Fundamentals of Engineering Physics 2019

Week 9.

6.2. Group Velocity

* Superposition of two harmonic oscillations to give amplitude-modulated oscillation :

* Let's consider a one dimensional string extending from $z=0$ to $+\infty$ again,

For $D(t) = A\cos(\omega_1 t) + A\cos(\omega_2 t)$, $= \psi(0,t)$, (2)

$$= A_{mod}(t) \cdot \cos(\omega_{Avg} t) \quad (3)$$

where $A_{mod}(t) = 2A\cos(\omega_{mod} t)$; modulated amplitude (4)

with $\omega_{mod} \doteq \frac{1}{2}(\omega_1 - \omega_2)$, $\omega_{avg} = \frac{1}{2}(\omega_1 + \omega_2)$

* In a linear, homogeneous system, each term in Eq. (2) will lead to different (independent) travelling wave respectively given by $\psi_1(z,t)$ and $\psi_2(z,t)$.

$*$ $\psi(0,t) = D(t) = A\cos(\omega_1 t) + A\cos(\omega_2 t)$ (5)

$\psi_1(0,t)$ $\qquad$ $\psi_2(0,t)$

$\downarrow$ ← independently → $\downarrow$

as we learned earlier $\qquad \psi_1(z,t) \qquad \psi_2(z,t)$

$$\psi_1(z,t) = A\cos(\omega_1 t - k_1 z), \quad \psi_2(z,t) = A\cos(\omega_2 t - k_2 z).$$

$*$ Then, using the principle of superposition,

$$\psi(z,t) = \psi_1(z,t) + \psi_2(z,t) = A\cos(\omega_1 t - k_1 z) + A\cos(\omega_2 t - k_2 z) \qquad (6)$$

$$\boxed{\psi(z,t) = A_{mod}(z,t)\cos\left(\omega_{Avg} t - k_{Avg} z\right)}, \qquad (7)$$

with

$$A_{mod}(z,t) = 2A\cos(\omega_{mod} t - k_{mod} z), \qquad (8)$$

with $\omega_{mod} = \frac{1}{2}(\omega_1 - \omega_2)$, $\quad k_{mod} = \frac{1}{2}(k_1 - k_2)$ (9)

$\omega_{Avg} = \frac{1}{2}(\omega_1 + \omega_2)$, $\quad k_{Avg} = \frac{1}{2}(k_1 + k_2)$ (10)

Modulation Velocity

* Suppose $|\omega_1 - \omega_2| \ll \omega_1 + \omega_2$ and $|k_1 - k_2| \ll k_1 + k_2$,

The modulation wave crest [a place where $A_{mod}(z,t) = 1$] will travel with a velocity which keeps "$(\omega_{mod} t - k_{mod} z)$" constant.

i.e. $$\omega_{mod}\, dt - k_{mod}\, dz = 0 \qquad (11),$$

$$\therefore \quad v_{mod} = \frac{dz}{dt} = \frac{\omega_{mod}}{k_{mod}} = \frac{\omega_1 - \omega_2}{k_1 - k_2} \qquad (12),$$

Now, $\omega = \omega(k)$; a dispersion relation.

By Taylor expanding Eq.(12) for small $\omega_1 - \omega_2$ and $k_1 - k_2$,

$$v_{mod} = \frac{\omega(k_1) - \omega(k_2)}{k_1 - k_2} = \cdots = \frac{d\omega}{dk} \qquad (15)$$

