Fundamentals of Engineering Physics 2019

Week 7.

4. Electric Currents

- Current density :

$$\vec{J} = \sum_k n_k q_k \vec{u}_k \qquad (3)$$

n_k : number density

q_k : charge

k : different kind of particles

$\vec{u}_k$: (mean) velocity of species k.

esu/sec-cm² in CGS

ampere/m² in SI.

See Fig 4.1.

If electrons are the only current carriers,

$$\vec{J}_e = -e N_e \underline{\vec{u}_e} \qquad (5)$$

average electron velocity

Steady Currents and Charge Conservation

- $I = \int_S \vec{J} \cdot d\vec{a}$: current flowing through any surface S.

For steady current, $\vec{J}$ remains constant in time everywhere.

- If S is enclosing some volume V, $I = 0$ because otherwise, the charge will accumulate indefinitely inside that volume. But charge cannot be created.

$$\oint_S \vec{J} \cdot d\vec{a} = \int_V \operatorname{div} \vec{J} \, dv = 0 , \quad \therefore \boxed{\operatorname{div} \vec{J} = 0}$$

- Now, if $\vec{J}$ can be time-dependent,

$$\oint_S \vec{J} \cdot d\vec{a} = -\frac{d}{dt}\left(\int_V \rho \, dv\right)$$

rate at which charge is leaving V, total charge inside V.

Charge Conservation

$$\Rightarrow \quad \int_V \operatorname{div} \vec{J}\, dv = -\int_V \frac{\partial \rho}{\partial t}\, dv \quad , \quad \text{for arbitrary volume } V ,$$

$$\therefore \quad \boxed{\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{J} = 0} \qquad (9)$$

Continuity Equation for charge density,

which describes the conservation of charge.

cf.

$$\boxed{\frac{\partial n}{\partial t} + \operatorname{div} (n \vec{u}) = 0}$$

Continuity Equation for number density

which describes the conservation of particle number.

Electrical Conductivity and Ohm's Law

Ohm's Law : $\vec{J} = \underline{\sigma} \vec{E}$ (10)

"electrical conductivity"

it depends on the material being considered, and its state such as temperature.

σ is high for good conductors (metals)
low for " insulators.

CGS unit for σ is "sec^{-1}".

Note that this is an <u>empirical law</u>, not a fundamental governing equation such as Newton's law or Maxwell's equation.

* Properties of σ (conductivity) depend on the material and strength of $\vec{E}$.

- $\vec{J} = \sigma \vec{E}$, with scalar "σ" which is independent of $\vec{E}$, is valid for isotropic material and sufficiently small $|\vec{E}|$.

- For anisotropic material, $\overleftrightarrow{\sigma}$ is a tensor quantity and $\vec{J} = \overleftrightarrow{\sigma} \cdot \vec{E}$ i.e.,

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}.$$

- For sufficiently strong $|\vec{E}|$, σ becomes a function of $|\vec{E}|$ and the Ohm's law becomes a nonlinear relation between $\vec{J}$ and $\vec{E}$.

