

Fusion Plasma Theory I. 2019

Week 4

6.4. Two-fluid Equations

II-10.

- * There are always at least two species in any neutral plasma.
- * In momentum balance equation for each species, transfer of momentum between different species due to collisions should be taken into account.
- * Rate at which momentum per unit volume is gained by species " α " due to collisions with species β is given by

$$\vec{R}_{\alpha\beta} = -m_\alpha n_\alpha \gamma_{\alpha\beta} (\vec{u}_\alpha - \vec{u}_\beta) \quad (6.21)$$

where $\gamma_{\alpha\beta}$: collision frequency : the rate at which the momentum of species α is transferred to species β .

- * Now, the momentum balance equation should be :

$$m_\alpha n_\alpha \left(\frac{\partial}{\partial t} \vec{u}_\alpha + (\vec{u}_\alpha \cdot \vec{\nabla}) \vec{u}_\alpha \right) = n_\alpha q_\alpha (\vec{E} + \vec{u}_\alpha \times \vec{B}) - \vec{\nabla} \cdot \vec{P}_\alpha - \sum_{\beta} \vec{R}_{\alpha\beta} \quad (6.22)$$

* From momentum conservation,

$$\vec{R}_{\beta\alpha} = -\vec{R}_{\alpha\beta},$$

This implies

$$\rightarrow m_\alpha n_\alpha v_{\alpha\beta} = m_\beta n_\beta v_{\beta\alpha}.$$

6.5. Plasma Resistivity

- * Collisions between electrons and ions in a plasma will impede the acceleration of electrons in response to an electric field applied along (or in the absence of) a magnetic field.
- * Neglecting small electron inertia (m_e) and assuming homogeneous electron pressure and velocity,

$$"0 = -n_e e E_{||} + R_{e||} " \quad \text{where } R_{e||} = -m_e n_e V_{ei} (u_{e||} - u_{i||}) \quad (6.24)$$

$$* \text{ Current density: } \underline{j_{||}} = -n_e e (u_{e||} - u_{i||}), \quad (6.26)$$

- * From Eqs (6.24) and (6.25), we obtain

$$E_{||} = -\frac{m_e \nu_{e_i}}{e} (u_{e||} - u_{i||}) = \frac{m_e \nu_{e_i}}{n_e e^2} j_{||}^* = \eta j_{||} \quad (6.26)$$

Taking into account of velocity distribution of electrons,

$$\nu_{e_i} \rightarrow \langle \nu_{e_i} \rangle \text{ and}$$

$$\eta = \frac{m_e \langle \nu_{e_i} \rangle}{n_e e^2} \quad (6.27)$$

Then,

$$\vec{R}_{ei} = -m_e n_e \langle \nu_{e_i} \rangle (\vec{u}_e - \vec{u}_i) = \eta n_e e \vec{j} \quad (6.28)$$

- * Note that Eq (6.28) can also be applied to the direction $\perp \vec{B}$:

$$\eta_{\perp} \approx 2 \eta_{||}$$

because electron distribution can be distorted significantly from Maxwellian due to $E_{||}$ in $||$ direction to \vec{B} .

- * Resistivity of fusion plasma is extremely low, lower than that of copper!

Ch.7. Relation between fluid equations and g.c. drifts.

7.1 Diamagnetic Drift.

* Ignoring inertia and collisions, the momentum balance relation is

$$\boxed{nq(\vec{E} + \vec{u} \times \vec{B}) \approx \vec{\nabla} \cdot \vec{P}} \quad (7.2)$$

By taking $\vec{B} \times$ of Eq. (7.2) and expanding the triple product, we obtain

$$\vec{u}_\perp = \frac{\vec{E} \times \vec{B}}{B^2} + \frac{\vec{B} \times (\vec{\nabla} \cdot \vec{P})}{nq B^2}$$

ExB drift - Diamagnetic Drift

(7.5)

* We did NOT find a 'diamagnetic' guiding-center drift.

