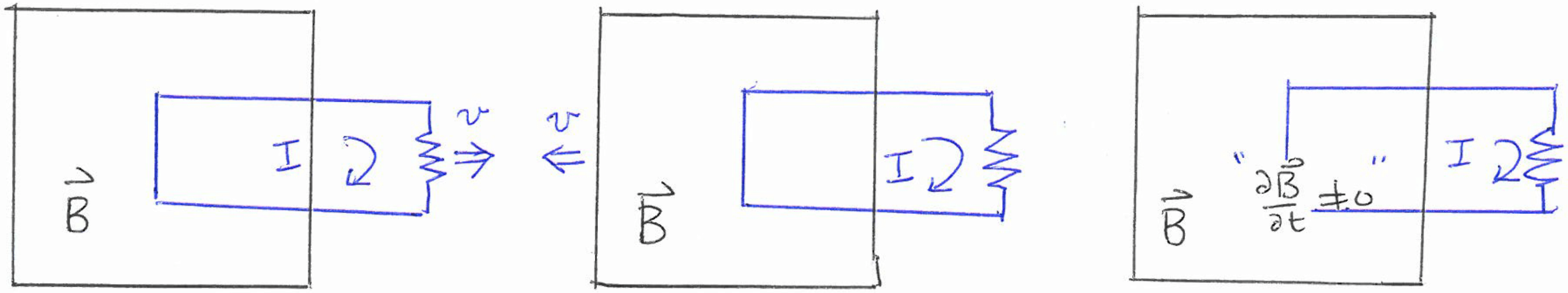
Fundamentals of Engineering Physics 2019

Week 8.

7. Electromagnetic Induction



1. A loop of wire got pulled through a magnetic field

2. The magnet moved to the left

3. Both loop and magnet at rest. $\vec{B}$ changed in time.

" A current flowed in the loop, as a result."
(in all 3 cases).

Why? Lorentz force on charges in the loop.
$\vec{F} = \frac{1}{c}\vec{v}\times\vec{B}$,

No Lorentz force, → ?
But 1. and 2. are equivalent according to relativity (or Galilean invariance)

→ ? ↓ Electric Field?

Faraday's Law of Induction

- $$\mathcal{E} = \oint_C \vec{E} \cdot d\vec{s} = -\frac{1}{c}\frac{d}{dt}\Phi = -\frac{1}{c}\frac{d}{dt}\int_S \vec{B}\cdot d\vec{a}, \qquad (25)$$

  where C is a closed, stationary curve.
  S is a surface spanning C.

- Since $\oint_C \vec{E}\cdot d\vec{s} = \int_S \text{curl } \vec{E}\cdot d\vec{a}$ from Stokes' theorem,

$$\boxed{\text{curl } \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{B}} \qquad (30)$$

* A changing magnetic field induces an electric field!

[Units] in CGS : $\dfrac{1}{\text{cm}} \dfrac{\text{Stat volts}}{\text{cm}} = \dfrac{\text{sec}}{\text{cm}} \dfrac{\text{gauss}}{\text{sec}}$.

9. Maxwell's Equations and EM Waves

- Let's consider time-dependent $\vec{E}$ field,

  - $\mathrm{div}\,\vec{E} = 4\pi\rho$ (1) Gauss's law

  - $\mathrm{div}\,\vec{J} = -\dfrac{\partial\rho}{\partial t}$ (2) charge conservation

  must be satisfied.

- For stationary $\vec{J}$; $\mathrm{curl}\,\vec{B} = \dfrac{4\pi}{c}\vec{J}$ (3) Ampère's Law

  But Eq.(3) implies that

  $$\mathrm{div}\,\vec{J} = \frac{c}{4\pi}\,\mathrm{div}\,\mathrm{curl}\,\vec{B} = 0\ !$$

  in contradiction to Eq.(2) for time-dependent system.

∴ Eq.(3) should be modified for time- " .

Displacement Current

- curl $\vec{B} = \frac{4\pi}{c}\vec{J} + (?)$

- Faraday's law states that changing magnetic field "$-\frac{1}{c}\frac{\partial}{\partial t}\vec{B}$" is accompanied by an electric field "curl $\vec{E}$".