Ohm's law for Macroscopic System

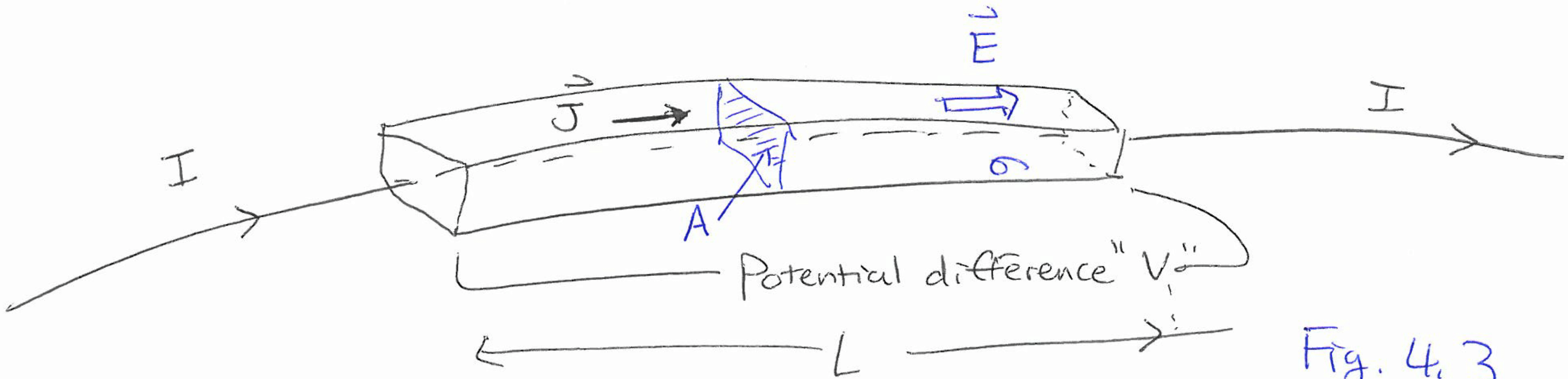


Fig. 4. 3.

Obviously,

$$J = \frac{I}{A} \quad \text{and} \quad E = \frac{V}{L},$$

Now, $J = \sigma E$ can be written as

$$I = \frac{V}{R} \qquad (11)$$

with $R = L/A\sigma$: Resistance. (SI unit for R is Ohm.).

Physics of Electrical Conduction

- Charge carriers in a conducting material execute a random motion, Their distribution function (for $\vec{E}=0$) is approximately a Maxwellian

$$f(v_x, v_y, v_z) \propto \exp\left[-\frac{(v_x^2+v_y^2+v_z^2)}{2v_{Th}^2}\right].$$

Their direction is random and the distribution is isotropic.

- When $\vec{E}$ field is applied, charged particles drift in that direction (for $q>0$) or in the opposite direction (for $q<0$).

For typical parameters of interest, that drift speed is much slower than the average thermal speed v_{Th}.

- When charged particles are accelerated by $\vec{E}$ field, the average speed during an acceleration phase "τ" is given by

$$v_d = \frac{1}{2}\left(\frac{qE}{m}\right)\cdot\tau \qquad (a)$$

- Meanwhile, the charge carriers suffer collisions with other particles in the system on the average once every collision time τ_{coll}. After a collision, the charged particle starts to move in a different direction. (no longer parallel to $\vec{E}$),

- Since $v_{th} \gg v_d$, $\tau_{coll} \cong \dfrac{\lambda_{mfp}}{v_{th}}$ mean-free-path.

(not λ_{mfp}/v_d)

(See Fig. 4, 7)

