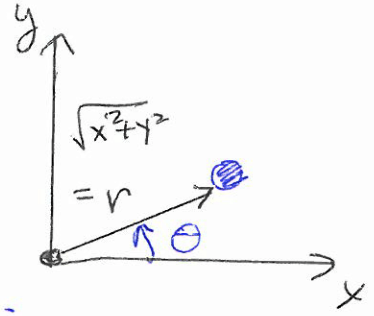
Total momentum is conserved (Compatible with Newton's 3rd Law).

⊛ What is the cyclic variable in this problem?

Transforming to the average and difference of x_1 and x_2, $x_+ \equiv \frac{x_1+x_2}{2}$, $x_- \equiv \frac{x_1-x_2}{2}$,

$\Rightarrow \mathcal{L} = m(\dot{x}_+^2 + \dot{x}_-^2) - V(x_-)$, x_+ is cyclic!

Eg 2., A particle in 2d plane. The potential depends only on the distance from the origin.



$$\mathcal{L} = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - V(x^2+y^2).$$

The system obviously has a rotational symmetry around the origin.

* This symmetry manifests itself more explicitly in polar coordinates.

" $x = r\cos\theta$ and $y = r\sin\theta$ ".

$$\rightarrow \mathcal{L} = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r^2)$$

* This Lagrangian is independent of θ, and θ is the cyclic variable.

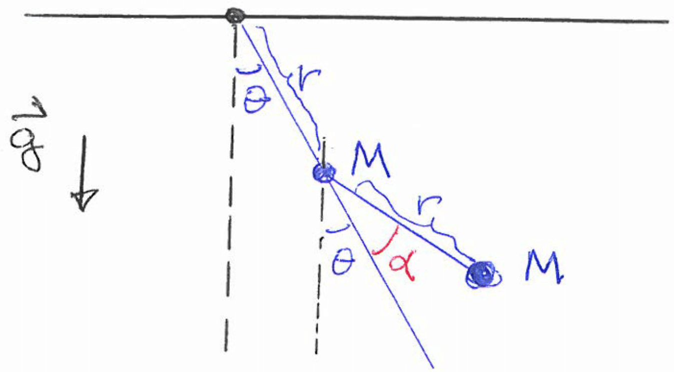
Its conjugate momentum, $p_\theta \equiv \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta}$ is conserved.

(*) For more complex problems, finding symmetries may not be straightforward. You may have to do some pattern recognition.

The double pendulum.

1. Choose some coordinates that uniquely specify the configuration of the components.
   - Keep them as simple as possible

   → Define θ and α.



2. Work out the kinetic energy;

   → Using cartesian coordinates is most straightforward.

Position of the 1st blob (x_1, y_1) ⇒ $x_1 = r\sin\theta$

$y_1 = r\cos\theta$

" 2nd blob (x_2, x_2) ⇒ $x_2 = r(\sin\theta + \sin(\alpha+\theta))$

$y_2 = r(\cos\theta + \cos(\alpha+\theta))$.

→ • Kinetic Energy of the 1st blob : $T_1 \Rightarrow \frac{1}{2} m r^2 \dot{\theta}^2$

• " 2nd blob : $T_2 = \frac{1}{2} m(\dot{x}_2^2 + \dot{y}_2^2) = \cdots = \frac{1}{2} m r^2 (\dot{\theta}^2 + (\dot{\theta} + \dot{\alpha})^2)$

$+ \cancel{\frac{1}{2}} m r^2 \dot{\theta}(\dot{\theta} + \dot{\alpha}) \cos\alpha$

3. Work out the Potential energy: it depends on y_1 and y_2 ⇒

- $V(\theta, \alpha) = -Mg\,[2\cos\theta + \cos(\theta+\alpha)]$.

