Fundamentals of Engineering Physics 2019

Week 6.

Energy of a System of Charges

- Consider the "Work" which must be done on the system to bring some charged bodies into a particular arrangement.

* Consider two charged bodies first:

$$W = \int \text{force} \times \text{distance} = \int_{\infty\ \text{initial distance}}^{r_{12}\ \text{final distance}} \frac{q_1 q_2 (-dr)}{r^2} = \frac{q_1 q_2}{r_{12}} \quad (3)$$

(See Fig. 1.4.)

- W is "independent" of the path of approach!

$\because\ \vec{F} \cdot d\vec{s} = F\,dr$ (only radial component of $d\vec{s}$ matters)

(See Fig 1.5.)

∼ electric forces are "conservative".

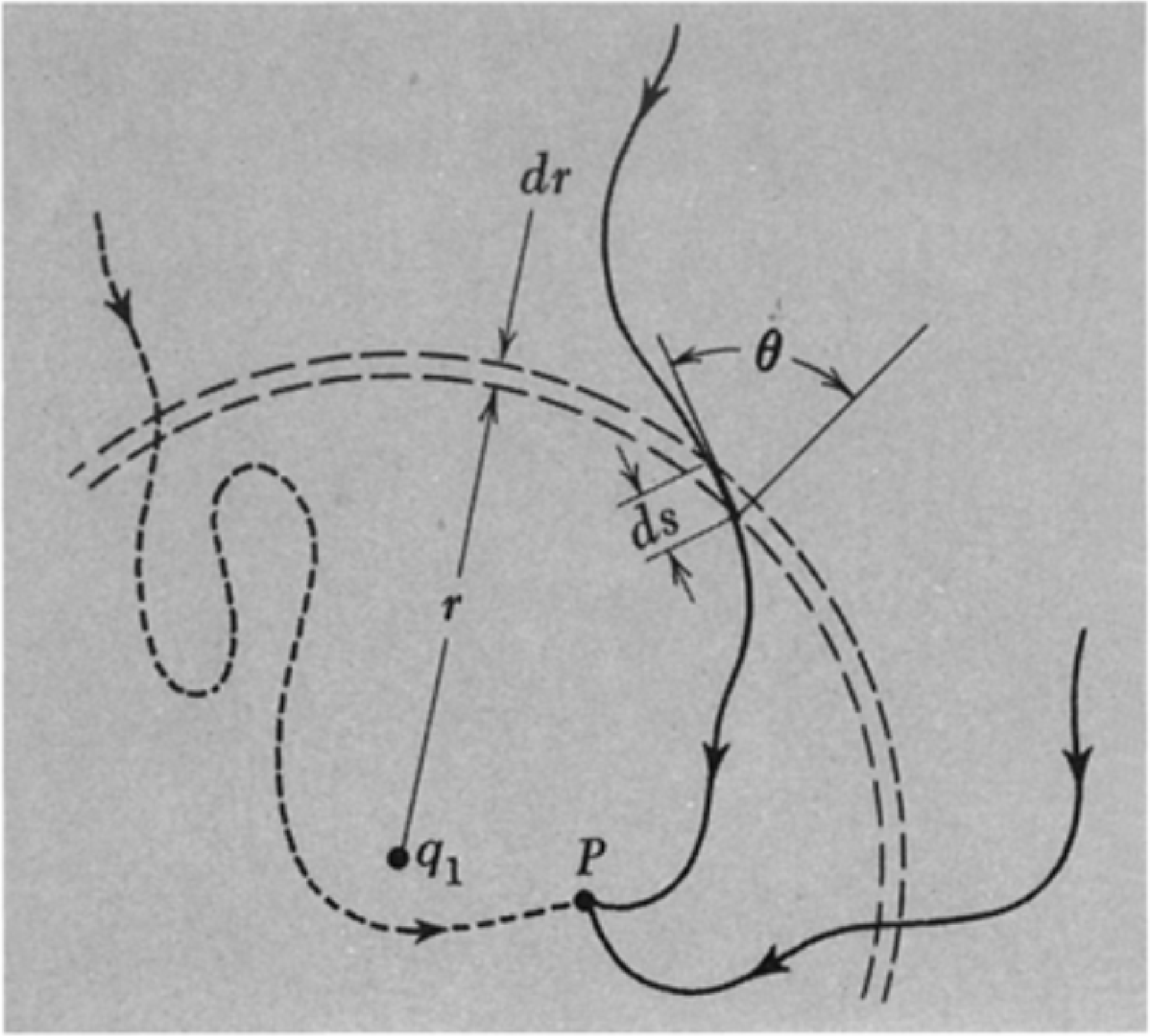


FIGURE 1.5
Because the force is central, the sections of different paths between $r + dr$ and r require the same amount of work.

* Now, bring the third charge q_3 from infinity

(See Fig. 1.4. (c))

$$W_3 = -\int_{\infty}^{P_3} \vec{F}_3 \cdot d\vec{s} = -\int (\vec{F}_{31} + \vec{F}_{32}) \cdot d\vec{s}$$

from superposition principle.

$$= -\int \vec{F}_{31} \cdot d\vec{s} - \int \vec{F}_{32} \cdot d\vec{s} \qquad (5)$$

$$= \frac{q_1 q_3}{r_{31}} + \frac{q_2 q_3}{r_{32}}. \qquad (6)$$

$\therefore$ The total work done in assembling this arrangement of three charges, U, is.

"Electrical Potential Energy"

$$U = \frac{q_1 q_2}{r_{12}} + \frac{q_2 q_3}{r_{23}} + \frac{q_3 q_1}{r_{31}} \qquad (7)$$

- Symmetric w.r.t. q_1, q_2 and q_3.
- independent of the order in which charges are assembled.