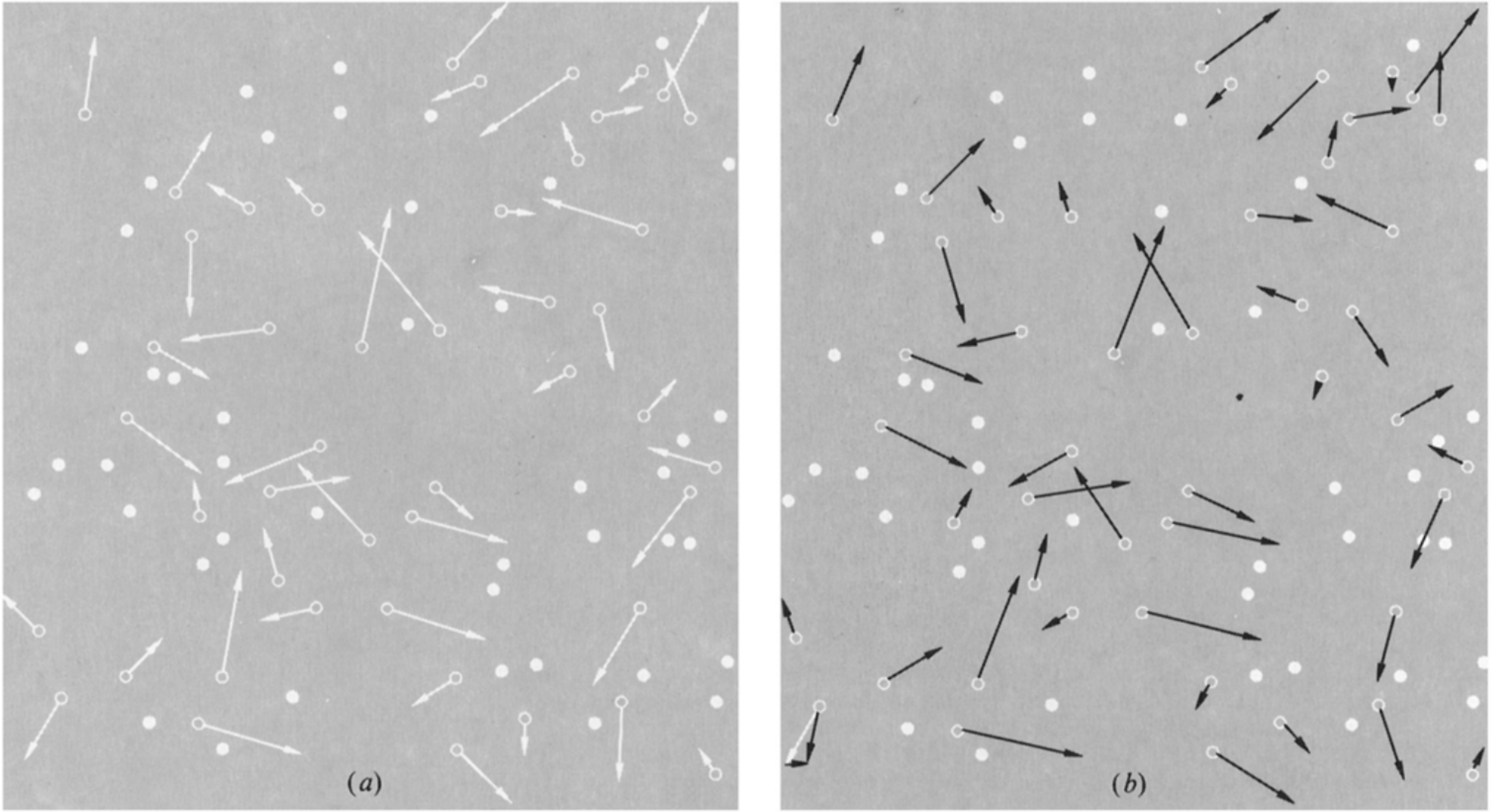


FIGURE 4.7
(a) A random distribution of electrons and positive ions with about equal numbers of each. Electron velocities are shown as vectors and in (a) are completely random. In (b) a drift toward the right, represented by the velocity vector →, has been introduced. This velocity was added to each of the original electron velocities, as shown in the case of the electron in the lower left corner.

- Then, $v_d = \frac{1}{2}\left(\frac{qE}{m}\right)\tau_{coll} \cong \frac{1}{2}\left(\frac{qE}{m}\right)\frac{\lambda_{mfp}}{v_{th}}$.

- $$J \cong n q v_d \simeq \underline{\frac{n q^2 \lambda_{mfp}}{2 m v_{th}}} \cdot E \;!$$

a crude estimate of σ

-(independent of E).

⊛ Note that this is a consequence of considering both Newton's law (acceleration) and statistical properties (distribution of many particles and random and collisions) of the system.

6. The Magnetic Field

- $$\vec{F} = q\left(\vec{E} + \underline{\frac{\vec{v}}{c} \times \vec{B}}\right) \quad (1)$$

  Lorentz Force

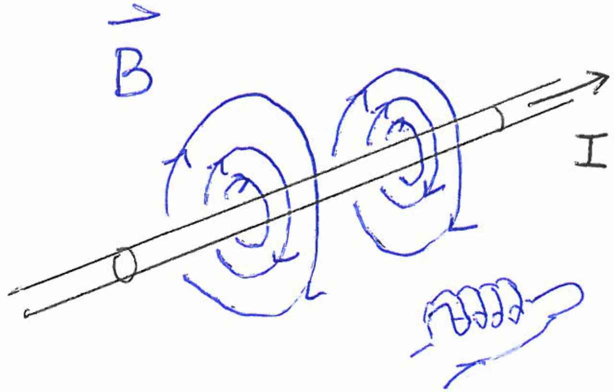
  (In CGS units, B in Gauss, q in esu, F in dyne)

  * Very long illustration from the special relativity and the charge invariance on Chapter. 5.

