

Fusion Plasma Theory I. 2019

Week 12

"Lorentz-Gas" Approximation

(*) Eq (13.12) is a good approximation for $Z \gg 1$ because $\nu_{ei} \gg \nu_{ee}$ in that case.

✓ In this approximation, the electron speed does not change

$\Rightarrow \left(\frac{\partial f_e}{\partial t} \right)_{\text{coll.}} = 0$ for any f_e which is isotropic in \mathbf{v}

✓ In general $\left(\frac{\partial f_e}{\partial t} \right)_{\text{coll.}} = 0$ for Maxwellian f_e for any valid collision operator, \therefore " implies

Thermodynamic equilibrium among the pfls.

(*) In spherical coordinates, $v_z = v \cos \theta$, $v_x = v \sin \theta \cos \phi$, $v_y = v \sin \theta \sin \phi$.

$$\Rightarrow \left(\frac{\partial f_e}{\partial t} \right)_{\text{coll.}} = \frac{n_i Z^2 e^4 \ln \Lambda}{8\pi \epsilon_0^2 m^2 v^3} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial f_e}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} f_e \right] \quad (13.14)$$

Plasma Resistivity in the Lorentz-Gas Approx. IV-7.

A small electric field \vec{E} will cause electrons to accelerate at a rate $-\frac{e\vec{E}}{m}$, so we have the following relation:

$$f_e(\vec{v}, t) = f_e(\vec{v} + e\vec{E}\Delta t/m, t - \Delta t) \quad (13.16)$$

∴ The time rate of change in f_e due to \vec{E} is given by

$$\lim_{\Delta t \rightarrow 0} \frac{f_e(\vec{v}, t) - f_e(\vec{v}, t - \Delta t)}{\Delta t} = \frac{e\vec{E}}{m} \cdot \frac{\partial f_{e0}}{\partial \vec{v}} = \left(\frac{\partial f_e}{\partial t}\right)_{\vec{E}} \quad (13.17)$$

In steady state, this change should be balanced by the collisional drag from the ions, i.e.,

$$-\frac{e}{m} \vec{E} \cdot \frac{\partial f_{e0}}{\partial \vec{v}} = \left(\frac{\partial f_e}{\partial t}\right)_{\text{coll.}} = \left(\frac{\partial f_{ei}}{\partial t}\right)_{\text{coll.}} \quad (13.19)$$

Here, we expanded $f_e(\vec{v})$ around $f_{e0}(\vec{v}) = \text{Maxwellian}$, i.e.,

$$f_e(\vec{v}) = f_{e0M}(\vec{v}) + f_{es}(\vec{v}) \text{ and used the fact that } \left(\frac{\partial f_{e0M}}{\partial t}\right)_{\text{coll}} = 0.$$

* Taking $\vec{E} = E \hat{z}$, f_{e1} will be symmetric w.r.t. the azimuthal velocity angle about z -axis, (ϕ is an ignorable coordinate),

∴ Only the 1st term on the RHS of (13.14) contributes from

$\left(\frac{\partial f_{e1}}{\partial t}\right)_{\text{coll.}}$ in Lorentz-gas approximation:

$$\frac{e E v f_{e0}}{T_e} \cos \theta = \frac{n_i Z^2 e^4 \ln \Lambda}{8 \pi \epsilon_0^2 m^2 v^3} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial f_{e1}}{\partial \theta} \quad (13.20)$$

$$\Rightarrow f_{e1} = - \frac{4 \pi \epsilon_0^2 m^2 E v^4 f_{e0}}{n_i Z^2 e^3 T_e \ln \Lambda} \cos \theta \quad (13.21)$$

Distortion of f_e from Maxwellian due to E , constrained by collisions.