* $$\boxed{v_g = \frac{d\omega}{dk}} \qquad \text{Group Velocity} \qquad (16)$$

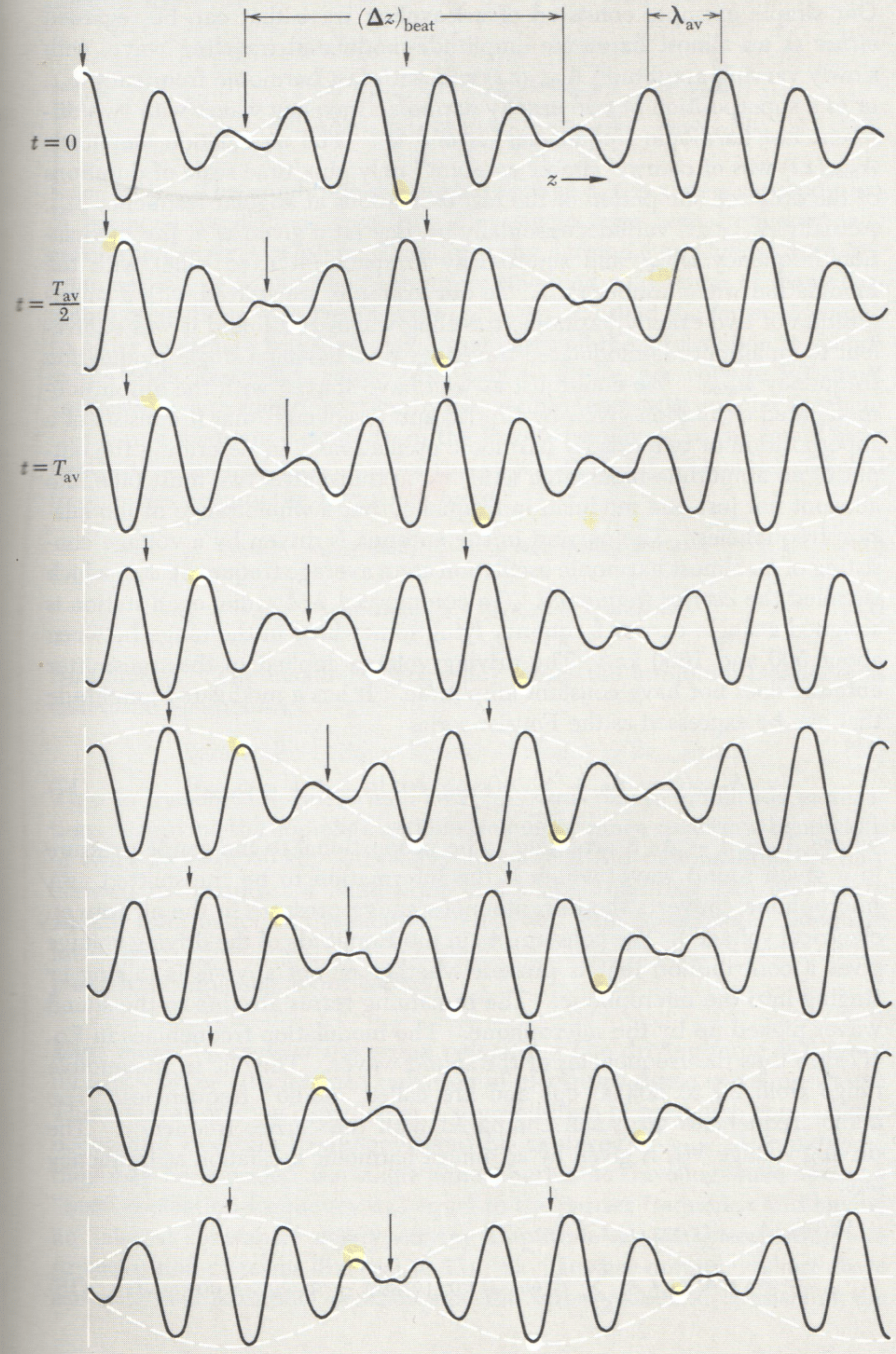

***Fig. 6.1** Group velocity. The arrows follow the beats, which travel at the group velocity v_g. The white circles follow individual wave crests, which travel at the average phase velocity v_{av}.*

draw this

of "envelope";

of

~~sc~~crests, troughs

$\frac{1}{4}$ wl.

then packet prop. by

$\frac{1}{4}$ wl.