*** Then, why not changing electric field can give rise to a magnetic field?

- $$\boxed{\text{curl } \vec{B} = \frac{4\pi}{c}\vec{J} + \underbrace{\frac{1}{c}\frac{\partial \vec{E}}{\partial t}}_{\text{displacement current.}}}$$ Proper modification of Ampère's law (9).

- Note that Eq.(9) is now compatible with the continuity equation $\text{div}\,\vec{J} = -\frac{\partial \rho}{\partial t}$ and $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$.

Maxwell's Equations

- $\mathrm{curl}\ \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$
- $\mathrm{curl}\ \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J}$
- $\mathrm{div}\ \vec{E} = 4\pi\rho$
- $\mathrm{div}\ \vec{B} = 0$ (no magnetic monopole).

These equations are compatible with the special relativity (can be applied to $|\vec{V}|/c \leq 1$).

Electromagnetic Waves

- In empty space with $\rho = 0$ and $\vec{J} = 0$,

$$\boxed{\begin{aligned} \text{curl } \vec{E} &= -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \quad \text{div } \vec{E} = 0 \\ \text{curl } \vec{B} &= \frac{1}{c}\frac{\partial \vec{E}}{\partial t}, \quad \text{div } \vec{B} = 0 \end{aligned}} \tag{16}$$

We will learn later, in this class, that these equations can be reduced to the wave equations for $\vec{E}$ and $\vec{B}$.

* A particular example of the wave solutions satisfying Eq.(16) can be

$$\begin{cases} \vec{E} = \hat{z}\, E_0 \sin(y - vt) & (17) \\ \vec{B} = \hat{x}\, B_0 \sin(y - vt) & (18) \end{cases}$$

→ propagation in y direction with a speed "v".

• Properties of Electromagnetic Waves

- By substituting the expressions for $\vec{E}$ and $\vec{B}$ in (17) and (18) to Eqs (16), we find the following conditions that must be satisfied.

  - $v = \pm c$ and $B_0 = E_0$ (in CGS), (21)

$\Rightarrow$ ⊛ The field pattern travels with speed "c".

⊛ At every point in the wave at any instant in time, the electric and magnetic strengths are equal.

⊛ The electric field and magnetic field are perpendicular to one another and to the direction of propagation.

