Fusion Plasma Theory I. 2019

Week 1

UNIT 1. Single-Particle Motion

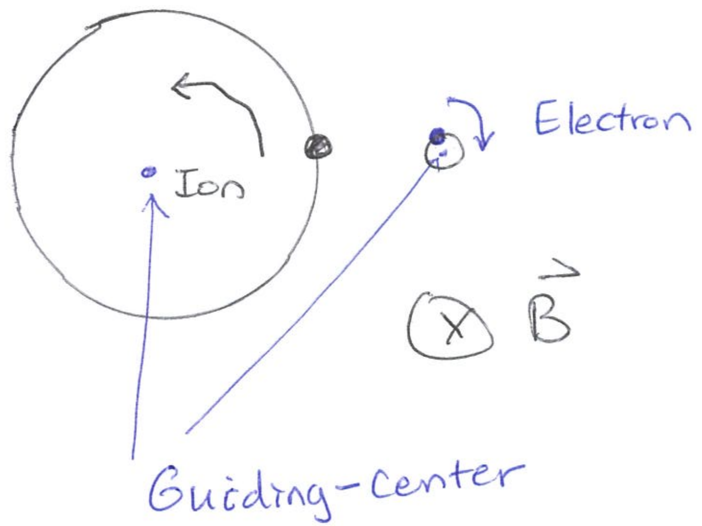
2. Particle drifts in uniform fields

2.1 GYRO-MOTION.

$$m \frac{d\vec{v}}{dt} = q\, \vec{v} \times \vec{B} \qquad (2,1)$$

for uniform $\vec{B}$ field and $\vec{E} = 0$.

It's straightforward to integrate Eq. (2,1)



- Larmor radius

  "$r_L = \dfrac{v_\perp}{\omega_c}$" $\propto (mW)^{1/2}$

- Cyclotron Frequency

  "$\omega_c = \dfrac{|q|B}{\text{"}m\text{"}}$".

Fundamental Scales in Magnetized Plasma

* Let l : spatial scale of interest.

  τ : temporal " .

* If $l \ll r_L$ and $\tau \ll \omega_c^{-1}$, the system can be regarded as "unmagnetized." ($\vec{B}$ field plays a minor role in dynamics)

* ~~If~~ $l \gg r_L$ and $\tau \gg \omega_c^{-1}$, the system is (strongly) magnetized.

  This is true for many examples in fusion plasma experiments.

* For earth's magnetosphere or many geophysical systems,

  $r_{Li} \gg l \gg r_{Le}$ and $\omega_{ci}^{-1} \gg \tau \gg \omega_{ce}^{-1}$

  i.e, ions are unmagnetized and electrons are magnetized.

2.2. E×B DRIFT

* Consider a uniform $\vec{B} = B\hat{z}$ and a uniform $\vec{E}$ in a different direction.

* $$m\frac{d\vec{v}}{dt} = q(\vec{E} + \vec{v}\times\vec{B}) \qquad (2.12)$$

Define $$\vec{u} \equiv \vec{v} - \frac{\vec{E}\times\vec{B}}{B^2} \qquad (2.13)$$

i.e., velocity in a frame moving with $\frac{\vec{E}\times\vec{B}}{B^2}$

Then,

* $$m\frac{d\vec{u}}{dt} = q\left(\hat{b}(\vec{E}\cdot\hat{b}) + \vec{u}\times\vec{B}\right) \qquad (2.16)$$

after substituting (2.13) to (2.12) and expanding the triple product $(\vec{E}\times\vec{B})\times\vec{B} = (\vec{E}\cdot\vec{B})\vec{B} - B^2\vec{E}$.