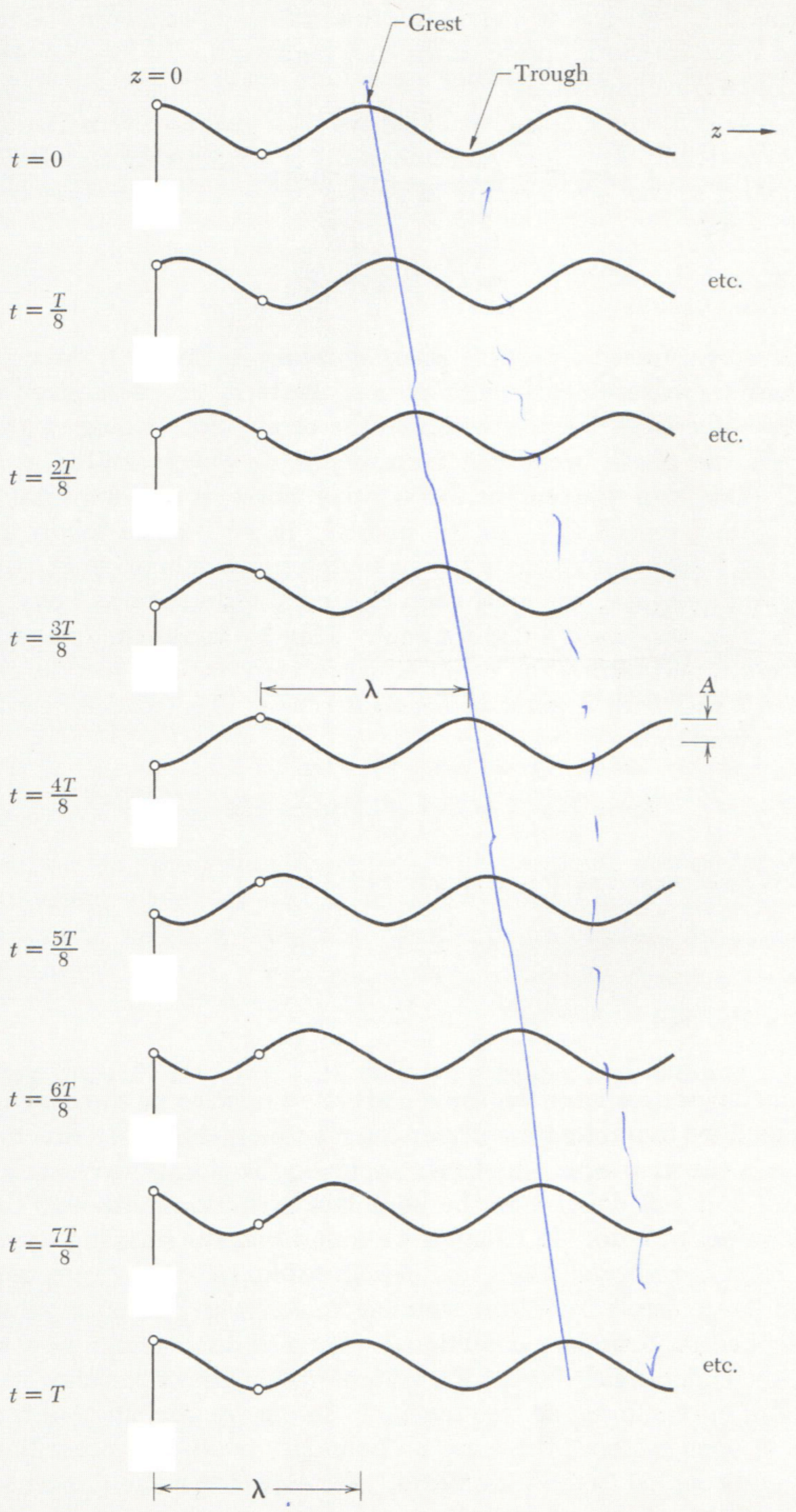

Fig. 4.1 Driving force at $z = 0$ describes harmonic motion of period T. Sinusoidal traveling wave propagates in $+z$ direction. The wavelength is λ. The phase velocity is $\lambda/T = \omega/k = \lambda\nu$. Every point on the string undergoes the same harmonic motion as that at $z = 0$, but at a later time.

Group Velocity and phase Velocity

* If we superpose more number of harmonic waves, the wave packet will be more localized in space, and the expression "$v_\varphi = \frac{d\omega}{dk}$" remains valid.