See Fig 9.7 for an illustration

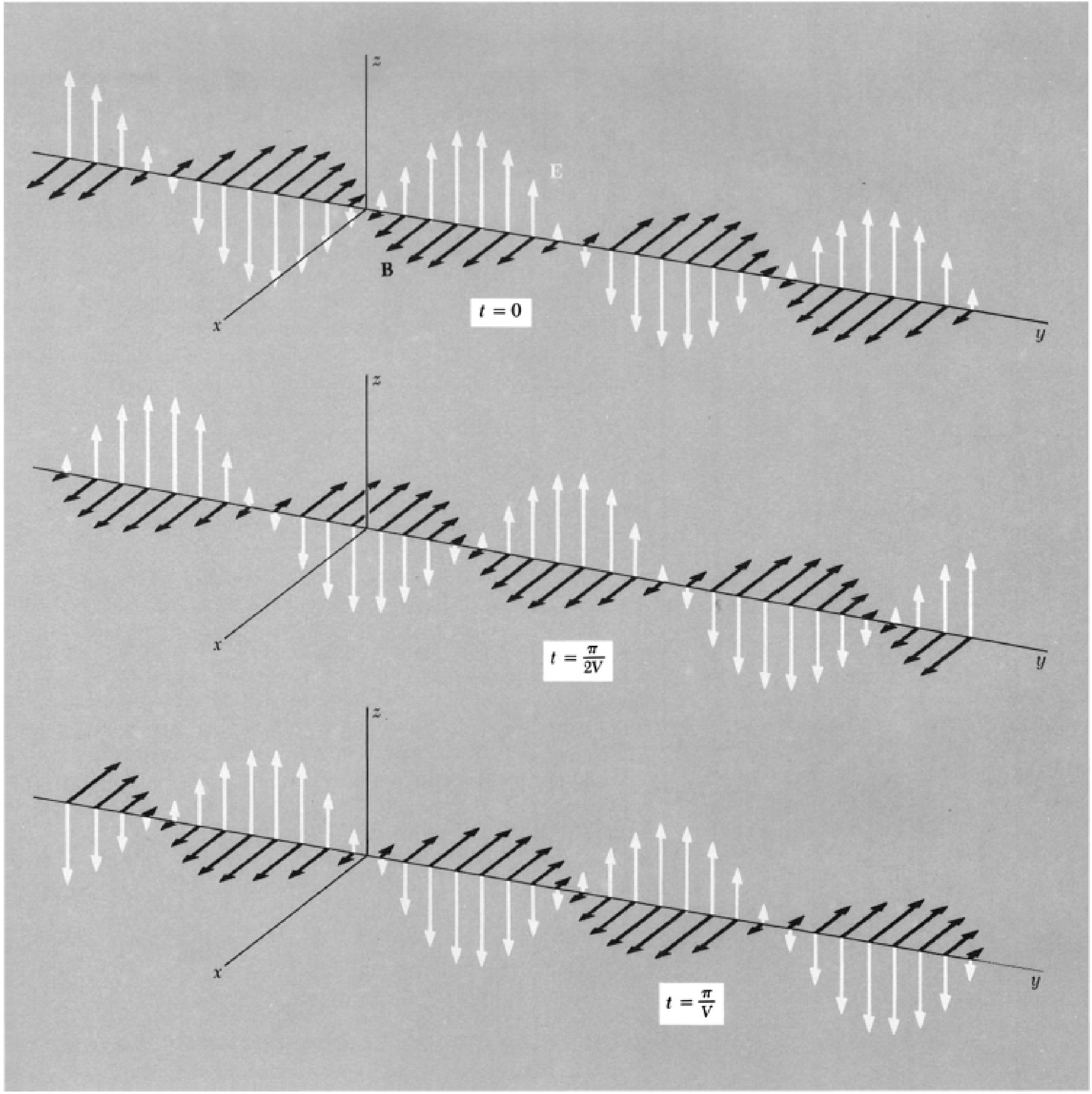


FIGURE 9.7
The wave described by Eqs. 17 and 18 is shown at three different times. It is traveling to the right, in the positive y direction.

Two counter-propagating Waves → Standing Wave

* $\vec{E}_1 = \hat{z} E_0 \sin \frac{2\pi}{\lambda}(y - ct)$, $\vec{B}_1 = \hat{x} E_0 \sin \frac{2\pi}{\lambda}(y-ct)$, (23)

is a solution of Eq. (16) (Maxwell's Eqns in empty space).

Note that

* $\vec{E}_2 = \hat{z} E_0 \sin \frac{2\pi}{\lambda}(y+ct)$, $\vec{B}_2 = -\hat{x} E_0 \sin \frac{2\pi}{\lambda}(y+ct)$, (24)

is also a solution of Eq. (16),

(23) : propagation in "$+y$" direction , (24): propagation in "$-y$" direction

* Since Eq (16) is linear equation in $\vec{E}$ and in $\vec{B}$,

the principle of superposition can be applied.

* ∴ $\vec{E} = \vec{E}_1 + \vec{E}_2 = \cdots = 2\hat{z}E_0 \sin\left(\frac{2\pi y}{\lambda}\right)\cos\left(\frac{2\pi ct}{\lambda}\right)$

$\vec{B} = \vec{B}_1 + \vec{B}_2 = \cdots = -2\hat{x} E_0 \cos\left(\frac{2\pi y}{\lambda}\right)\sin\left(\frac{2\pi ct}{\lambda}\right)$ (26)

is also a solution describing a standing wave!

* A standing wave's pattern is illustrated in Fig. 9.9. on page 337

$\boxed{\text{HomeWork}}$ Problems 9.5 and 9.8.