- Magnetic Field around a straight wire carrying curren I;

  $$\vec{B} = \hat{z}\,\frac{2I}{rc} \quad (5)$$

  " right-handed "



- Force per unit length between two wires;

  $$F = \frac{2 I_1 I_2}{c^2 r} \quad (7)$$

  (Currents in the same direction attract one another) Fig. 6.5

Ampère's Law

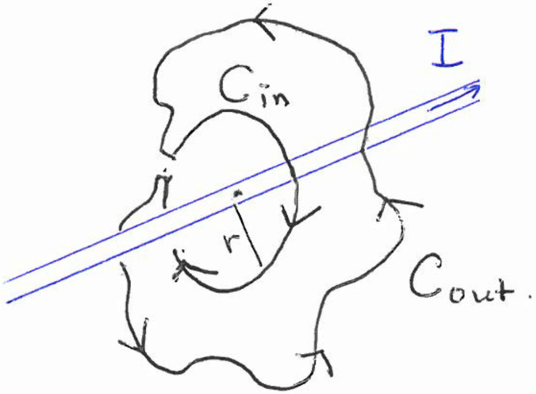


Fig 6.6.

$$\int_{C_{in}+C_{out}} \vec{B}\cdot d\vec{s} = 0 \qquad (a)$$

Since the segments $\perp \vec{B}$ do not contribute to the integral and $|\vec{B}| \propto \frac{1}{r}$.

$$\int_{C_{in}} \vec{B}\cdot d\vec{s} = 2\pi r \cdot \frac{2I}{rc} = \frac{4\pi}{c} I = \frac{4\pi}{c}\left\{\begin{array}{c}\text{current enclosed}\\ \text{by path}\end{array}\right\}. \qquad (b)$$

From (a),

$$\int_{C} \vec{B}\cdot d\vec{s} = \frac{4\pi}{c}\int_{S} \vec{J}\cdot d\vec{a} \qquad (c)$$

for any C !

- $$\int_C \vec{B} \cdot d\vec{s} \xrightarrow[\text{Stokes' theorem}]{} \int_S \text{curl } \vec{B} \cdot d\vec{a}$$

$$= \frac{4\pi}{c} \int_S \vec{J} \cdot d\vec{a} \qquad \text{from (c).}$$

Since this is satisfied for any S,

$$\boxed{\text{curl } \vec{B} = \frac{4\pi}{c} \vec{J}} \qquad (15)$$

at any point in space.

- This is true for time-independent $\vec{J}$ (implicitly assumed in derivation).

For any vector field $\vec{G}$, $\text{div} \cdot \text{curl } \vec{G} = 0$ (a mathematical identity).

- Eq. (15) implies $\text{div } \vec{J} = 0$ i.e. steady current.

Vector Potential

- Recall that electrostatic potential satisfying $+\vec{E} = -\vec{\nabla}\varphi$ has been found to be useful.

- For problems involving $\vec{B}$, we define
$$\vec{B} = \vec{\nabla} \times \vec{A} = \text{curl } \vec{A} \qquad (20)$$
and introduce the vector potential $\vec{A}$.

- Ampère's law becomes
$$\text{curl } \vec{B} = \boxed{\text{curl curl } \vec{A} = \frac{4\pi}{c} \vec{J}} \qquad (21)$$

- For any $\vec{G}$;
$$\text{curl curl } \vec{G} = \vec{\nabla} \times (\vec{\nabla} \times \vec{G})$$
$$= -\nabla^2 \vec{G} + \vec{\nabla}(\vec{\nabla} \cdot \vec{G}),$$

• div · grad   grad · div

"Laplacian"

- $\therefore \quad -\nabla^2 \vec{A} + \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) = \dfrac{4\pi}{c} \vec{J} \qquad (21'),$

Vector Potential

- We can choose $\vec{A}$ which satisfies

  "$\mathrm{div}\,\vec{A} = 0$" — "Coulomb gauge condition".

  to simplify the problem.