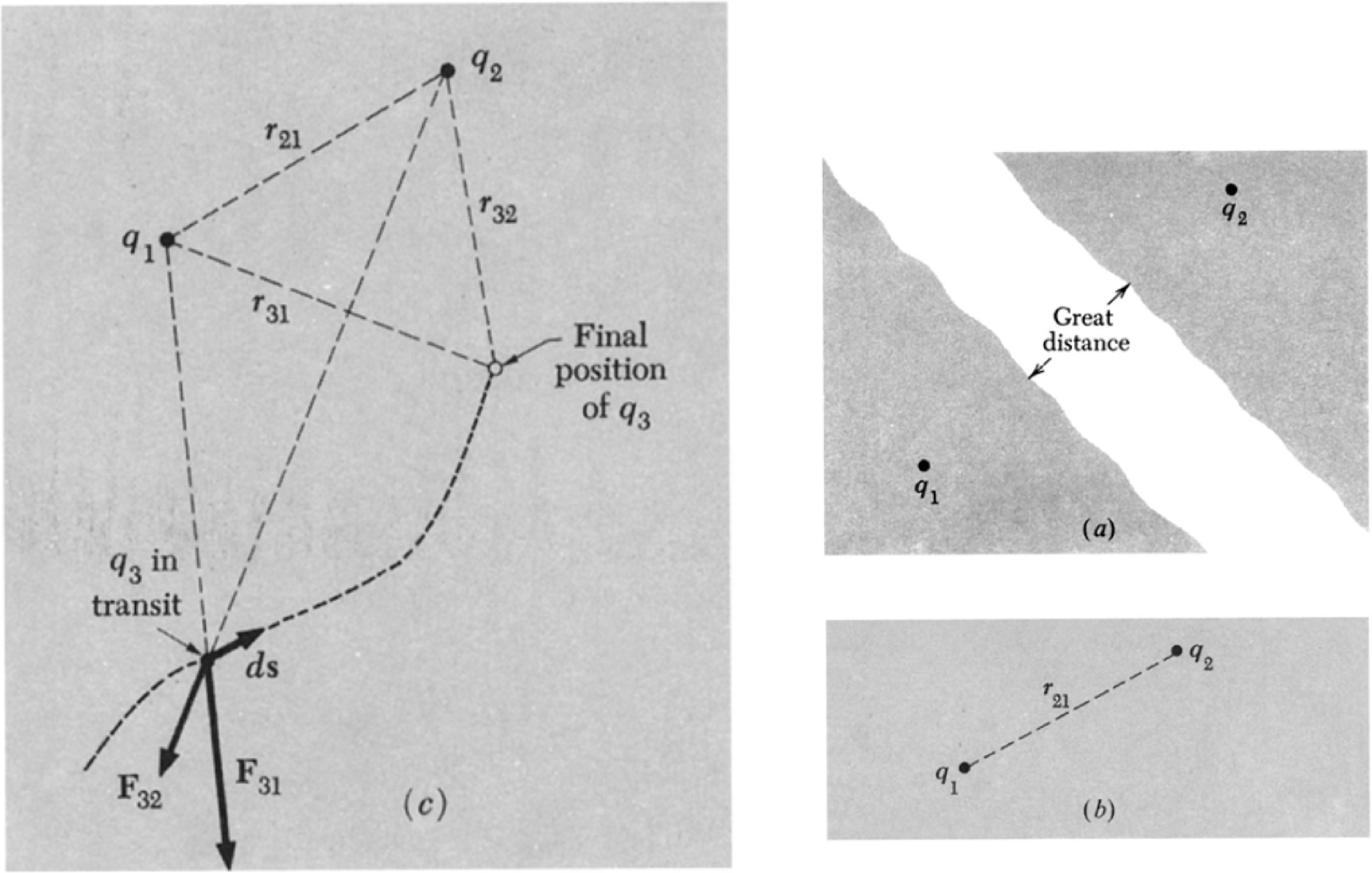


FIGURE 1.4

Three charges are brought near one another. First q_2 is brought in; then with q_1 and q_2 fixed, q_3 is brought in.

* The potential energy belongs to the configuration as awhole.

– Eg. 1. in Fig 1.6.

$$U = \frac{8(-2e^2)}{\frac{\sqrt{3}}{2}b} + \frac{12e^2}{b} + \frac{12e^2}{\sqrt{2}b} + \frac{4e^2}{\sqrt{3}b} = \frac{4.32e^2}{b} \quad (8)$$

(number of legs connecting two charges)

$U > 0$: Work had to be done on the system to assemble it.

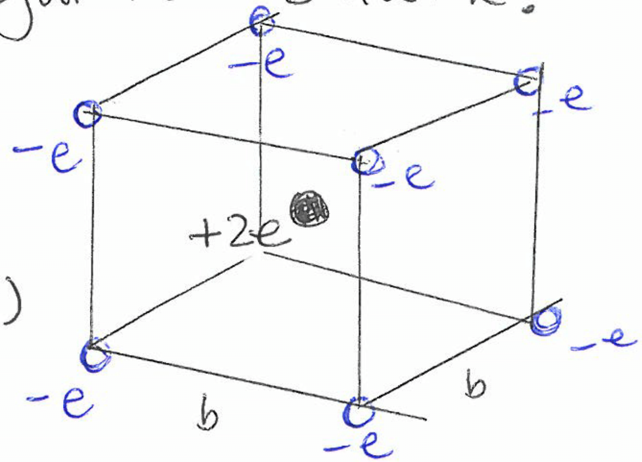


Fig. 1.6.

• For N bodies of charges :

$$U = \frac{1}{2}\sum_{j=1}^{N}\sum_{k\neq j}\frac{q_j q_k}{r_{jk}} . \quad (9)$$

– Eg. 2. "A Crystal Lattice" in Fig 1.7

$$U = \frac{1}{2}N\left(-\frac{6e^2}{a} + \frac{12e^2}{\sqrt{2}a} - \frac{8e^2}{\sqrt{3}a} + \cdots\right) = \cdots = \frac{-0.87Ne^2}{a} < 0. \quad (12)$$

"Work would have to done to take the crystal apart into individual ions."