* For some examples, $v_g < v_\varphi$, for some other, $v_g > v_\varphi$ and sometimes $v_g = v_\varphi$ depending on the dispersion relation $\omega = \omega(k)$. Both a signal (information) and energy ~~are~~ travel at the "Group velocity", not at the phase velocity.

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* Eg.,

① Sound Wave: $\omega = \sqrt{\frac{\gamma P_0}{\rho_0}}\, k$, $\rightarrow$ $v_\varphi = \frac{\omega}{k} = \sqrt{\frac{\gamma P_0}{\rho_0}}$

$v_g = \frac{d\omega}{dk} = \sqrt{\frac{\gamma P_0}{\rho_0}}$

② EM wave in vacuum: $\omega = ck$ $\rightarrow$ $v_\varphi = v_g = c$.

③ EM wave in plasma (eg. ionosphere): $\omega^2 = \omega_p^2 + c^2k^2$
(plasma wave)

$\rightarrow$ $v_\varphi = \frac{\omega}{k} = \sqrt{c^2 + \omega_p^2/k^2} \geq c$, but

$v_g = c^2/v_\varphi \leq c$.

⊛ Electromagnetic Waves – Revisited.

⊛ Maxwell's equation in vacuum:

$$\frac{\partial}{\partial t}\vec{E} = c\,\vec{\nabla}\times\vec{B}, \quad \frac{\partial}{\partial t}\vec{B} = -c\,\vec{\nabla}\times\vec{E} \qquad (77.a;b)$$

$$\vec{\nabla}\cdot\vec{E} = 0, \quad \vec{\nabla}\cdot\vec{B} = 0 \qquad (77.c;d)$$

⊛ Classical Wave Equation:

$$\frac{\partial^2}{\partial t^2}\vec{E} = \frac{\partial}{\partial t}\left(c\,\vec{\nabla}\times\vec{B}\right) = c\,\nabla\times\frac{\partial}{\partial t}\vec{B} = c\,\nabla\times\left(-c\,\vec{\nabla}\times\vec{E}\right)$$

apply $\frac{\partial}{\partial t}$ to (77.a)

(77.b)

$$= c^2\left(\nabla^2\vec{E} - \vec{\nabla}(\cancel{\vec{\nabla}\cdot\vec{E}})\right) \qquad (79,a)$$

expanding triple product

(77.c)

$$\boxed{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = \nabla^2\vec{E}} \qquad (79.a)$$

This consists of 3 separate PDE's for E_x, E_y and E_z.

Similar procedure will lead to

$$\frac{1}{c}\frac{\partial^2}{\partial t^2}\vec{B} = \nabla^2\vec{B}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

in Cartesian coordinate.

Properties of Electromagnetic Plane Waves in vacuum

1. There is a unique propagation direction which can be taken to be along z.

2. None of the components of $\vec{E}$ or $\vec{B}$ depends on either of transverse coordinates x and y.

3. We can show that E_z and B_z can be taken to be zero. (Read page 356-357), i.e.,

   " Electromagnetic ~~Wa~~ Plane Waves are transverse waves".

   i.e. $\vec{E} \perp \hat{z}$ and $\vec{B} \perp \hat{z}$.

4. $|\vec{E}| = |\vec{B}|$ in cgs unit.

   $\vec{E} \perp \vec{B}$

   and $\vec{E} \times \vec{B}$ is in the direction of $\vec{k}$, $(\hat{z})$.

⊛ "De Broglie" Waves

* Louis de Broglie proposed that moving objects (with mass) have wave characteristics as well as well-accepted particle nature.

  – This precedes (1924) an experimental demonstration (1927).

  cf. Particle property of light waves has been discovered in 1905.