* Then, it's straight forward to calculate the current density in z -direction:

$$\begin{aligned} \dot{j}_z &= -e \int f_{e1}(v \cos \theta) d^3 v = (\text{const}) \int_0^\infty v^7 f_{e0} dv \int_0^\pi \cos^2 \theta \sin \theta d\theta, \\ &= \frac{32 \pi^{1/2} \epsilon_0^2 E}{m^{1/2} Z e^2 \ln \Lambda} (2 T_e)^{3/2} \quad (13.22) \end{aligned}$$

∴ From $\dot{j}_z = \eta E$ and Eq. (13.22), we obtain

$$\eta_{\text{Lorentz}} = \frac{m^{1/2} z e^2 \ln \Lambda}{32 \pi^{1/2} \epsilon_0^2 (2T_e)^{3/2}} \approx \frac{1}{3.4} \eta_{\text{fluid}} \text{ of Eq. (11.30).}$$

* The lower resistivity arises from the dominant contribution of high-velocity electrons in carrying current in the Lorentz-gas approx.

* If e-e collision is included in the analysis, it turns out

$$\eta_{\text{SP}} \approx 1.7 \eta_{\text{Lorentz}} \approx \frac{1}{2} \eta_{\text{fluid}}$$

[L. Spitzer and R. Harm P. Rev. '53]

Homework

Problem 13.1. on page 224

Problem 13.2. on page 227

Ch. 14. Collisions of Fast Ions in a Plasma

(*) Fast Ions in Fusion Plasmas.



↳ particles with ~ 3.5 MeV

born with an isotropic distribution in \vec{v}

2. Neutral Beam Injection for Plasma Heating.

$E_b \sim 100$ keV presently

1 MeV needed for reactors.

"highly directed Beam" (anisotropic in \vec{v})

3. Ion Cyclotron Heated Ions

mostly in \perp direction to \vec{B} .

14.2. Slowing-down of Beam Ions due to Collisions with electrons

⊗ Consider $v_{Ti} \ll \underline{V_b} \ll v_{Te}$.

of Beams with M_b and Z_b .

with Maxwellian background plasma electrons.

⇒ In the frame moving with the beam ions, the situation is similar to the case considered in Ch 11. i.e., electrons colliding with more massive (essentially stationary) ions.

In this case, the plasma electrons can transfer "momentum" to the beam ions, but not much energy.

The momentum gained by the beam will be exactly opposite to the velocity of beam ions. ⇒ "slow down".

⊗ Momentum Conservation $\Rightarrow -M_b \Delta \vec{V} = m_e \Delta \vec{v}$

⊗ Change in beam energy ; $\Delta W_b = \frac{M_b}{2} (|\vec{V} + \Delta \vec{V}|^2 - V^2) \approx M_b \vec{V} \cdot \Delta \vec{V}$

Energy Conservation $\Rightarrow \Delta W_e = \frac{1}{2} m_e |\Delta \vec{v}|^2 = \frac{m_e}{2} \left(\frac{M_b}{m_e} \right) |\Delta \vec{V}|^2 = \frac{M_b^2}{2m} |\Delta \vec{V}|^2$
 $= -\Delta W_b.$ (14.2)

$\therefore -M_b V \Delta V_{||} = \frac{M_b^2}{2m} |\Delta \vec{V}|^2 = \frac{M_b^2}{2m} [(\Delta V_{||})^2 + (\Delta V_{\perp})^2]$ (14.3)

$\Rightarrow \cancel{M_b V \Delta V_{||}} \frac{M_b}{2} (\Delta V_{\perp})^2 < m V |\Delta V_{||}| \ll M_b V |\Delta V_{||}|$

∴ Energy in beam ion \perp velocity components due to collisional deflection \ll Energy decrease due to slowing-down without change of direction

In addition, from (14.3) $|\Delta V_{||}| < \frac{2m}{M_b} V$, i.e.,

beam ion momentum loss $\sim O\left(\frac{m}{M_b}\right)$ · original momentum

" energy loss $\sim O\left(\frac{m}{M_b}\right)$ " energy.

$$\rightarrow \Delta V_{\perp} < \frac{2m}{M_b} V \quad \text{i.e., deflection through an angle } \sim \mathcal{O}\left(\frac{m}{M_b}\right)$$

$$\therefore \Delta W_{\perp} = \mathcal{O}\left(\left(\frac{m}{M_b}\right)^2\right) \cdot W.$$

In summary, the force of the background electrons on the beam ions is mostly in the nature of a "frictional drag."

