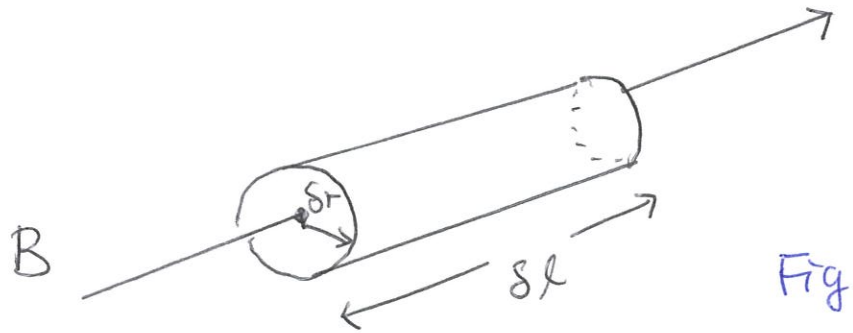
Fusion Plasma Theory I. 2019

Week 2

3.3. Conservation of μ : static $\vec{B}$ field



(r, θ, z)

Locally $\vec{B} = B\hat{z}$

Fig 3.4.

* Consider a differential cylindrical volume around a magnetic field line somewhere in the system where the $\vec{B}$ fields are converging (eg. Fig 3.3, see the next slide).

* $\vec{\nabla}\cdot\vec{B} = 0 \rightarrow$ apply Gauss's law to the differential volume.

$$\pi(\delta r)^2 \delta l \frac{dB}{dz} + 2\pi \delta r \delta l \langle B_r \rangle = 0 \qquad (3.17)$$

Flux through the ends, the sides

$$\therefore \langle B_r \rangle = -\frac{\delta r}{2}\frac{dB}{dz} \qquad (3.18).$$