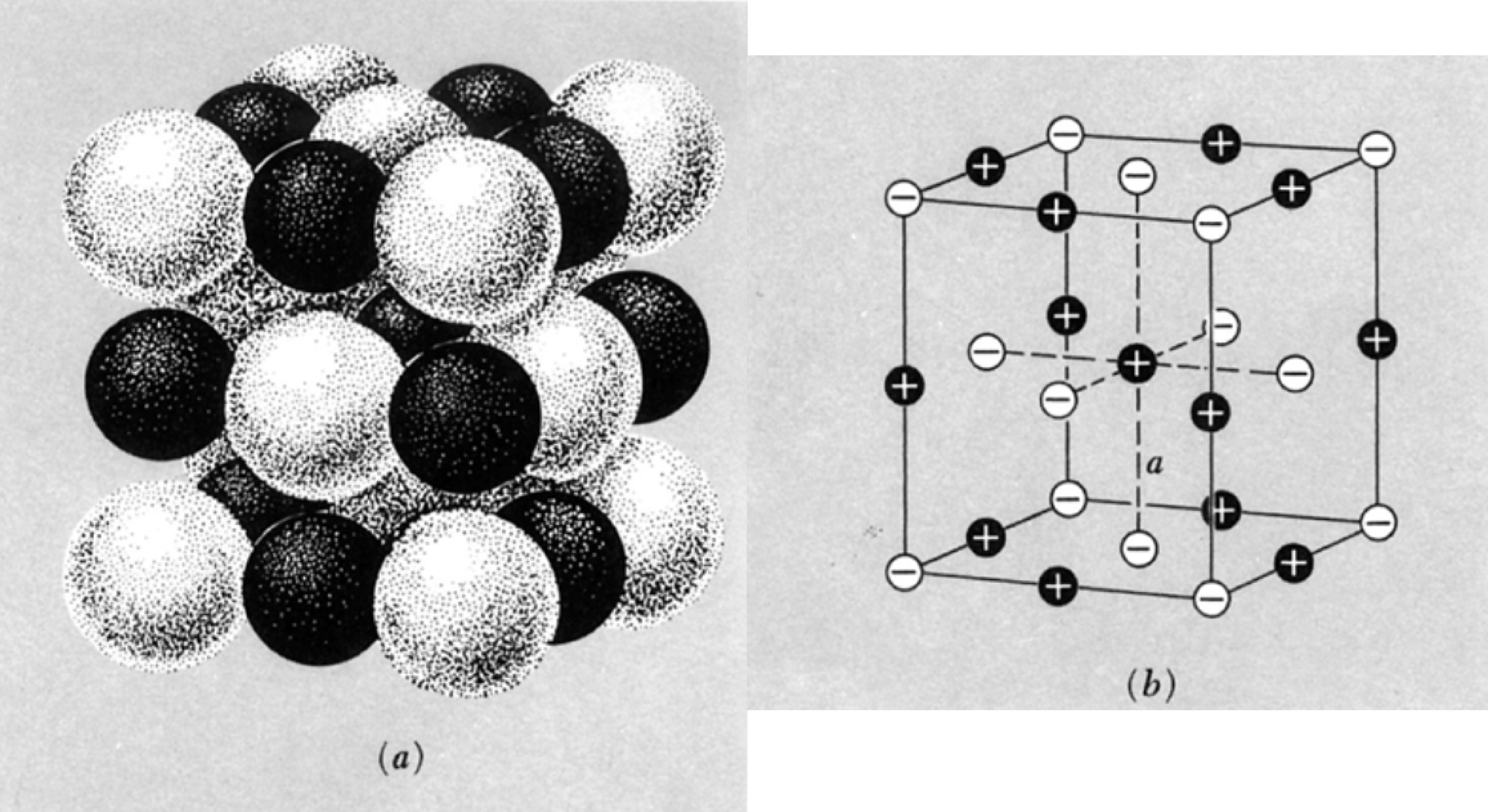


FIGURE 1.7
A portion of a sodium chloride crystal, with the ions Na^+ and $C1^-$ shown in about the right relative proportions (*a*), and replaced by equivalent point charges (*b*).

The Electric Field

* Consider some arrangement of charges, $q_1, q_2, \cdots, q_N$, and the force on another charge q_0 they exert.

$$\vec{F}_0 = \sum_{j=1}^{N} \frac{q_0 q_j \hat{r}_{0j}}{r_{0j}^2} = q_0 \sum_{j=1}^{N} \frac{q_j \hat{r}_{0j}}{r_{0j}^2} \qquad (13)$$

* Electric Field ;

$$\vec{E}(x,y,z) = \sum_{j=1}^{N} \frac{q_j \hat{r}_{0j}}{r_{0j}^2} \qquad (14)$$

at the point (x, y, z).

- charges $q_1, q_2, \cdots q_N$ are the sources of the field.

* There are at least two different ways to illustrate the Electric Field ; using arrows Fig 1.9, or lines. Fig. 1.11,

* Charge Distributions :

$$\vec{E}(x,y,z) = \sum_{j=1}^{N} \frac{q_j \hat{r}_{0j}}{r_{0j}^2} \xrightarrow[\text{limit}]{\text{continuum}} \int \frac{\rho(x',y',z') \hat{r}\, dx'dy'dz'}{r^2} \qquad (15)$$

Electric Field at (x,y,z) produced by Sources at (x',y',z')

where,

$\hat{r}$: unit vector from (x',y',z') to (x,y,z).

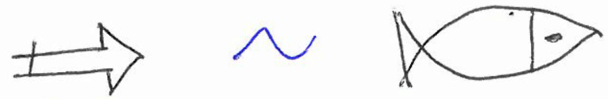
$$r^2 = (x-x')^2 + (y-y')^2 + (z-z')^2 .$$

Flux

$$\Phi = \sum_{\text{all } j} \vec{E}_j \cdot \vec{a}_j \; . \longrightarrow \int_{\substack{\text{entire}\\ \text{Surface}}} \vec{E} \cdot d\vec{a}$$

See Fig. 1.13 and 1.14.