* Physical Meaning of the Wave Function : $\psi(\vec{x}, t)$

  " The probability of experimentally finding the body described by the wave function ψ at the point $\vec{x}$ at the time t is proportional to the value of $|\psi|^2$ there at t. "

⇝ Erwin Schrödinger → "Quantum Mechanics."

two years later

Dispersion Relation for de Broglie Waves

⊛ Consider a ptl in 1-d described by a wave function,

$$\psi(z,t) = f(z)\, e^{-i\omega t} \qquad (1)$$

If the potential energy of the ptl is constant in z, the medium is homogeneous and f(z) can be expressed as a sinusoidal function of kz ;

$$\psi(z,t) = \{A \sin(kz) + B \cos(kz)\}\, e^{-i\omega t} \qquad (2)$$

* $E = \frac{p^2}{2m} + V$, (3) for a particle

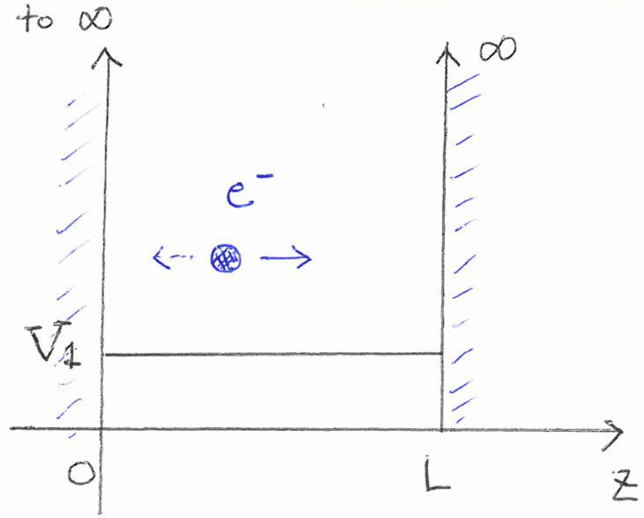
From "E" = ℏ"ω" ; Bohr frequency condition

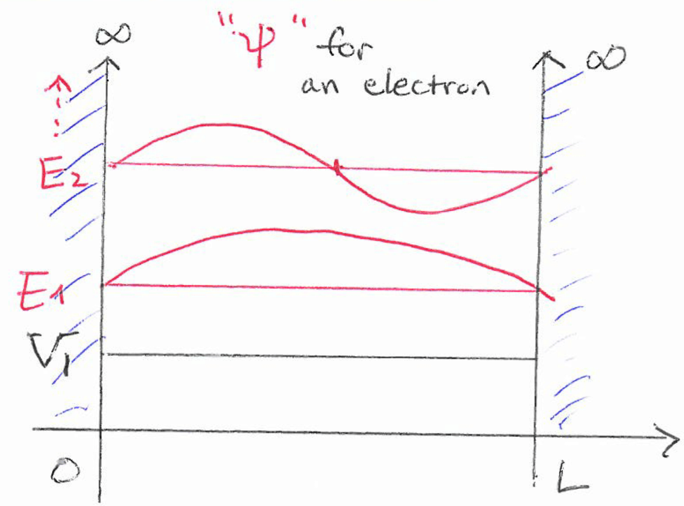
"p" = ℏ"k" ; de Broglie wavenumber relation

$$\boxed{\hbar\omega = \frac{\hbar^2 k^2}{2m} + V,} \qquad (4)$$

Dispersion relation for de Broglie waves.

Electrons in a Box





According to classical mechanics, an electron's kinetic energy is given by "$\frac{p^2}{2m} = E - V_1$" (5)

and, E can be <u>any</u> value $\geq V_1$.

According to quantum mechanics, an electron is described by a wave function $\psi(z,t)$ of the de Broglie wave.

The allowed energy values are given by

"$E = \hbar\omega = \frac{\hbar^2 k^2}{2m} + V_1$"

and only certain values are allowed corresponding to "k" values satisfying Boundary Conditions.

Quantization of k and E from Boundary Conditions

(*) $|\psi|^2_{(\vec{x})}$: Probability of finding the electron at $\vec{x}$;

$\Rightarrow \psi = 0$ at $z = 0$, and L. (Infinite potential wells are impenetrable)

$\therefore \psi(z,t) = e^{-i\omega t} A \sin(kz)$, with $kL = \pi, 2\pi, \cdots, n\pi$ (6)

"Quantization of k" $\rightarrow$ $\boxed{k_n = \frac{n\pi}{L}}$, $n = 1, 2, \cdots$ (7)

Since the probability of finding an e^- inside the box (somewhere) is "1";

$$\int_0^L dz |\psi(z,t)|^2 = 1 \quad \Rightarrow \quad |A| = \sqrt{\frac{2}{L}} \quad \text{and} \quad A = \sqrt{\frac{2}{L}} e^{i\alpha}$$