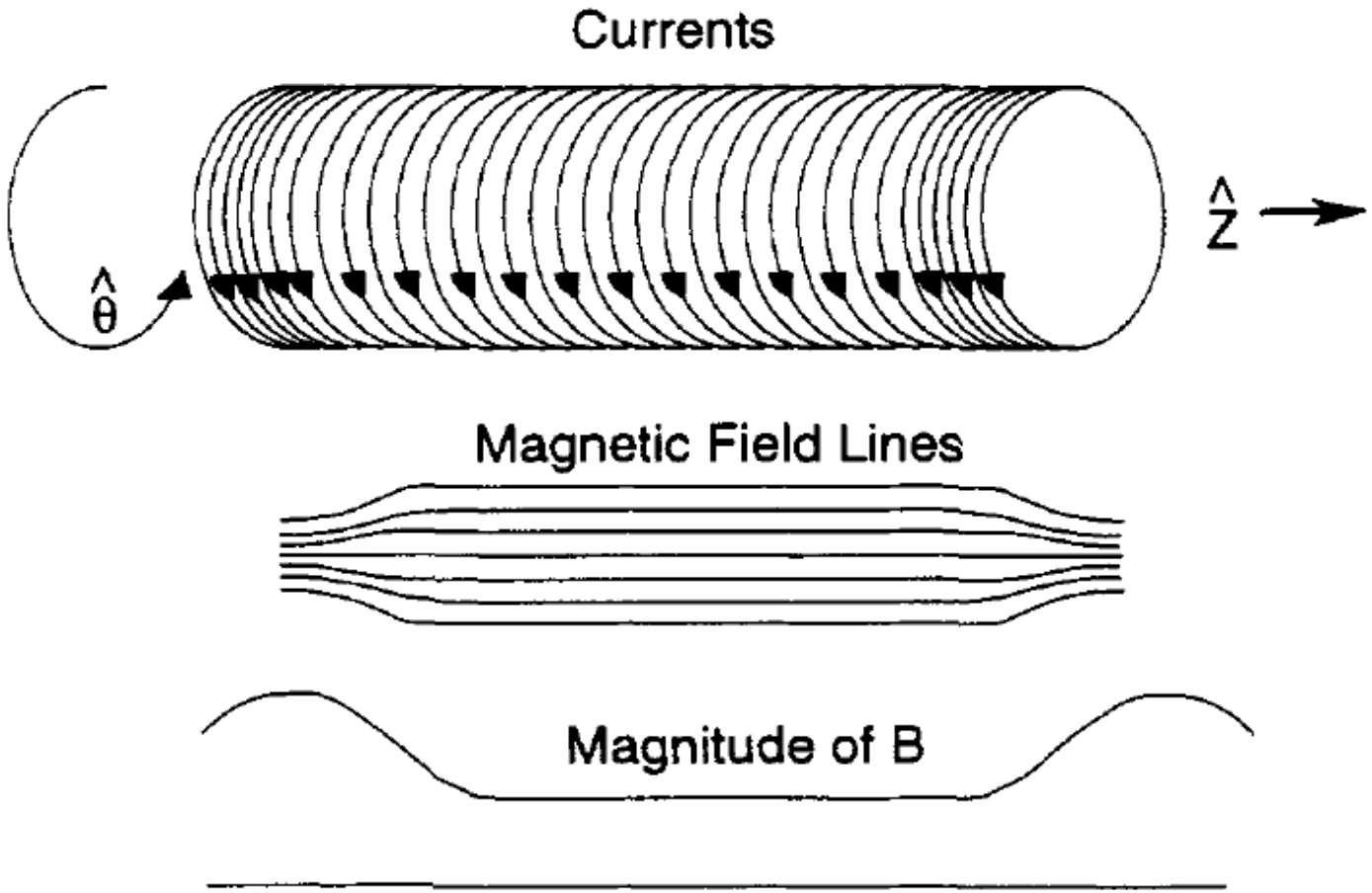


Figure 3.3. Currents in a solenoidal winding and the resulting 'mirror' magnetic fields inside the solenoid, shown schematically.

* Now, suppose that δr is chosen to be r_L of a particle whose guiding center lies on the axis of the cylinder.

* The Lorentz force on the particle is

$$\vec{F}_{mag} = q\,\vec{V}_\perp \times \langle B_r \rangle \hat{r} \rightarrow \text{along } \hat{b} \cong \hat{z}.$$

$$\therefore \langle F_\parallel \rangle = -\frac{|q|\,V_\perp^2}{2\cdot\omega_c}\frac{dB}{dz} = -\frac{W_\perp}{B}\frac{dB}{dz} = -\mu\frac{dB}{dz} \qquad (3.19)$$

The force in the direction opposite to the field gradient for both electrons and ions.

"Mirror Force."

* General Expression :

$$\boxed{m\frac{dV_\parallel}{dt} = -\mu\frac{dB}{ds}} \qquad (3.20)$$

s : distance along the field.

* Multiplying Eq. (3.20) with $v_{\parallel} = \frac{ds}{dt}$, we obtain

$$\frac{d}{dt}\left(\frac{m v_{\parallel}^2}{2}\right) = -\mu \frac{dB}{ds}\frac{ds}{dt} = -\mu \frac{dB}{dt} \qquad (3.21).$$

* In the presence of static (time-independent!) $\vec{B}$ field, the kinetic energy of a single particle should be conserved.

$$\therefore \frac{d}{dt}\left(\frac{m v_{\parallel}^2}{2} + \frac{m v_{\perp}^2}{2}\right) = \frac{d}{dt}\left(\frac{m v_{\parallel}^2}{2} + \mu B\right) = 0. \qquad (3.22).$$

* From Eqs. (3.21) and (3.22), we ~~conclude~~ conclude that

$$\boxed{\frac{d\mu}{dt} = 0} \qquad (3.24).$$

Invariance of "μ": The velocity component $v_{\perp}$ increases as the particle moves along $\vec{B}$ field into a region of higher B so as to keep "$W_{\perp}/B$" constant. $\rightarrow$ $v_{\parallel}$ decreases.