Consider a number of fishes one can catch.



Rate of flow of water thru the frame per unit time

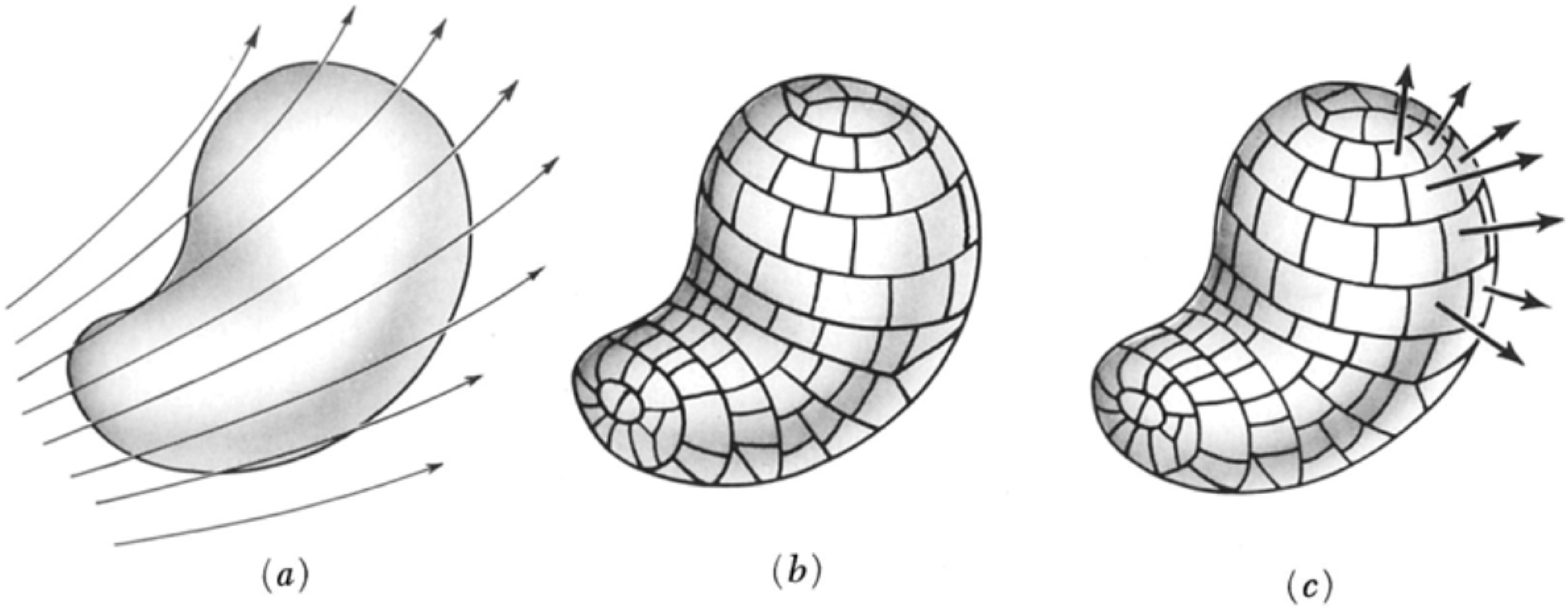


FIGURE 1.13
(*a*) A closed surface in a vector field is divided (*b*) into small elements of area. (*c*) Each element of area is represented by an outward vector.

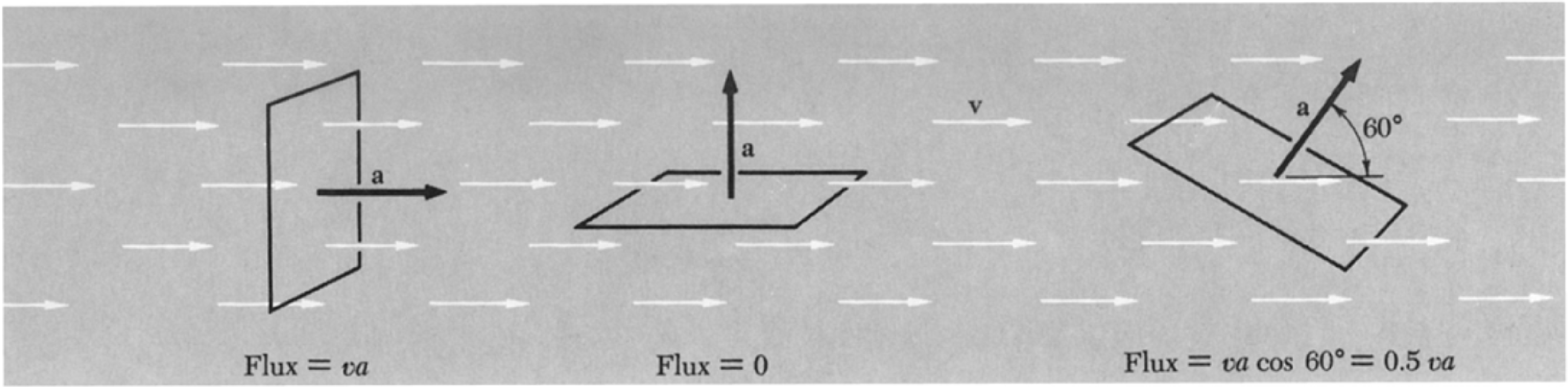


FIGURE 1.14
The flux through the frame of area **a** is **v** · **a**, where **v** is the velocity of the fluid. The flux is the volume of fluid passing through the frame, per unit time.