*** The next subject is "WAVES."

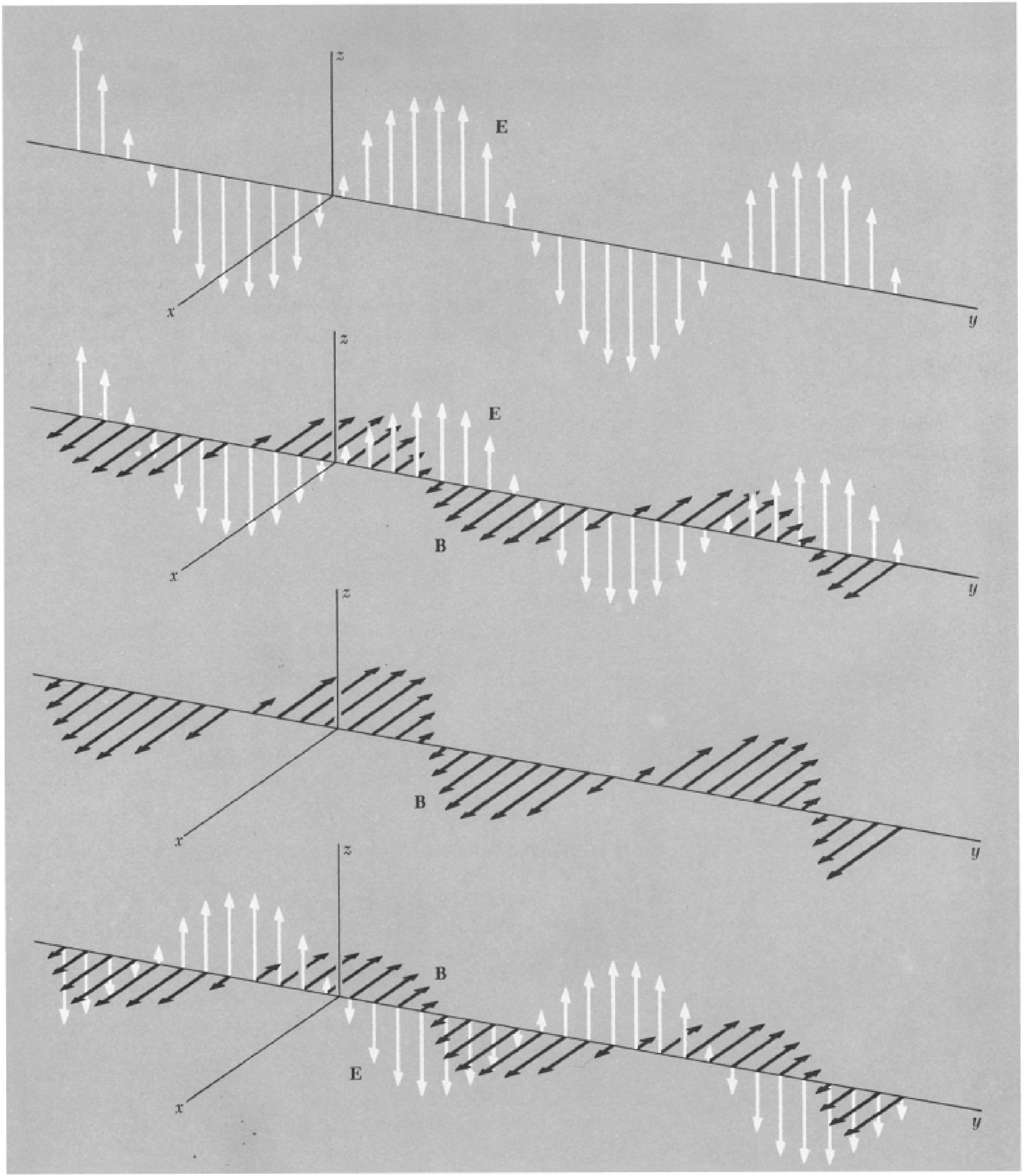


FIGURE 9.9
A standing wave, resulting from the superposition of a wave traveling in the positive y direction (Eq. 23) and a similar wave traveling in the negative y direction (Eq. 24). Beginning with the top figure, the fields are shown at four different times, separated successively by one-eighth of a full period.

Waves : Ch.4. Travelling Waves

⊛ Harmonic Travelling waves in one dimension and phase velocity.

— Consider a one-dimensional system consisting of a continuous, homogeneous string stretching from $z=0$ to infinity,

— We wish to find a displacement $\psi(z,t)$ of a moving part located at position z, $0 \le z < \infty$.

— Let the displacement of the string at $z=0$ be given by

$$\psi(0,t) = D(t) = A\cos(\omega t). \qquad (2)$$

— Then, the displacement will travel with a constant velocity from $z=0$ to $z \to$ infinity.

— $\psi(z,t)$ will describe a harmonic travelling wave.