* We can further decompose Eq. (2.16) into // component and ⊥ component (to $\vec{B}$)

E x B Drift

- $$m \frac{du_{\parallel}}{dt} = q E_{\parallel}, \quad \text{where} \quad u_{\parallel} \equiv \vec{u}\cdot\hat{b}, \quad E_{\parallel} \equiv \vec{E}\cdot\hat{b}.$$

$\rightarrow$ $$v_{\parallel} = \frac{q E_{\parallel}}{m}\cdot t + v_{\parallel i} \qquad (2.19).$$

acceleration along $\vec{B}$.   initial value.

- $\perp$ component ; $$m \frac{d\vec{u}_{\perp}}{dt} = q\,\vec{u}_{\perp}\times\vec{B}, \qquad (2.20)$$

where $\vec{u}_{\perp} \equiv \vec{u} - u_{\parallel}\hat{b}$.

This equation describes a gyromotion as we have seen before.

$\therefore$ Guiding-center does <u>not</u> move in the frame moving with $\frac{\vec{E}\times\vec{B}}{B^2}$.

$\rightarrow$ • $$V_{gc} = v_{\parallel}\hat{b} + \frac{\vec{E}\times\vec{B}}{B^2} \equiv v_{\parallel}\hat{b} + \underline{\underline{\vec{v}_E}} \qquad (2.21)$$

in the Lab frame

ExB drift

I.-5.

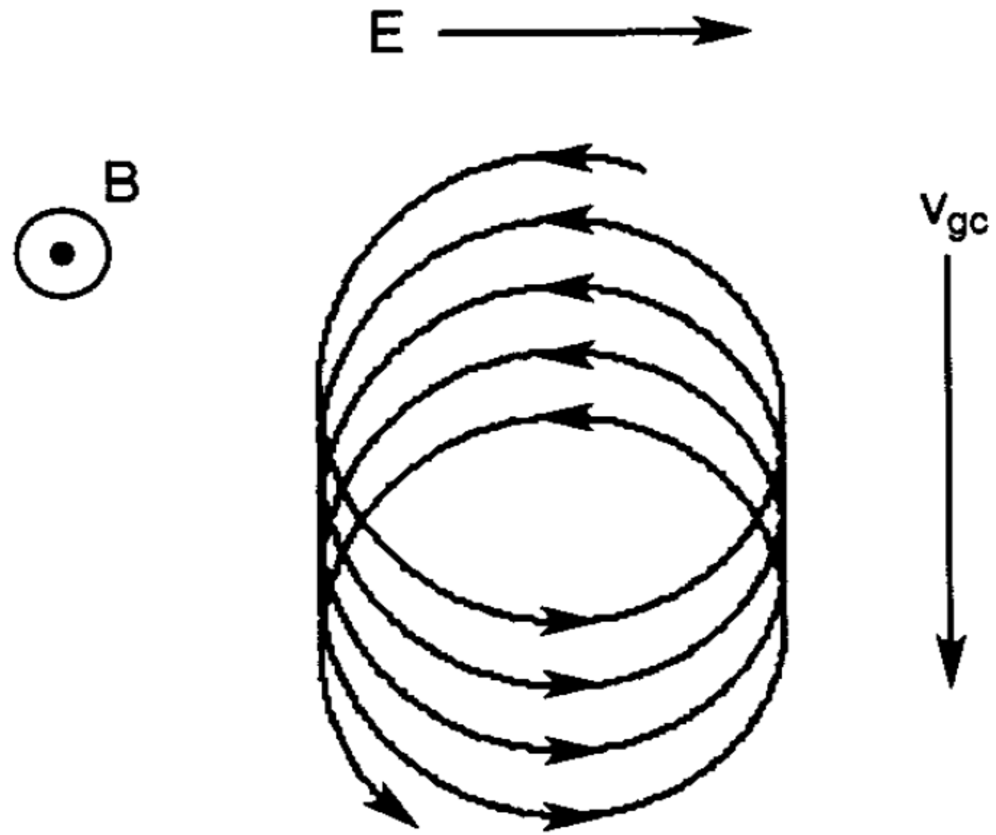


Figure 2.2. Electron $\mathbf{E} \times \mathbf{B}$ drift motion. The half-orbit on the left-hand side is larger than that on the right, because the electron has gained energy from the electric field. The dot indicates that the magnetic field faces out of the page.