Gauss's Law

* Consider a single point charge "q" at the origin:

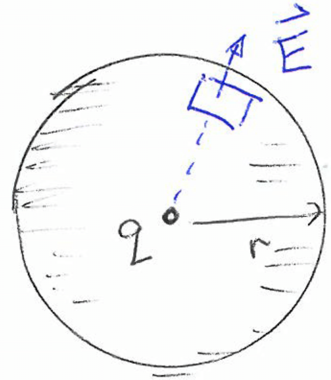


Fig 1.15

- Flux through the surface of a sphere of radius r:

$$\Phi = E \times (\text{area}) = \frac{q}{r^2} \times 4\pi r^2 = 4\pi q \quad (18)$$

- Flux through the surface of arbitrary shape:

  considering a patch $\vec{A}$ at a distance R from the point charge q.

  and compare fluxes thru $\vec{A}$ and thru a small patch $\vec{a}$ on the inner spherical surface.

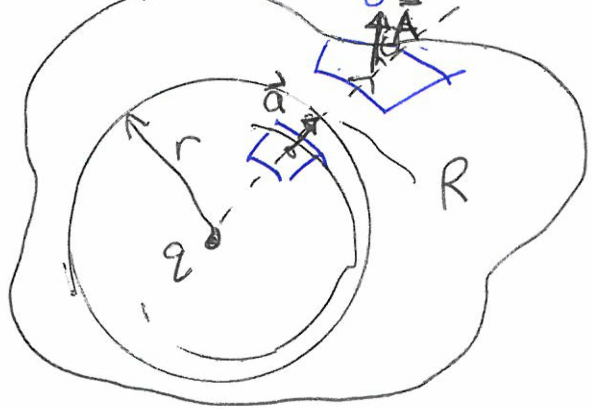


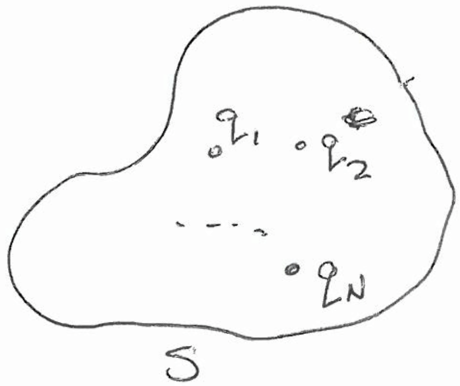
Fig 1.16.

$$\text{"}E_{(R)} A \cos\theta\text{"} = \left[E_{(r)}\left(\frac{r}{R}\right)^2\right] \cdot \left[a\left(\frac{R}{r}\right)^2 \frac{1}{\cos\theta}\right] \cos\theta = \text{"}E_{(r)}\, a\text{"}$$

thru outer patch — thru inner patch

$\therefore \quad \Phi = 4\pi q\ !$

* Flux through some surface S enclosing charges $q_1, q_2, \cdots q_N$;



$$\Phi = \int_S \vec{E} \cdot d\vec{a} = \int_S (\vec{E}_1 + \vec{E}_2 + \cdots \vec{E}_N) \cdot d\vec{a}$$

from principle of superposition !

$$= 4\pi \sum_{i=1}^{N} q_i \qquad (20)$$

Gauss's Law

"The flux of the electric field $\vec{E}$ through any closed surface, that is, the integral $\int \vec{E} \cdot d\vec{a}$ over the surface, equals 4π times the total charge enclosed by the surface."

$$\int_S \vec{E} \cdot d\vec{a} = 4\pi \sum_i q_i = 4\pi \int \rho \, dv . \qquad (21)$$

⊛ For systems with symmetry, one can calculate the $\vec{E}$ field more easily using the Gauss's law without using the Coulom's law followed by integrations, ~~for~~ instance using

$$\vec{E}(x,y,z) = \int \frac{\rho(x',y',z')\,\hat{r}\,dx'dy'dz'}{r^2} \qquad (15).$$

1. Field of a spherically symmetric charge distribution

See Fig 1.18 and 1.19

- Considering a spherical surface S_1 of radius r_1,

$$\Phi = E_1 \cdot 4\pi r_1^2 = 4\pi \cdot (\text{charge inside } S_1),$$

$$\therefore \quad E_1 = \frac{\text{charge inside } S_1}{r_1^2} \qquad (23).$$

2. Field of a line charge with the line charge density λ,

- Considering a "tin can" with radius r and Lengh L,

$$\Phi = E_r \cdot 2\pi r L = 4\pi \lambda L$$

See Fig 1.22

$$\therefore \quad E_r = 2\lambda / r \qquad (27).$$

FIGURE 1.18
A charge distribution with spherical symmetry.