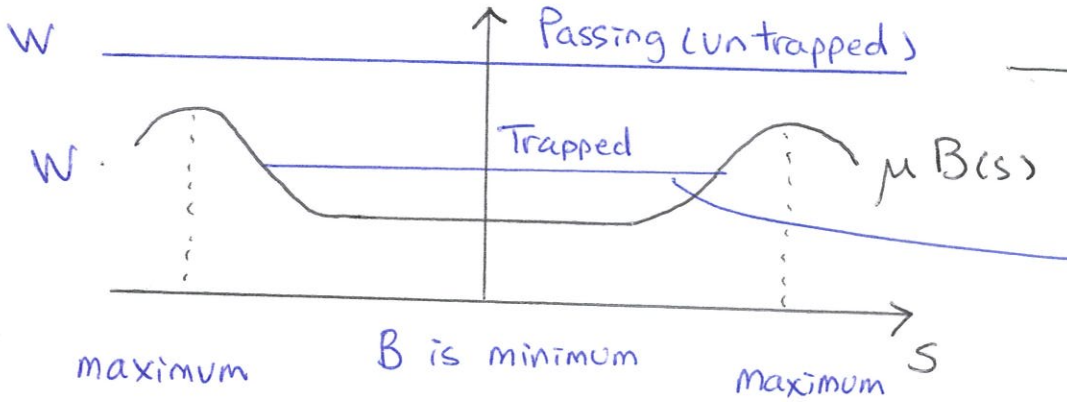
3.4. Magnetic Mirrors

* $$\frac{m v_{\parallel}^2}{2} = W - \underline{\mu B} \qquad (3.25)$$

origin : $\perp$ kinetic energy

acts as an effective potential energy as a ptl moves along $\vec{B}$,

because μ = constant and B is a function of position (varies along $\hat{b}$).



$\mu < \frac{W}{B_{max}}$.

$\mu > \frac{W}{B_{max}}$.

* For marginally trapped particles,

$$W_{\perp}\,(\text{midplane}) = \mu B_{min} = W B_{min}/B_{max},$$

$$\therefore\ W_{\parallel}\,(\ \text{"}\ )/W = (1 - B_{min}/B_{max}).$$

* $\rightarrow$ $v_\perp(\text{midplane})/v = (B_{min}/B_{max})^{1/2}$

$v_\parallel$ (")$/v = (1 - B_{min}/B_{max})^{1/2}$ (3.26)

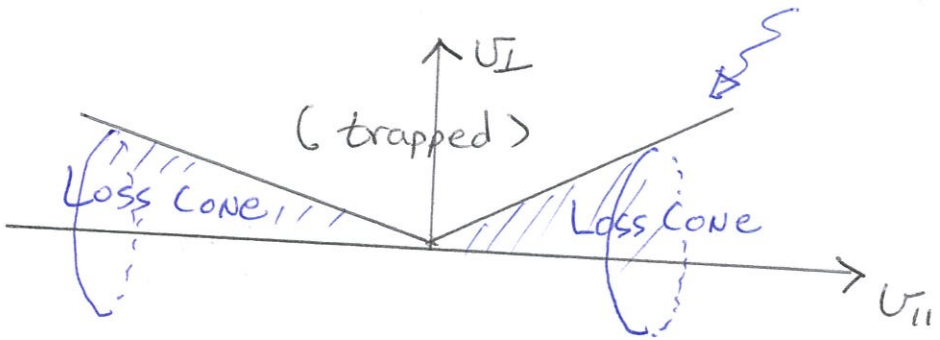


Fig. 3.5.

* Loss cone is independent of "q and m" of particles.

* But collisions can change the direction of ptls' velocity and scatter them into the loss cone.

* For $T_e \approx T_i$, electrons collide more frequently and will be lost preferentially.

* As a consequence, positive electric potential develops and confines the electrons with low-energy in the loss cone electrostatically.

* This electric field builds up to the point where it keeps the net outflux of electrons balanced with the slower outflux of ions.

  → Collisional scattering of ions sets the pace for particle loss.

* However, more energetic electrons will still escape over the "top" of the "electrostatic potential well".

* Therefore, electron thermal losses tend to dominate the energy balance of mirror-trapped plasmas.

Homework

Problem 3.3. on page 32, Problem 3.5. on page 41.

Ch.4. Particle Drifts in time-dependent Fields

4.1. Time-varying B field :

Consider a slowly time-varying B field with $\frac{\partial}{\partial t} \sim \omega \ll \omega_c$.

(Similar to weakly varying B field in space with $\frac{\partial}{\partial x} \sim k \ll \frac{1}{r_L}$).

* Faraday's Law : $\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} \xrightarrow[\text{theorem}]{\text{Stokes'}} \oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t}\int \vec{B} \cdot d\vec{S}$

* If we consider charged particles gyrating along $d\vec{l}$, from the right-hand rule, a particle will be accelerated for $\frac{\partial}{\partial t}\vec{B} > 0$ regardless of the sign of its charge.