2.3. Gravitational Drift

* $\vec{E}\times\vec{B}$ drift : $\vec{V}_E = \dfrac{\vec{E}\times\vec{B}}{B^2}$ independent of m and q !

We can repeat the same derivation for any other simple force on the charged particles.

$$\vec{V}_F = \left(\frac{\vec{F}}{q}\right)\times\vec{B}/B^2 \xrightarrow[\text{for } \vec{F}=m\vec{g}]{} \frac{m\vec{g}\times\vec{B}}{qB^2} \quad (2.23)$$

$\vec{E}$ is a particular example.

gravitational drift

" This expression will ~~be~~ turn out to be useful later when we consider nonuniform $\vec{B}$. "

- Typically very slow.

Ch. 3. Particle Drifts in ~~o~~ non-uniform $\vec{B}$

3.1. ∇B drift :

* We will show later that the magnetic moment

$\mu \equiv \dfrac{m v_\perp^2}{2B} = \dfrac{W_\perp}{B}$ is a conserved quantity.

(actually an adiabatic invariant).

We can check $\mu = I \cdot A$ with $I = \dfrac{|q|\,\omega_c}{2\pi}$

and

$\text{Area} = \pi r_L^2 = \dfrac{\pi v_\perp^2}{\omega_c^2}$.

indeed a magnetic moment.

* ∴ For guiding-center's motion, the $\perp$ energy

$W_\perp = \mu B$ acts like an effective potential,

with corresponding force $\vec{F}_{\nabla B} = -\mu \vec{\nabla} B$.

∇B Drift

* $$\vec{V}_F = \frac{\vec{F}}{q} \times \vec{B} / B^2$$

use $\vec{F} = \vec{F}_{\nabla B} = -\mu \vec{\nabla} B$.

to get

$$\boxed{\vec{V}_{\nabla B} = \frac{-\mu \vec{\nabla} B \times \vec{B}}{q B^2} = \frac{W_\perp}{q} \frac{\vec{B} \times \vec{\nabla} B}{B^3}} \qquad (3.9)$$

∇B drift

- Its direction depends on the sign of "q".

- It is independent of particles mass (for same $W_\perp$).

I-9.