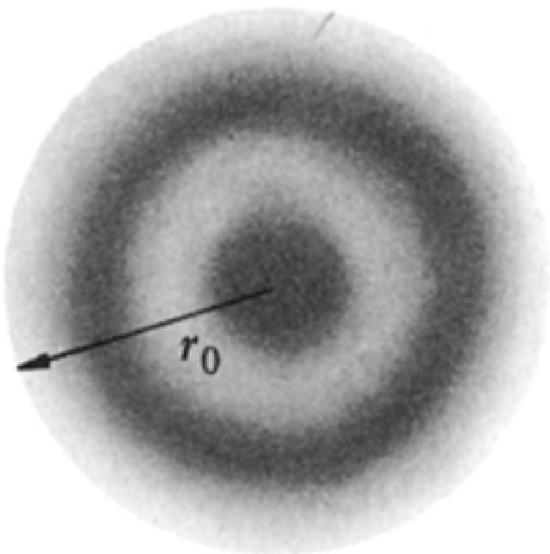
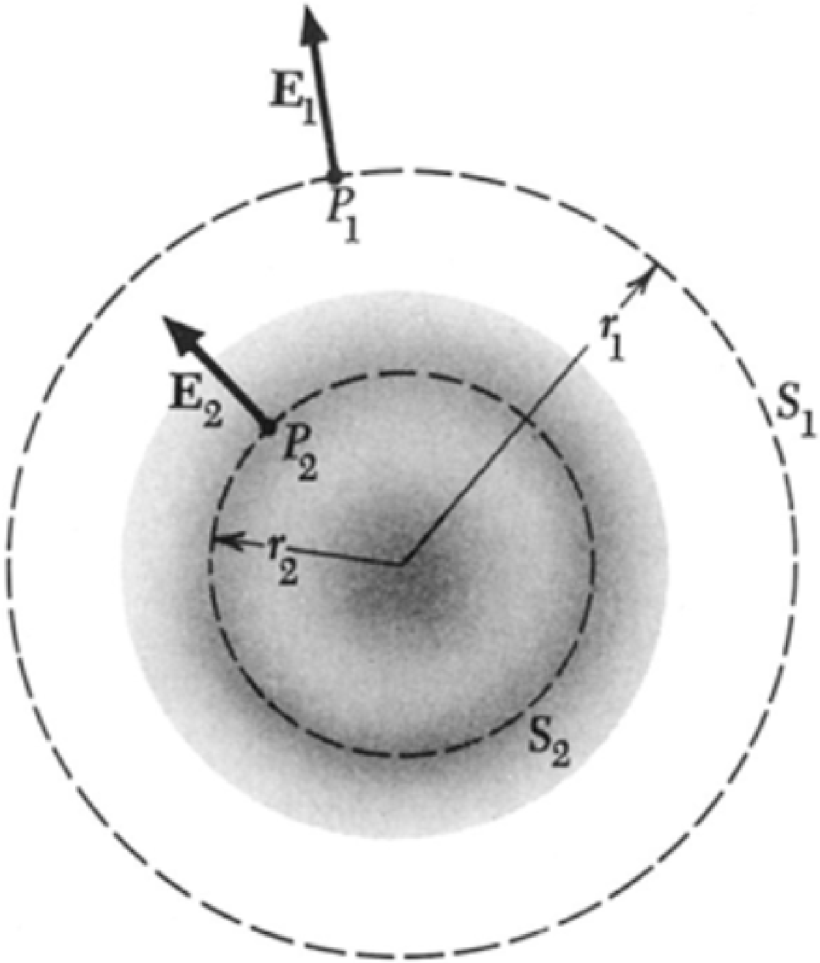


FIGURE 1.19
The electric field of a spherical charge distribution.



3. Field of an infinite flat sheet of charge.

with a surface charge density σ.

- Considering a cylinder in Fig. 1.23, and using the Gauss's law,

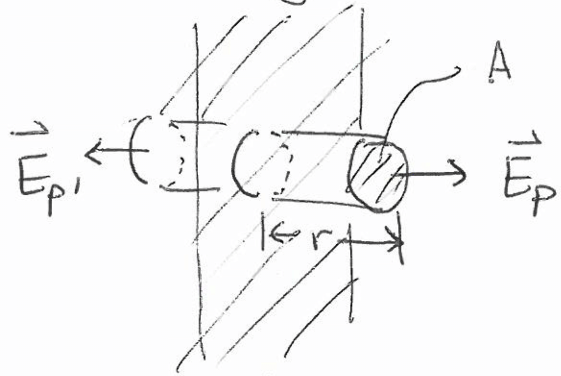


Fig. 1.23

$$A \cdot E_p + A E_{p'} = 4\pi \cdot \sigma A$$

Since this system has a reflectional (mirror) symmetry,

$E_p = E_{p'}$.

$$\therefore \quad E_p = 4\pi\sigma/2 = 2\pi\sigma \quad (28)$$

i.e, independent of distance "r" from the sheet.

$$\boxed{\text{Homework}}$$

Problems 1.5, 1.14, 1.17, 1.21

2. The Electric Potential

- Line Integral of the Electric Field is path-independent.

1. In the presence of a point charge source q,

$$\int_{P_1}^{P_2} \vec{E}\cdot d\vec{s} = \int_{r_1}^{r_2} \frac{q\,dr}{r^2} = q\left(\frac{1}{r_1}-\frac{1}{r_2}\right) \quad (2)$$

(Only the component of $d\vec{s}$ parallel to $\vec{E}$ contributes).

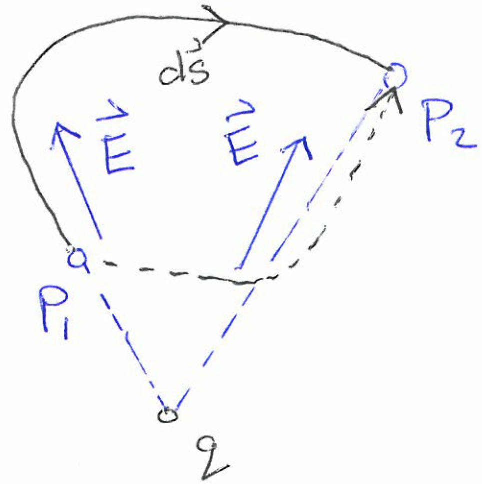


Fig. 2.2.

2. In the presence of many charge sources,
infinitely many
↓
continuously "

we can use the principle of superposition

$$\int_{P_1}^{P_2} \vec{E}\cdot d\vec{s} = \int_{P_1}^{P_2} \sum_i \vec{E}_i\cdot d\vec{s} = \sum_i \int_{P_1}^{P_2} \vec{E}_i\cdot d\vec{s}$$

(each integral for particular i is path-independent as shown in Eq. (2)).

∴ $\boxed{\text{The line integral } \int_{P_1}^{P_2}\vec{E}\cdot d\vec{s} \text{ for any electrostatic field } \vec{E} \text{ has the same value for all paths from } P_1 \text{ to } P_2}$ (4)

Potential Difference and Potential Function

- Potential Difference : $$\Phi_{21} = -\int_{P_1}^{P_2} \vec{E} \cdot d\vec{s} \qquad (6)$$

Then, $$\boxed{\vec{E} = -\vec{\nabla}\varphi(x, y, z)} \qquad (12)$$

with $P_2 = (x, y, z)$ and P_1 is the reference point.

Now,
We can obtain $\vec{E}$ once we know a scalar function $\varphi(x, y, z)$!