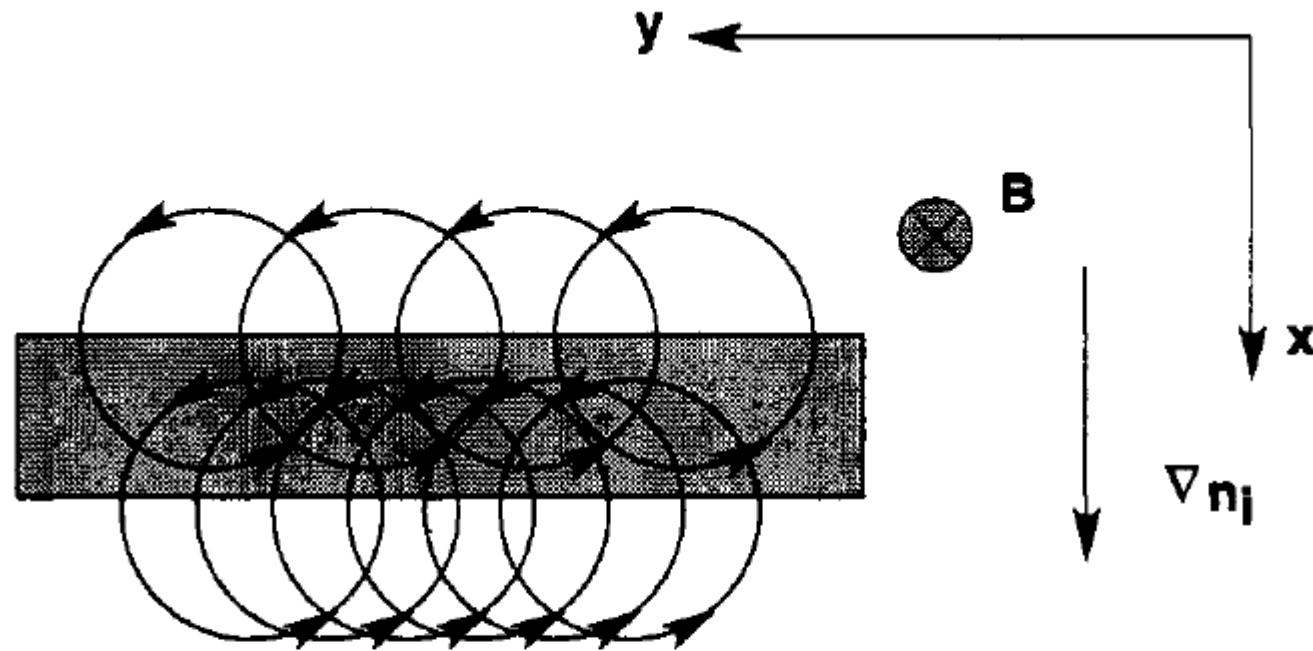
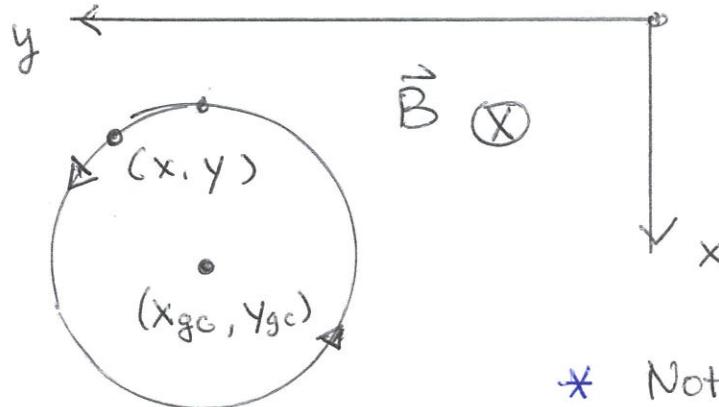


Figure 7.1. Larmor orbits of ions in the presence of a density gradient. In the shaded region there is a net current to the left, even though the guiding centers have no net motion.