* Indeed, $\frac{d}{dt}\langle W_\perp \rangle = q\langle \vec{v} \cdot \vec{E} \rangle = |q| v_\perp \frac{\pi (r_L)^2}{2\pi r_L} \frac{\partial B}{\partial t} = \frac{W_\perp}{B}\frac{\partial B}{\partial t} = \mu \frac{\partial B}{\partial t}$ (4.3)

$\therefore$ $W_\perp$ grows steadily as B increases. (Note that, Flux $= \pi r_L^2 B = \frac{2\pi m}{q^2} \cdot \mu$)

* $$\boxed{\frac{d\mu}{dt} = \frac{1}{B}\frac{dW_\perp}{dt} - \frac{W_\perp}{B^2}\frac{\partial B}{\partial t} = 0}$$ (4.4).

4.2. Adiabatic Compression

* A changing magnetic field will heat (or cool) a plasma as a consequence of magnetic moment "μ".

* For a cylindrical plasma in a solenoidal B field, if $\frac{\partial B_z}{\partial t} > 0$, $\frac{dW_\perp}{dt} > 0$, and plasma will be driven in towards the center of the solenoid (i.e., compressed away from the coils).

* Faraday's law: $\rightarrow$ $$2\pi r E_\theta = -\pi r^2 \frac{\partial B_z}{\partial t}. \qquad (4.9)$$

$$\therefore \frac{dr}{dt} = \vec{v}_E \cdot \hat{r} = \frac{E_\theta}{B_z} = -\frac{r}{2B_z}\frac{\partial B_z}{\partial t}. \qquad (4.10)$$

Frozen-in-flux!

* During this "adiabatic" compression ("$\omega \ll \omega_c$"), the amount of magnetic flux enclosed by the annulus is conserved.

$$\frac{d}{dt}(\pi r^2 B_z) = 2\pi r B_z \frac{dr}{dt} + \pi r^2 \frac{\partial B_z}{\partial t} = 0. \qquad (4.11)$$

4.3. Polarization Drift.

* Consider a uniform and strong $\vec{B}$ field and slowly time-varying $\vec{E}$ field $\perp \vec{B}$ with $\frac{\partial}{\partial t} \sim \omega \ll \omega_c$.

* $$m \frac{d\vec{v}}{dt} = q(\vec{E} + \vec{v} \times \vec{B}) \quad (1)$$

We learned that the lowest order guiding-center drift is given by $\vec{E} \times \vec{B}$ drift, i.e.,

$$\vec{v}^{(0)} = \vec{v}_E = \frac{\vec{E} \times \vec{B}}{B^2} \quad (2)$$

This makes $\vec{E}_\perp + \vec{v}^{(0)} \times \vec{B} = 0$.

* Writing Eq.(1) at each order, after formally expanding (with a small parameter, $\omega/\omega_c \ll 1$) $\vec{v} = \vec{v}^{(0)} + \vec{v}^{(1)} + \cdots$,

0-th order : $$0 = q(\vec{E}_\perp + \vec{v}^{(0)} \times \vec{B}) = 0. \quad (3)$$

1-st order : $$m \frac{d}{dt} \vec{v}^{(0)} = q(0 + \vec{v}^{(1)} \times \vec{B}), \quad (4)$$

* Inverting Eq.(4) by applying $\vec{B} \times (\cdots)$, we obtain the Polarization Drift.

$$\boxed{\vec{v}_\perp^{(1)} = \frac{m}{qB^2} \vec{B} \times \frac{d}{dt} \vec{v}^{(0)} = \frac{m}{qB^2} \frac{d}{dt} \vec{E}_\perp} \quad (4.15)$$

4.1. Review of Single Particle Motion in a Strong Magnetic Field

4.1.1. Adiabatic Invariant

When magnetic field varies in space smoothly (i.e. $\rho_i \ll L_B \equiv |\nabla \ln B|^{-1}$), we can identify approximate constants of motion.
"Adiabatic" in here means slow variation in time and space. This is well illustrated from the point of view of Quantum Mechanics (QM).
Let's consider a Simple Harmonic Oscillator (SHO): Schrödinger Equation is

$$H\psi(x) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega_0^2 x^2 \right)\psi(x) = E\psi(x)$$

Here the eigenvalues are

$$E = \hbar\omega_0\left(N + \frac{1}{2}\right)$$

where N is the quantum number ($N = 0, 1, 2, \ldots$).