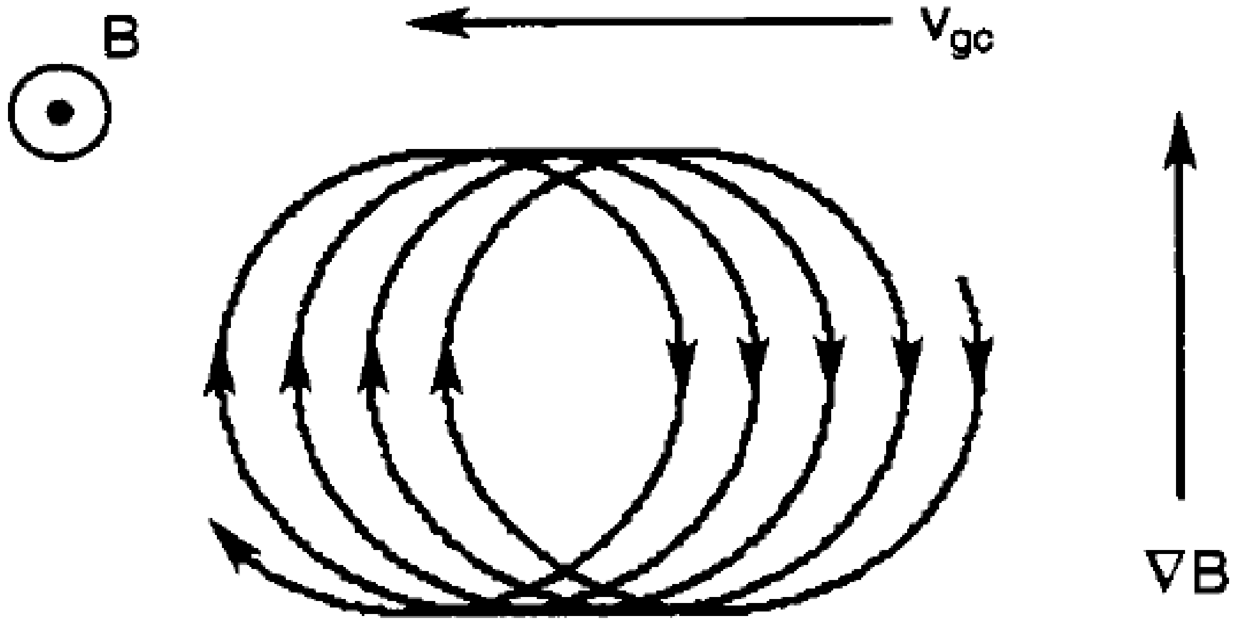


Figure 3.1. Ion ∇B drift motion. The combined effect of smaller gyro-orbits on the high-field side and larger gyro-orbits on the low-field side produces a net leftward drift of the guiding center. The dot indicates that the magnetic field faces out of the page.

3.2. Curvature Drift

* Now, consider a curved magnetic field.

A simple example is given in Fig 3.2.

In a rotating frame in θ-direction, a charged particle will feel a "centrifugal force."

$$\vec{F}_{cf} = \frac{m v_\parallel^2}{R_c}\hat{r} = m v_\parallel^2 \frac{\vec{R}_c}{R_c}, \qquad (3.10)$$

where $\vec{R}_c$ is the radius of curvature vector

in general, $\frac{\vec{R}_c}{R_c^2} = -(\hat{b}\cdot\vec{\nabla})\hat{b}$ (3.12).

* Once again, using $\vec{V}_F = \frac{\vec{F}\times\vec{B}}{qB^2}$ for any simple force,

$$\boxed{\vec{V}_{curv} = \frac{m v_\parallel^2}{qB^2}\frac{\vec{R}_c\times\vec{B}}{R_c^2} = \frac{2W_\parallel}{qB^2}\frac{\vec{R}_c\times\vec{B}}{R_c^2} = \frac{2W_\parallel}{qB^2}\vec{B}\times(\hat{b}\cdot\vec{\nabla})\hat{b}} \qquad (3.11)\ (3.13)$$

curvature drift

∇B and Curvature Drifts in vacuum Fields

* If $\vec{\nabla} \times \vec{B} = 0$, $(\vec{\nabla} B)_\perp = -B \vec{R}_c / R_c^2 = (\vec{B} \cdot \vec{\nabla}) \hat{b}$ (3.14)

(vacuum with no $\vec{J}$).

Then,
$$\vec{V}_{curv.} = \pm \frac{v_\parallel^2}{\omega_c} \frac{\vec{B} \times \vec{\nabla} B}{B^2} = \frac{2W_\parallel}{q} \frac{\vec{B} \times \vec{\nabla} B}{B^3} \quad (3.15)$$

$$\left(\text{cf. } \vec{V}_{grad} = \frac{W_\perp}{q} \frac{\vec{B} \times \vec{\nabla} B}{B^3} \right).$$

i.e. it has almost the same expression as $\vec{V}_{grad}$.

* For bi-Maxwellian (anisotropic) plasma, $\langle W_\parallel \rangle = T_\parallel / 2$ and $\langle W_\perp \rangle = T_\perp$.

$$\therefore \langle \vec{V}_{curv} + \vec{V}_{grad} \rangle = \frac{T_\parallel + T_\perp}{q} \frac{\vec{B} \times \vec{\nabla} B}{B^3} \quad (3.16)$$