$$\Rightarrow \boxed{-\nabla^2 \vec{A} = \frac{4\pi}{c}\vec{J}} \qquad (27')$$

- Each component of Eq. (27') is formally the same as

$$-\nabla^2 \varphi = \frac{4\pi}{c}\rho \qquad (28)$$

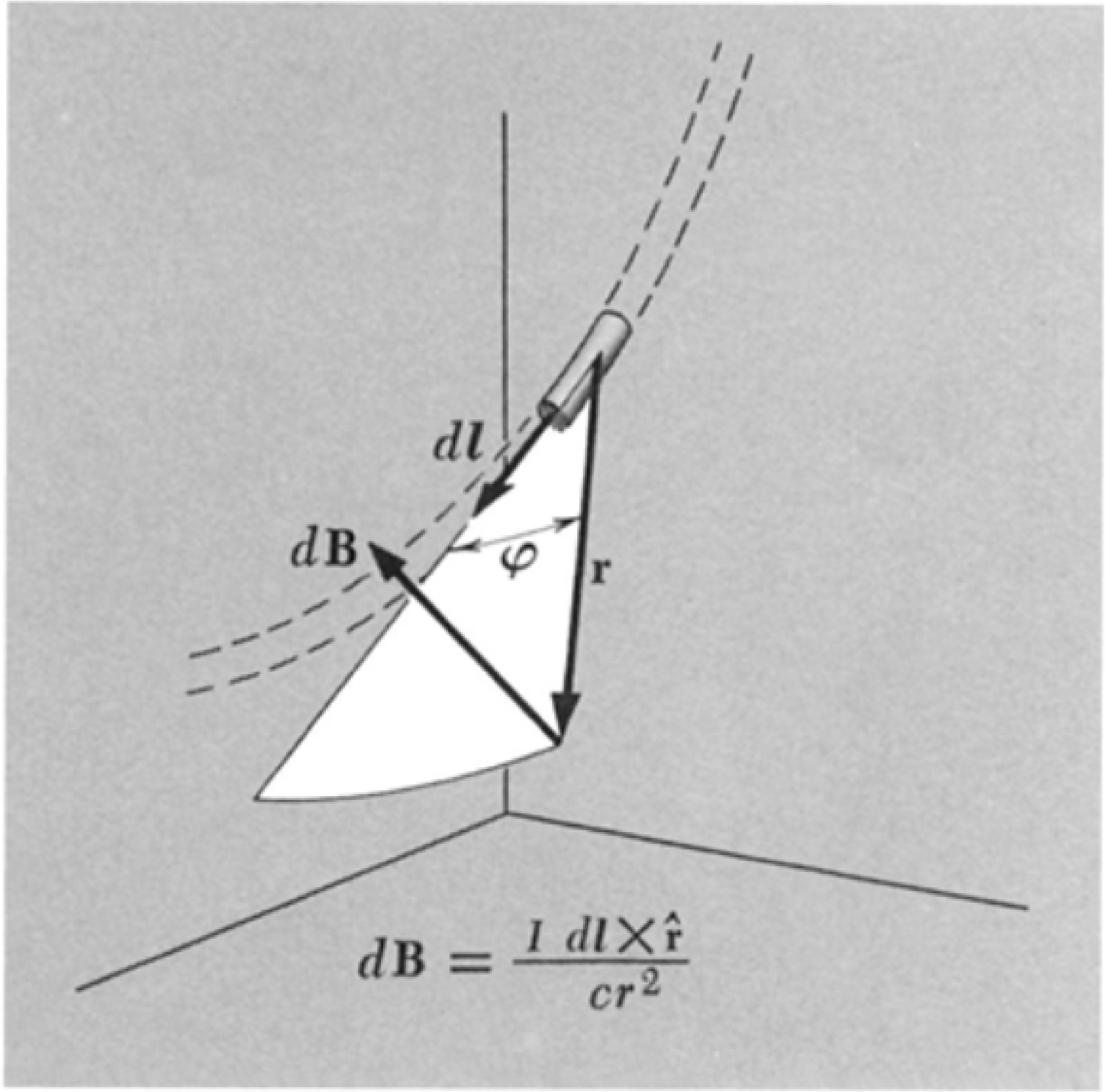
i.e. the Poisson's equation!

- Since the solution of Eq. (28) is $\varphi(x, y, z) = \int \frac{\rho(x', y', z')\, dx'dy'dz'}{r}$,

  the solution of Eq. (27') is

$$\boxed{\vec{A}(x, y, z) = \frac{1}{c}\int \frac{\vec{J}(x', y', z')}{r}\, dx'dy'dz'} \qquad (30).$$

FIGURE 6.14
The field of any circuit can be calculated by using this relation for the contribution of each circuit element.