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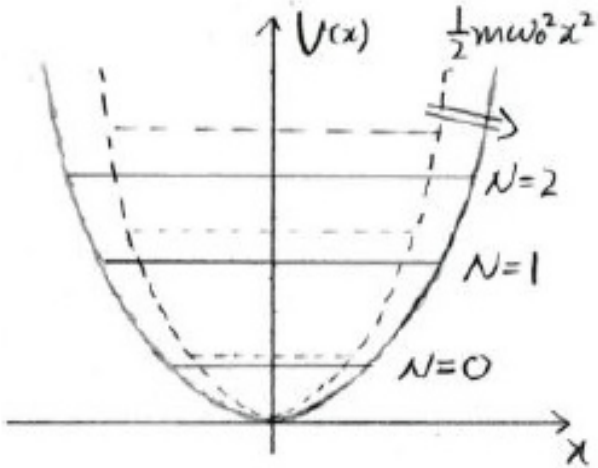


Suppose that potential well characterized by $\omega_0(t)$ is changed very slowly in the time with a scale τ ($\tau \gg 1/\omega_0$).
In this adiabatic process, what remains constant is "N" (eigenstates are preserved).
While the energy (eigenvalue) changes in time. "N" is an example of adiabatic invariant.

In classical limit, $N = E/\omega_0$.

4.1.2. Adiabatic Invariant in Classical Limit

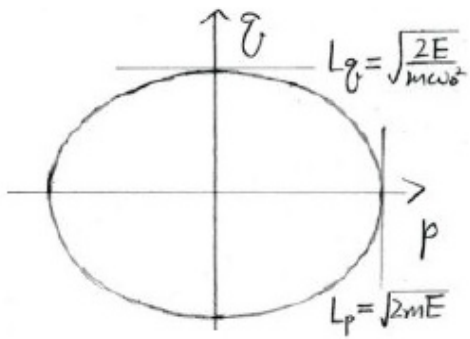
In Classical Mechanics (CM), the Hamiltonian of SHO is given by

$$H(p,q) = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 q^2 = E$$

Adiabatic invariant in CM is related to the conservation of the volume in the phase space for appropriate action-angle variables.

$$I = \oint dq\ p$$

This is also called "Action Invariant".



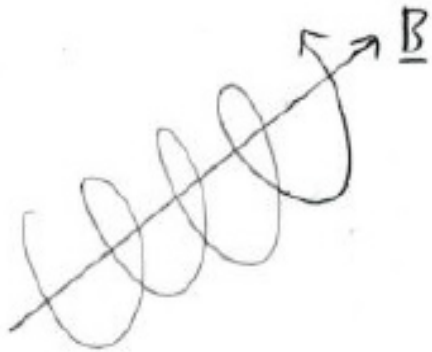
For SHO the action invariant is

$$I = \pi L_q L_p = \pi\sqrt{2mE\frac{2E}{m\omega_0^2}} = 2\pi\frac{E}{\omega_0}$$

Thus except for a numerical factor 2π, we recover $N = E/\omega_0$ from SHO in QM. (The useful formula $N = E/\omega_0$ represents the "Duality of Wave and Particles".) This illustration of geometric meaning of action invariant can be extended to quasi-periodic motion (recall $\tau \gg 1/\omega_0$).

4.1.3. Gyromotion in Slowly Varying Magnetic Field

Consider the gyrating motion of charged particles in slowly varying magnetic field in time $(1/\omega)$ and space (L_B).



For this gyration, the corresponding action invariant is the magnetic moment (the 1st adiabatic invariant).

- Energy corresponding to gyration: $E_\perp = mv_\perp^2/2$
- Frequency corresponding to gyration: $\Omega_c = eB/mc$

$$\Rightarrow \mu \propto \frac{1}{2} m v_\perp^2 / \left(\frac{eB}{mc}\right) \propto \left(\frac{v_\perp^2}{2B}\right) \quad \text{This is not an exact constant}$$

What is the error or precision of the statement?
What is the expansion parameter or smallness parameter describing this motion?
If $\epsilon_\omega \equiv \omega/\Omega_c \ll 1$ and $\epsilon_B \equiv \rho_i/L_B \ll 1$,
the adiabatic invariant is good up to any order!

$$\text{Error} = \mathcal{O}\Big(\exp\left(-\frac{\text{const}}{\epsilon_B}\right),\ \exp\left(-\frac{\text{const}}{\epsilon_\omega}\right)\Big)$$