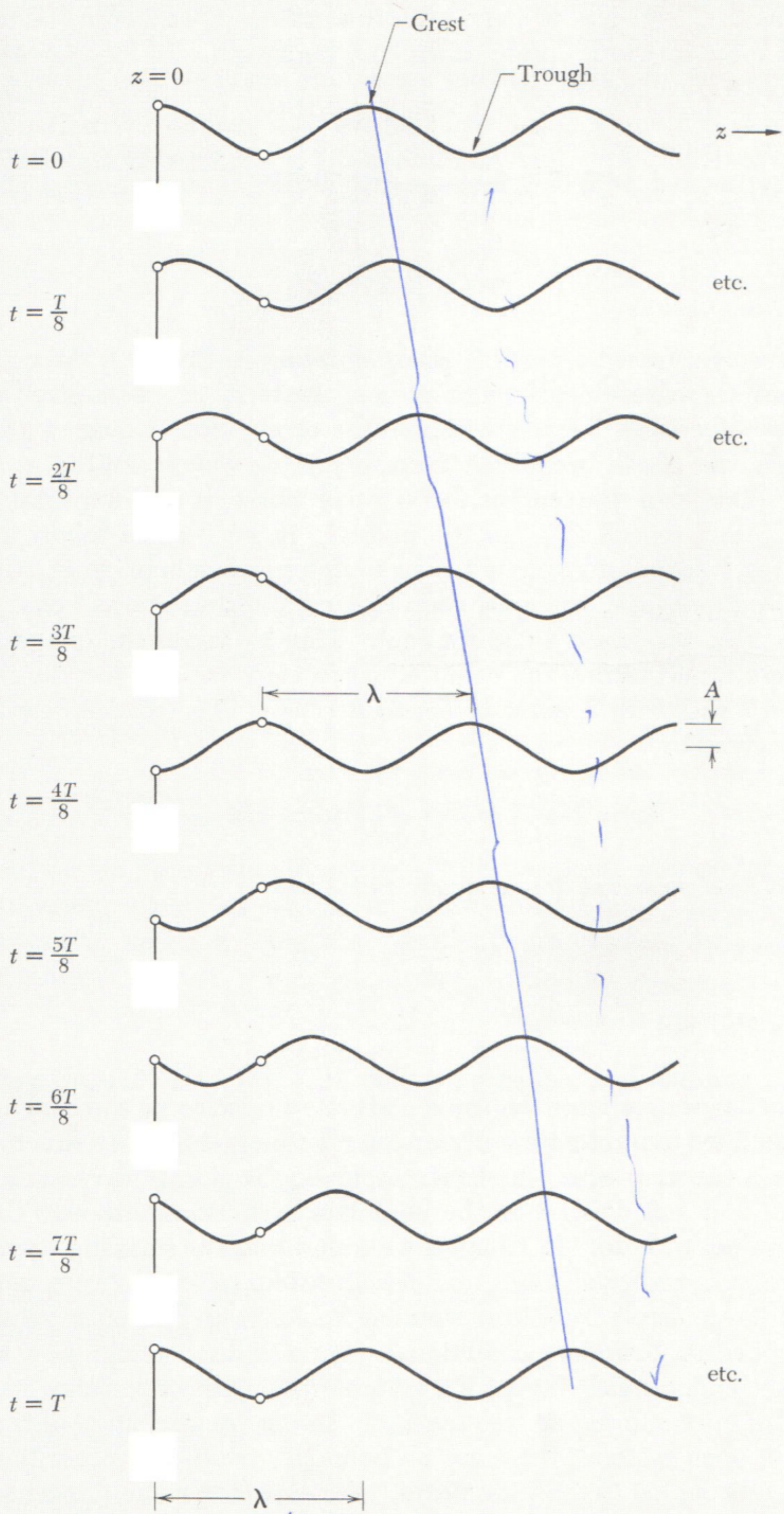


Fig. 4.1 Driving force at $z = 0$ describes harmonic motion of period T. Sinusoidal traveling wave propagates in $+z$ direction. The wavelength is λ. The phase velocity is $\lambda/T = \omega/k = \lambda\nu$. Every point on the string undergoes the same harmonic motion as that at $z = 0$, but at a later time.

* Since the motion of a moving part at position z at time t must be the same as that of the moving part at $z=0$ at the earlier time t', with t and t' related by

$$t' = t - \frac{z}{v_\varphi} , \qquad (3)$$

where v_φ is called the "phase velocity."