⊛ Momentum Conservation:

$$n_b M_b \frac{d\langle \underline{\vec{V}} \rangle}{dt} = -m \int \frac{d\langle \underline{\vec{v}} \rangle}{dt} \underline{f}_e(\underline{v}) d^3v \quad (14.6)$$

mean velocity of beam ions. assume Maxwellian

⊛ From Ch. 11., we can deduce: $\frac{d\langle \underline{\vec{v}} \rangle}{dt} = -\nu_{eb} (\underline{\vec{v}} - \underline{\vec{V}})$ (14.7)
(in the Laboratory frame)

where

$$\nu_{eb} = \frac{n_b Z_b^2 e^4 \ln \Lambda}{4\pi \epsilon_0^2 m^2 |\underline{\vec{v}} - \underline{\vec{V}}|^3} \quad (14.8)$$

Using (14.6),
$$\frac{d}{dt} \langle \vec{V} \rangle = \frac{Z_b^2 e^4 \ln \Lambda}{4\pi \epsilon_0^2 m M_b} \int \frac{\vec{v} - \vec{V}}{|\vec{v} - \vec{V}|^3} f_e(v) d^3v \quad (14.9)$$

Note that
$$\frac{\vec{v} - \vec{V}}{|\vec{v} - \vec{V}|^3} = \frac{\partial}{\partial \vec{V}} \frac{1}{|\vec{v} - \vec{V}|}$$

(This identity ~~is~~ has appeared in the context of Coulomb force or gravity),

Then,
$$\frac{d}{dt} \langle \vec{V} \rangle = - \frac{Z_b^2 e^4 \ln \Lambda}{4\pi \epsilon_0^2 m M_b} \frac{\partial}{\partial \vec{V}} I(\vec{V}) \quad (14.10)$$

where

$$I(\vec{V}) = - \int \frac{f_e(v) d^3v}{|\vec{v} - \vec{V}|} = - 2\pi \int \frac{f_e(v) v^2 dv \sin \theta d\theta}{(v^2 + V^2 - 2vV \cos \theta)^{1/2}}$$

in spherical
coordinate

$$\dots = - \frac{2\pi}{V} \int_0^\infty dv v f_e(v) \{ -|v-V| + v+V \}$$

see Fig 14.1.

$$\therefore I = -4\pi \int_0^\infty v f_e(v) dv - \frac{4\pi}{V} \int_0^V v^2 f_e(v) dv$$

and

$$-\frac{\partial I}{\partial \vec{V}} = -\frac{4\pi \vec{V}}{V^3} \int_0^V v^2 f_e(v) dv.$$

For Maxwellian $f_e(v)$ and $V \ll v_{Te}$, we obtain

$$\frac{d}{dt} \langle \vec{V} \rangle = - \frac{2^{1/2} n_e Z_b^2 e^4 m^{1/2} \ln \Lambda}{12 \pi^{3/2} \epsilon_0^2 M_b T_e^{3/2}} \vec{V} \quad (14.12)$$

(slowing down time)⁻¹: independent of $|\vec{V}|$!

depends on "T_e".

Taking $M_b \vec{V}$ of (14.12), we obtain

$$\frac{d}{dt} W_b = - \frac{2^{1/2} n_e Z_b^2 e^4 m^{1/2} \ln \Lambda}{6 \pi^{3/2} \epsilon_0^2 M_b T_e^{3/2}} W_b. \quad (14.13)$$