Biot-Savart Law

$$\boxed{d\vec{B} = \frac{I\, d\vec{l} \times \hat{r}}{c r^2}} \qquad (38).$$

- This could have been derived directly from Eq (35), noting that $\vec{\nabla}$ or $\vec{\nabla}\times$ applies to the position of the field (x, y, z), not positions of the source (x', y', z'), and an identity $\vec{\nabla}\left(\frac{1}{r}\right) = -\hat{r}/r^2$.

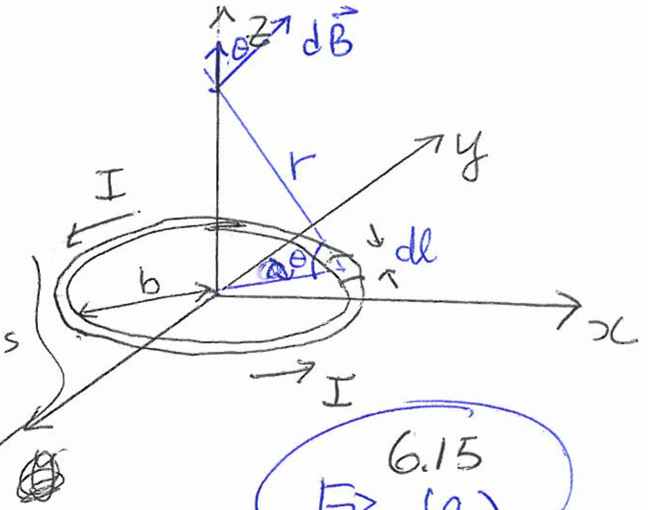
i.e.,

$$d\vec{B} = \vec{\nabla} \times \frac{I d\vec{l}}{cr} = -\frac{I}{c} d\vec{l} \times \vec{\nabla}\left(\frac{1}{r}\right) = -\frac{I}{c} d\vec{l} \times \left(-\frac{\hat{r}}{r^2}\right)$$

$$= \frac{I\, d\vec{l} \times \hat{r}}{c r^2} \qquad (39).$$

Fields of Rings

- Each element of the ring of length dl contributes to a $d\vec{B} \perp \vec{r}$.



6.15 Fig (a)

- From cylindrical symmetry, $\vec{B}$ on the z-axis should point in the z-direction.

So, it suffices to calculate dB_z only.

$$dB_z = \frac{I\,dl}{cr^2}\cos\theta = \frac{I\,dl}{cr^2}\frac{b}{r} \qquad (40)$$

- Integrating over the whole ring, $\oint dl = 2\pi b$.

$$\therefore \quad B_z = \frac{2\pi b^2 I}{cr^3} = \frac{2\pi b^2 I}{c(b^2+z^2)^{3/2}} \qquad (41)$$

— $\vec{B}$ field on axis

Fields of Coils

- From
$$dB_z = \frac{Ib}{cr^3}\, d\ell, \quad (40)$$
See Fig. 6.16.

i.e., $\vec{B}$ field from a segment of a ring "$d\ell$".

For a coil illustrated in Fig 6.16, $\quad d\ell = \dfrac{r\,d\theta}{\sin\theta}$,

and $\quad r = b/\sin\theta$,

$$\rightarrow \quad dB_z = \frac{2\pi b^2}{cr^3}\,\frac{I n r\, d\theta}{\sin\theta} = \frac{2\pi I n}{c}\sin\theta\, d\theta,$$

where "n" is the number of turns in the winding per unit length along the cylinder,

- Carrying out the integration from θ_1 to θ_2,
$$B_z = \frac{2\pi I n}{c}\int_{\theta_1}^{\theta_2}\sin\theta\, d\theta = \frac{2\pi I n}{c}\left(\cos\theta_1 - \cos\theta_2\right) \quad (44)$$
$$\rightarrow \frac{4\pi I n}{c} \quad \text{(for infinitely long cylinder).}$$

• Change in $\vec{B}$ across a Current Sheet

* Surface current density $\mathcal{J}$ (in esu/sec-cm):

(Recall the volume current density J is in unit esu/sec-cm^2).