* Thus,

$$\psi(z,t) = \psi(0,t') = A\cos(\omega t') = A\cos\left(\omega\left(t - \frac{z}{v_\phi}\right)\right)$$

$$= A\cos\left(\omega t - \frac{\omega}{v_\varphi} z\right) \qquad (4)$$

I.e., harmonic oscillation in time at fixed z, and sinusoidal " in space at fixed t.

* We can recognize that k (wave number) $= \frac{\omega}{v_\phi}$ by writing Eq. (4) as

$$\psi(z,t) = A\cos(\omega t - kz) \qquad (6)$$

∴ $\boxed{\text{Phase Velocity}}$

$$v_\varphi = \frac{\omega}{k}. \qquad (8a)$$

Since $\omega = 2\pi\nu$,

$k = \frac{2\pi}{\lambda}$,

ω: angular frequency,
(ν: frequency, λ: wavelength)

$$v_\varphi = \lambda\nu \qquad (8b)$$

and $T = 1/\nu$,
T: period.

$$v_\varphi = \lambda/T \qquad (8c).$$

The argument "$\omega t - kz$" in Eq. (6) is called the phase, $\varphi(z,t)$.

* Then, we can notice that "the point of constant phase" (e.g., the wave crest (maximum of $\cos\varphi(z,t)$) or the wave trough (minimum of ")), travels with a speed v_φ!

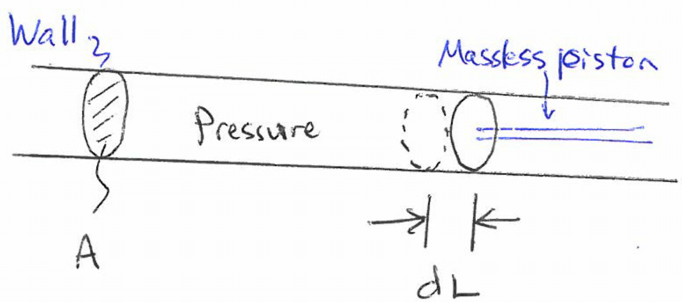
$$d\varphi = \left(\frac{\partial\varphi}{\partial t}\right)dt + \left(\frac{\partial\varphi}{\partial z}\right)dz = \omega\,dt - k\,dz = 0, \qquad (10)$$

$$\hookrightarrow v_\varphi \equiv \left(\frac{dz}{dt}\right)_{[d\varphi=0]} = \omega/k. \qquad (11),$$

• Phase velocity of Sound Waves – Newton's Model

* Suppose air is confined in a closed cylindrical container.

→ the air acts like a compressed spring extending along the cylinder [see Fig. 4.12 on page 165].



* the air exerts a force on the piston given by

$$F = pA$$

∴ For a small displacement of piston "dL",

* $$dF = A\,dp = A\left(\frac{dp}{dV}\right)_0 A\,dL$$

(For a spring, $dF = -K_L\,dL$, with K_L the spring constant).

$$v^2 = \frac{\text{Tension}}{\rho_0\ (\text{line mass density})} = \frac{K_L L_0}{\rho_0\,(\text{line})} \quad — (31).$$

* Since $K_L = -A^2 \left(\frac{dP}{dV}\right)_g$ and ρ_0 (line) $L_0 = \rho_0$ (volume) $A L_0$,

spring air

$$\boxed{v^2 = -\frac{V_0 \left(\frac{dP}{dV}\right)_0}{\rho_0}}$$

"Velocity of Sound". (33)

- Newton further used the Boyle's law ; $pV = p_0 V_0$

$$\rightarrow \quad v_{Newton} = \sqrt{\frac{P_0}{\rho_0}} , \qquad (36)$$

- But, the Boyle's law applies to the isothermal process (at constant temperature).

* "Adiabatic gas law" which applies ~~with~~ to processes with "no heat exchange". is a better assumption,

$$pV^\gamma = p_0 V_0^\gamma \qquad (40).$$

* Since $V_0 \left(\frac{dP}{dV}\right)_0 = -\gamma P_0$, $\gamma = 5/3$.

$$\rightarrow \quad \boxed{v_{Sound} = \sqrt{\frac{\gamma P_0}{\rho_0}}} \qquad (41)$$