α = const.

$$\therefore \quad \psi_n(z,t) = \sqrt{\frac{2}{L}}\, e^{-i(\omega t - \alpha)} \sin k_n z \qquad (10)$$

⊛ Quantization of Energy Level

From the dispersion relation of de Broglie wave, (Eq.(4)),

$$\omega_n = \omega_0 + \frac{\hbar k_n^2}{2m}, \quad \omega_0 \equiv \frac{V_1}{\hbar}$$

* the corresponding electron's energy :

$$\boxed{E_n = V_1 + \frac{\hbar^2 k_n^2}{2m} = V_1 + \frac{\hbar^2 \left(\frac{n\pi}{L}\right)^2}{2m}},$$

where $n = 1, 2, 3, \cdots$ (12)

only a set of discrete values allowed for energy, i.e, "quantized"!

called the "Eigenvalues",

"$\Psi_n(z,t)$" is called the eigenfunction.

(~ standing wave for this example of infinite wall.

⊛ Phase and Group Velocities of de Broglie Waves

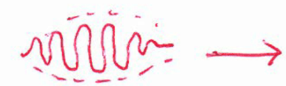
Consider an electron of energy E in a constant potential V,

→ it satisfies the dispersion relation:

$$\omega = \frac{\hbar k^2}{2m} + \frac{V}{\hbar}$$

– particle: 

- The phase velocity is:



$$v_\varphi(k) = \frac{\omega}{k} = \frac{\hbar k}{2m} + \frac{V}{\hbar k} .$$

– wave packet:

- If we use the de Broglie wave number relation $p = \hbar k$,

$$v_\varphi = \frac{1}{2}\frac{p}{m} + \frac{V}{p}$$, where p is a momentum of a ptl.

???

- On the other hand, the group velocity is: $v_g = \left(\frac{d\omega}{dk}\right)$

$$\text{"} \quad v_g = \frac{d\omega}{dk} = \frac{\hbar k}{m} = \frac{p}{m} \quad \text{"} .$$

→ the same as the ptl's velocity!

Wave Equation for de Broglie Waves

* Classical Wave Eqn for EM waves: (light waves or photon with "$m=0$")

"$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = \nabla^2 \vec{E}$".

This cannot describe the de Broglie wave for a massive ~~particle~~ object.

* $\psi(z,t) = e^{-i\omega t}\left(A e^{ikz} + B e^{-ikz}\right)$, in a region of constant potential. (1)

" Dispersion Relation is an outcome of the underlying Wave Equation."

— We can deduce the Wave Equation from the dispersion relation.

$$\hbar\omega = \frac{\hbar^2 k^2}{2m} + V,$$

— Since "$i\frac{\partial}{\partial t}\psi = \omega\psi$" and "$-\frac{\partial^2}{\partial z^2}\psi = k^2\psi$" for ψ in Eq. (1)

The wave function ψ of the de Broglie wave satisfies.

$$\boxed{i\hbar\frac{\partial}{\partial t}\psi(z,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi(z,t) + V\psi(z,t)} \quad (7)$$

Schrödinger Equation

⊛ Equation (7) can be generalized to inhomogeneous potential $V(\vec{x})$ in three-dimensions :

$$\boxed{i\hbar \frac{\partial}{\partial t} \psi(\vec{x},t) = -\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{x},t) + V(\vec{x}) \psi(\vec{x},t)}$$

Time-dependent Schrödinger Equation. A.

⊛ When the energy of an object does not vary in time, one can express the time-dependence of $\psi(\vec{x},t)$ as $\psi(\vec{x},t) = \psi(\vec{x}) e^{-i\omega t}$
$= \psi(\vec{x}) e^{-i(\frac{E}{\hbar})t}$

⊛ Then by removing the common factor "$e^{-i(\frac{E}{\hbar})t}$" from Eq(A), we obtain

$$* \quad \boxed{E \psi(\vec{x}) = -\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{x}) + V(\vec{x}) \psi(\vec{x})} \qquad B.$$

Time-independent Schrödinger Equation.