* Potential of a charge distribution

$$\varphi(x, y, z) = \int_{\text{All Sources}} \frac{\rho(x', y', z')\, dx'dy'dz'}{r} \qquad (15)$$

In most cases, this integral is easier to perform than that using the Coulomb's law.

For instance, potential of a long charged wire with line density λ is given by $\varphi = -2\lambda \ln r + \text{const.} \qquad (17)$

$$\vec{E} = -\vec{\nabla}\varphi = \frac{2\lambda}{r}\hat{r} \qquad (18)$$

Potential of a Uniformly Charged Disk.

- $\varphi(0,y,0) = \int \frac{dq}{r} = \int \frac{2\pi\sigma\, s\, ds}{\sqrt{y^2+s^2}}$

$$= 2\pi\sigma \left[\sqrt{y^2+s^2}\right]_{s=0}^{s=a} \qquad (19)$$

$$= 2\pi\sigma \left(\sqrt{y^2+a^2} - |y|\right) \qquad (20, 21).$$

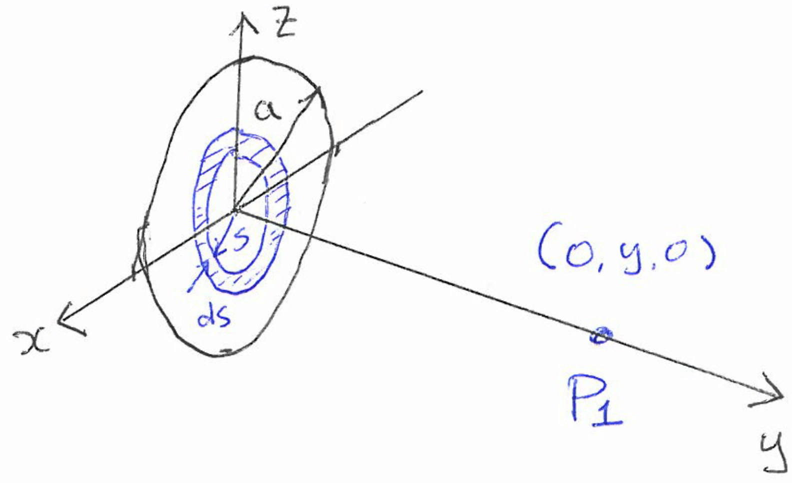


Fig. 2.6.

(In the Limit of very large y ($\gg a$), $\varphi(0,y,0) \to \frac{(\pi a^2)\sigma}{y}$, for $y \gg a$ (23)

i.e., "$(\pi a^2)\sigma$" acts like a point charge source.)

- $\vec{E} = -\vec{\nabla}\varphi = -\hat{y}\frac{\partial}{\partial y}\left\{2\pi\sigma\left(\sqrt{y^2+a^2} - |y|\right)\right\}$

$$= \hat{y}\, 2\pi\sigma\left[1 - \frac{|y|}{\sqrt{y^2+a^2}}\right]. \qquad (26)$$

- This procedure is easier than computing $\vec{E}$ directly.

Divergence and Gauss's Theorem

- Divergence of a Vector Function; $\vec{F}$:

$$\boxed{\operatorname{div} \vec{F} \equiv \lim_{V_i \to 0} \frac{1}{V_i} \int_{S_i} \vec{F} \cdot d\vec{a}_i} \quad (33)$$

↑ a scalar

See Fig 2.12, and illustration on page 56-57.

- **Gauss's Theorem (Divergence Theorem)**

$$\boxed{\int_S \vec{F} \cdot d\vec{a} = \int_V \operatorname{div} \vec{F}\, dv} \quad (36)$$

for any vector field $\vec{F}$ and any closed surface "S" enclosing a volume "V".

This is a mathematical theorem.

* We can apply this to the Gauss's law, "$\int_S \vec{E} \cdot d\vec{a} = 4\pi \int_V \rho\, dV$".

Then, $\int_V \operatorname{div} \vec{E}\, dv = \int_V 4\pi\rho\, dv$. Since this is true for any "V",

→ $\boxed{\operatorname{div} \vec{E} = 4\pi\rho}$ Gauss's Law in differential form, (→ "Poisson Equation.")

Gradient, Divergence, and Laplacian

- Scalar "G" : $\vec{F} = \underline{\vec{\nabla}} G$: Vector "$\vec{F}$".

  gradient

- Scalar "A" : $A = \underline{\text{div}}\, \vec{F} \equiv \vec{\nabla} \underset{=}{\bullet} \vec{F}$: "

  divergence

- In Cartesian coordinate, $\vec{\nabla} G = \hat{x} \frac{\partial G}{\partial x} + \hat{y} \frac{\partial G}{\partial y} + \hat{z} \frac{\partial G}{\partial z}$.

  For $\vec{F} = (F_x, F_y, F_z)$, $\text{div}\, \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$.

- Laplacian ; "divergence of gradient" (of a scalar)

  $$\text{div}\cdot\text{grad}\, G = \text{div}\left(\hat{x} \frac{\partial G}{\partial x} + \hat{y} \frac{\partial G}{\partial y} + \hat{z} \frac{\partial G}{\partial z}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) G$$

  in Cartesian coordinate.

  $$\vec{\nabla} \cdot \vec{\nabla} G \equiv \underline{\underline{\nabla}}^2 G \quad \left(\neq \text{not } (\text{grad})^2 \right)$$

  Laplacian.

Poisson's Equation.

* Gauss's Law in differential form :

$$\mathrm{div}\,\vec{E} = 4\pi\rho$$

$$\mathrm{div}(-\mathrm{grad}\,\varphi) = -\vec{\nabla}\cdot\vec{\nabla}\varphi = -\nabla^2\varphi$$

$$\boxed{\nabla^2\varphi = -4\pi\rho}$$: Poisson's Equation (50).

Physical content is the same as the Gauss's Law, But it ~~~~ relates ~~the~~ a scalar "ρ" to another scalar φ directly.