* Note that particle position and guiding center position are related by

$$x = x_{gc} - \frac{v_y}{\omega_c} \quad (7.7)$$

* The mean y-directed (fluid) drift, u_y , at x due to particles in dx is

$$\boxed{n u_y dx = \int v_y f(x, \vec{v}) d^3 v dx} \quad (7.6)$$

* We can also define a distribution function of guiding centers $f_{gc}(x_{gc}, \vec{v})$, where \vec{v} remains a particle velocity.

$$\begin{aligned}
 * \quad f(x, \vec{v}) dx &= f_{gc}(x_{gc}, \vec{v}) dx_{gc} \\
 &= f_{gc}\left(x + \frac{v_y}{\omega_c}, \vec{v}\right) \underbrace{\left(\frac{dx_{gc}}{dx}\right) dx}_{=1 \text{ for uniform } \vec{B}}, \\
 &= f_{gc}\left(x + \frac{v_y}{\omega_c}, \vec{v}\right).
 \end{aligned} \tag{7.8}$$

$$\begin{aligned}
 \therefore u_y &= \frac{1}{n} \int v_y f_{gc}\left(x + \frac{v_y}{\omega_c}, \vec{v}\right) d^3v \\
 &\quad \text{Taylor expand around } x, \\
 &\approx \frac{1}{n} \int v_y \left(f_{gc}(x, \vec{v}) + \frac{v_y}{\omega_c} \frac{\partial f_{gc}(x, \vec{v})}{\partial x} \right) d^3v
 \end{aligned} \tag{7.10}$$