See Fig 6.21 on page 232,

* One can show that the jump in the component $\vec{B}$ tangential to the surface and perpendicular to $\vec{\mathcal{J}}$ across the surface is given by

$$B_z^+ - B_z^- = \frac{4\pi}{c}\mathcal{J} \qquad (46).$$

Homework

Probs, 6.4, 6.8, 6.14, 6.25 and 6.26,

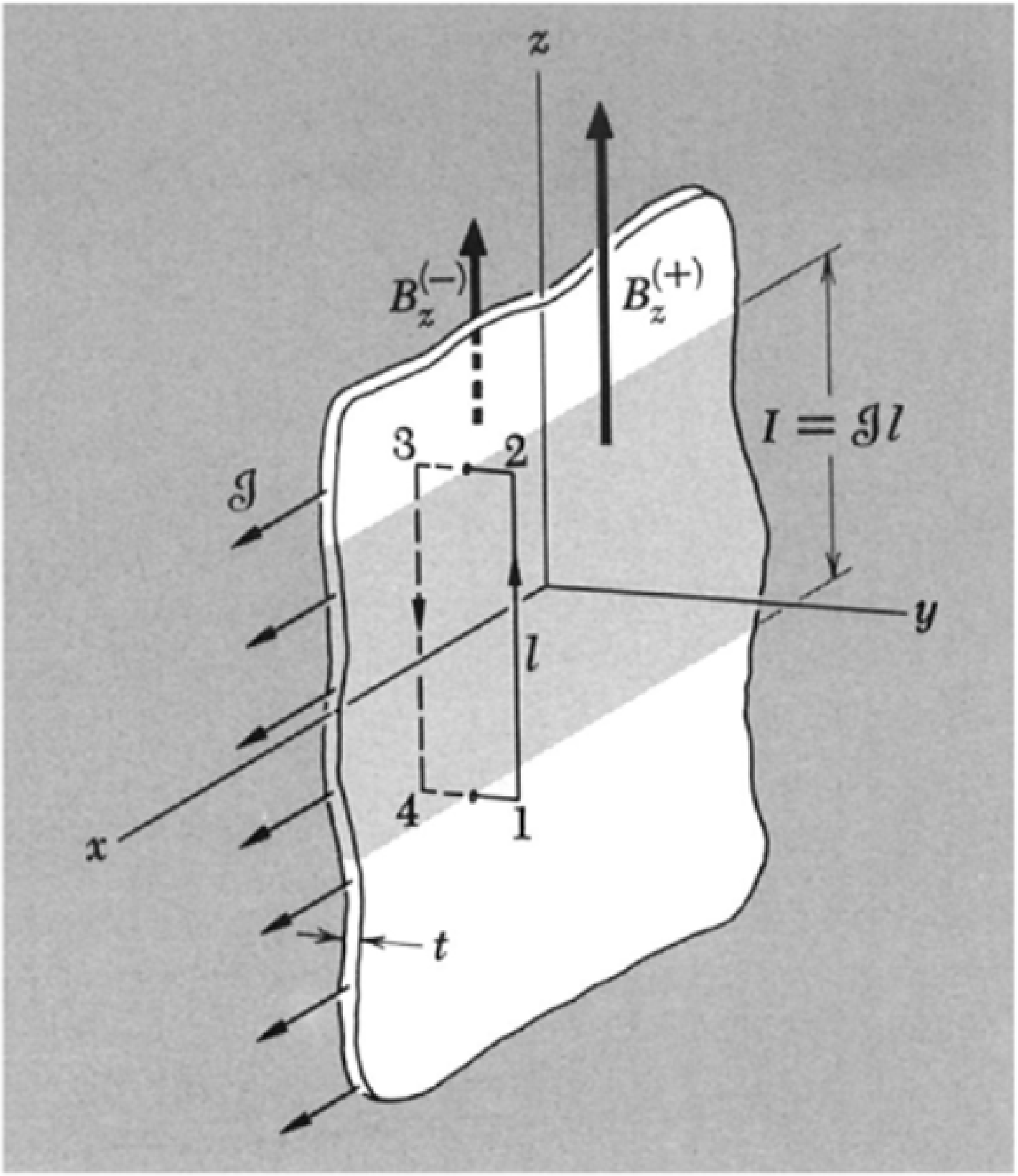


FIGURE 6.21
At a sheet of surface current there must be a change in the parallel component of **B** from one side to the other.