* In cylindrical coordinates (ρ, θ, z) ;

$$\underset{\text{grad.}}{\vec{\nabla} G} = \hat{\rho}\frac{\partial G}{\partial\rho} + \hat{\theta}\frac{1}{\rho}\frac{\partial G}{\partial\theta} + \hat{z}\frac{\partial G}{\partial z}, \qquad \underset{\text{(div)}}{\vec{\nabla}\cdot\vec{F}} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\rho) + \frac{1}{\rho}\left(\frac{\partial F_\theta}{\partial\theta}\right) + \left(\frac{\partial F_z}{\partial z}\right);$$

$$\underset{\text{Laplacian}}{\nabla^2 G} = \mathrm{div}\cdot\mathrm{grad}\,G = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}G \ne \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}\right)G$$

Curl of a Vector Function

- $$(\mathrm{Curl}\,\vec{F})\cdot\hat{n} = \lim_{a_i \to 0} \frac{\Gamma_i}{a_i} = \lim_{a_i \to 0} \frac{\int_{C_i} \vec{F}\cdot d\vec{l}}{a_i} \qquad (61)$$

See Fig 2.21, 22

Stokes' Theorem:

$$\int_C \vec{F}\cdot d\vec{l} = \int_S \mathrm{curl}\,\vec{F}\cdot d\vec{a} \qquad (64)$$

where C is the curve which bounds a surface.

- In Cartesian Coordinates,

$$\mathrm{Curl}\,\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \hat{x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{y}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right).$$

* For electrostatic field $\vec{E}$, $\vec{E} = -\vec{\nabla}\varphi$,

then

$$\text{curl } \vec{E} = -\vec{\nabla}\times\vec{\nabla}\varphi = 0, \quad \text{everywhere.} \tag{75}$$

(Note that curl∘grad of any scalar field = 0 : mathematical identity).

◎ One can see <u>not</u> all of vector fields in Fig 2.30 can be an electrostatic field $\vec{E}$.

Homework

Problems 2.8, 2.11, 2.12, and 2.16.

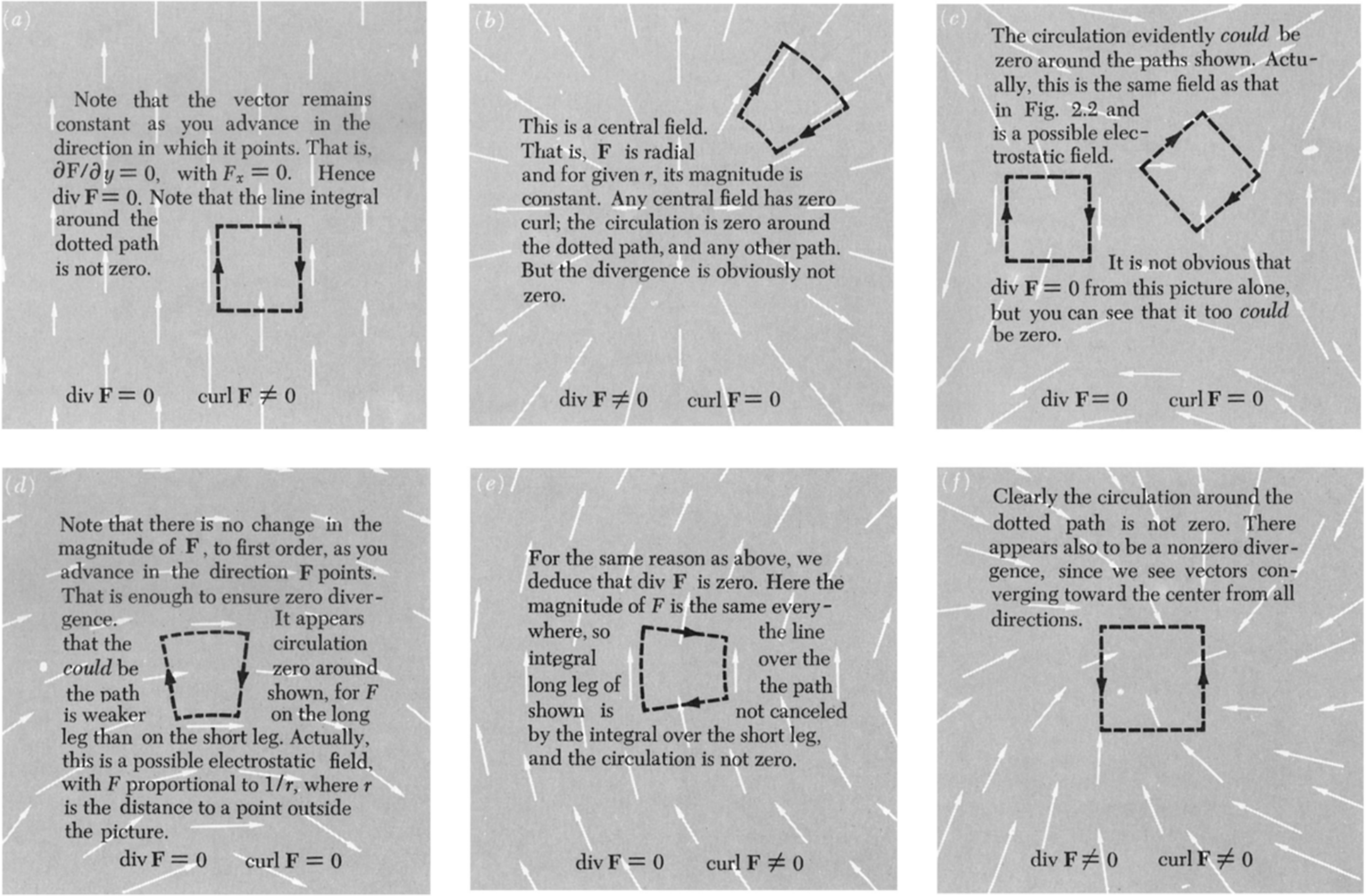


FIGURE 2.32
Discussion of Fig. 2.30.