4. Determine the Lagrangian: $\mathcal{L} = T - V$

- $$\mathcal{L} = \frac{1}{2} m r^2 \left(\dot\theta^2 + (\dot\theta + \dot\alpha)^2\right) + m r^2 \dot\theta(\dot\theta + \dot\alpha)\cos\alpha + mg\left[2\cos\theta + \cos(\theta+\alpha)\right].$$

5. Work out the Euler-Lagrange equations for each degree of freedom. This procedure involves finding the conjugate momenta for each coordinate, $P_i \equiv \dfrac{\partial \mathcal{L}}{\partial \dot q_i}$

⊛ Note that the double pendulum problem has a rotational symmetry for $g = 0$.

Homework

Ex. 5 on page 140

Ex. 6. on page 143

and

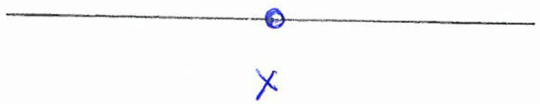
Ex. 7 on page 144.

Lecture 8: Hamiltonian Mechanics

⊛ Time-Translation Symmetry:

— We learned that "symmetry" and "conservation law" go together.

eg. 1,



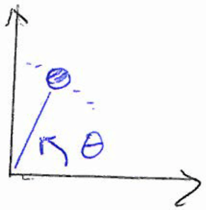
If $\mathcal{L}$ is independent of x,

i.e., translational symmetry (of position)

$\Downarrow$

(linear) momentum P_x is conserved.

— eg. 2,



If $\mathcal{L}$ is independent of θ,

i.e., rotational symmetry

$\Downarrow$

angular momentum P_θ is conserved.

⊛ Then, what is the underlying Symmetry behind the Energy Conservation ?

*** $\boxed{\text{Time-Translation Symmetry}}$

* How is it reflected in the Lagrangian formulation of mechanics?

- So far, we have considered Lagrangians in the form

$$\mathcal{L}(q_i, \dot{q}_i).$$

- In this case, the value of "$\mathcal{L}$" changes in time, but only through the changes in $q_i(t)$ and $\dot{q}_i(t)$

- "t" is not considered as an independent variable for $\mathcal{L}$.

Time-Translation Invariance

⊛ A system is time-translation invariant if there is no explicit time dependence in its Lagrangian.

⇒ As we can show later, the total energy of such system is conserved.

⊛ This applies to any Lagrangian of the form "$\mathcal{L}(q_i, \dot{q}_i)$".

cf. Time-dependent System.

Eg 1. Particle attached to a (low quality) spring.

~~~ : the spring constant k decays in time ; $k = k(t)$.

$$\Rightarrow \mathcal{L} = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k(t) x^2 ,$$

This Lagrangian is in the form $\mathcal{L}(x, \dot{x}, t)$ with an explicit time dependence! ⇒ Energy is not conserved.
~~~

Temporal Evolution of Lagrangian.

⊛ Let's consider a Lagrangian $\mathcal{L}(q_i, \dot{q}_i, t)$, then

$$\frac{d}{dt}\mathcal{L}(q_i, \dot{q}_i, t) = \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{d}{dt}\dot{q}_i \right) + \frac{\partial \mathcal{L}}{\partial t}$$

through changes in values of q_i and $\dot{q}_i$

from explicit time-dependence

from the E-L eqn, $\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}p_i$, and

from the definition of p_i, $\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \equiv p_i$

$$= \sum_i \left(\frac{dp_i}{dt}\cdot\frac{dq_i}{dt} + p_i \frac{d\dot{q}_i}{dt} \right) + \frac{\partial \mathcal{L}}{\partial t} = \sum_i \frac{d}{dt}\left(p_i \frac{dq_i}{dt}\right) + \frac{\partial \mathcal{L}}{\partial t}$$

$$= \frac{d}{dt}\left(\sum_i p_i \frac{dq_i}{dt} \right) + \frac{\partial \mathcal{L}}{\partial t}.$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\sum_i p_i \left(\frac{dq_i}{dt}\right) - \mathcal{L} \right) = -\frac{\partial \mathcal{L}}{\partial t}}$$