$$\rightarrow \boxed{u_y = \frac{1}{nqB} \frac{dP}{dx}} \tag{7.11}$$

7.4. Diamagnetic Drift in Non-uniform B Fields

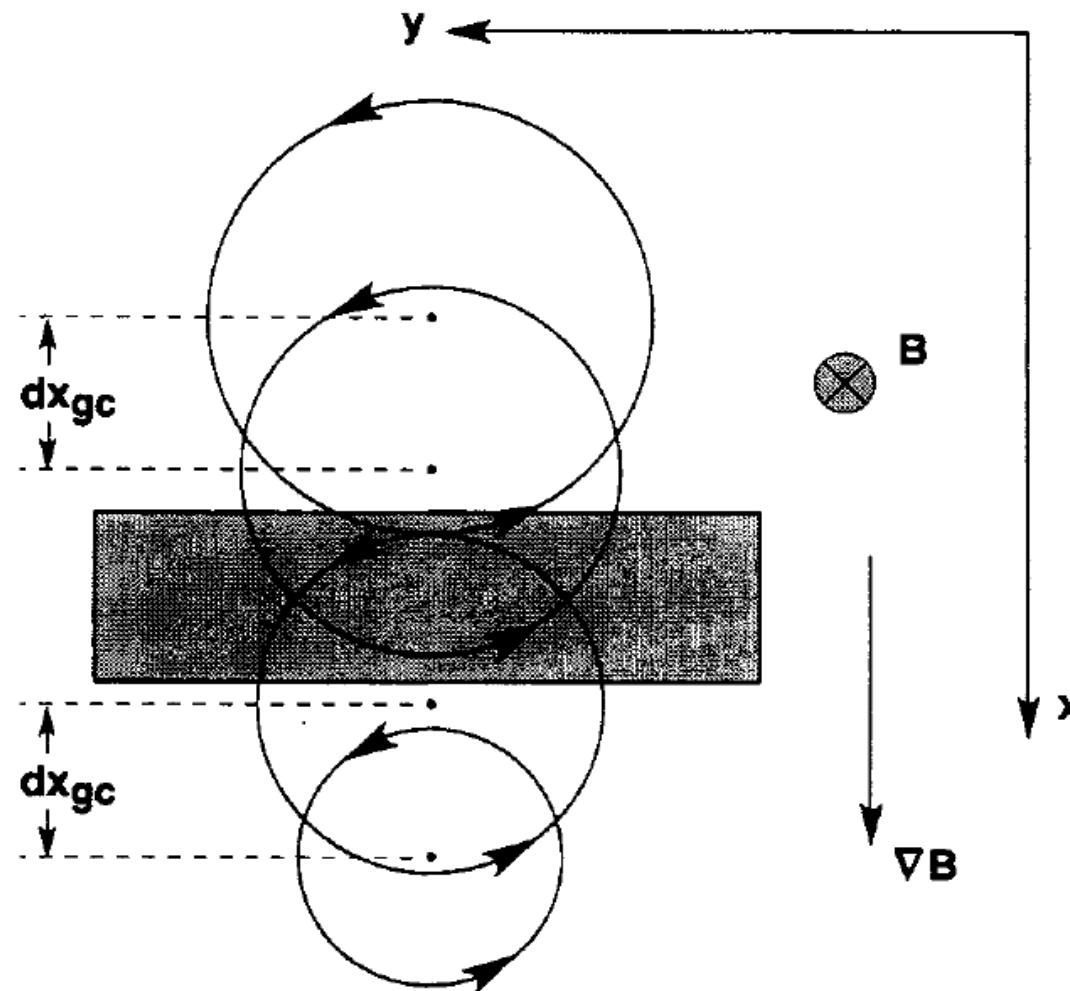


Figure 7.3. Larmor orbits of ions in the presence of a field gradient. For guiding centers with equal spacing dx_{gc} , there are more particles with $v_y < 0$ falling in the shaded region, and fewer particles with $v_y > 0$, leading to a net current to the right.

* Since $x = x_{gc} - \frac{v_y}{\omega_c(x_{gc})} dx_{gc}$, (7. ~~23~~)

$$dx = \left(1 + \frac{v_y}{\omega_c} \frac{1}{B} \frac{dB}{dx}\right) dx_{gc} \quad (7.24)$$

$$\begin{aligned} \therefore f(x, \vec{v}) &= f_{gc}(x + \frac{v_y}{\omega_c}, \vec{v}) \frac{dx_{gc}}{dx} \\ &= f_{gc}(x + \frac{v_y}{\omega_c}, \vec{v}) (1 + \frac{v_y}{\omega_c} \frac{1}{B} \frac{dB}{dx})^{-1} \\ &\approx (f_{gc} + \frac{v_y}{\omega_c} \frac{df_{gc}}{dx}) (1 - \frac{v_y}{\omega_c} \frac{1}{B} \frac{dB}{dx}) \\ &\approx f_{gc} + \frac{v_y}{\omega_c} \frac{df_{gc}}{dx} - \frac{v_y}{\omega_c} \frac{1}{B} \frac{dB}{dx} f_{gc} \end{aligned} \quad (7.25)$$

* Then, $u_y = \frac{1}{n} \int v_y f(x, \vec{v}) d^3v$

$$u_y = \frac{1}{nqB} \frac{dP}{dx} - \frac{I}{qB^2} \frac{dB}{dx}$$

(7.27)

for Maxwellian.

Average velocity in the frame in which guiding centers have no motion

- * On the other hand, the average DB drift for a Maxwellian distribution function ($\langle v_{\perp}^2 \rangle_2 = T_m$) is

$$v_{Dy} = \frac{\langle v_{\perp}^2 \rangle_2}{w_c B} \frac{dB}{dx} = \frac{I}{qB^2} \frac{dB}{dx} \quad (7.28)$$

- * So the total fluid velocity in the laboratory frame can be obtained by adding Eq(7.27) to (7.28),,

$$u_y = \frac{1}{nqB} \frac{dP}{dx}$$

(7.29).

- * A similar discussion regarding the curvature drift is given on pg. 108 - 109. Make sure you understand it.



Reading Assignment .

7.6. Parallel Pressure Balance

II-20.

- * Since electrons can respond to a perturbation along the B field (say a 'hump' in n) more quickly than ions do,

$$\delta \approx e n_e \nabla_{||} \phi - \nabla_{||} P_e , \quad (7.43)$$

which leads to

$$n_e \propto \exp(-e\phi/T_e) \quad (7.44)$$

for uniform T_e .

"Boltzmann relation"

- * Then, electric field pulls the ions in the same direction as their own pressure gradient, i.e.,

$$m_i n_i \ddot{u}_{i||} = -n_i e \nabla_{||} \phi - \nabla_{||} P_i \approx -(T_e + T_i) \nabla_{||} n \quad (7.46)$$

- * Boltzmann relation is much more efficient than using Poisson's equation to find a relation between n_e and ϕ in many cases.