

# Fusion Plasma Theory I. 2019

Week 5

# Single-fluid Magnetohydrodynamics

## 8.1. MHD equations:

\* Consider a quasi-neutral ( $n_e \approx Z n_i$ ) two species plasma in the presence of a strong  $\vec{B}$  field. Let's consider  $Z=1$  for simplicity

\* MHD model treats the plasma as a ~~two~~ "single fluid" with

$$- \rho = n_i M_i + n_e m_e \approx n (M + m) \approx n M \quad \text{mass density} \quad (8.1)$$

$$- \sigma = (n_i - n_e) e \quad \text{charge " " } \quad (8.2)$$

$$- \vec{u} = (n_i M \vec{u}_i + n_e m \vec{u}_e) / \rho \approx \vec{u}_i + \left(\frac{m}{M}\right) \vec{u}_e \quad \text{mass velocity} \quad (8.3)$$

$$- \vec{j} = e (n_i \vec{u}_i - n_e \vec{u}_e) \approx n \cdot e (\vec{u}_i - \vec{u}_e) \quad \text{current density} \quad (8.4)$$

Eqs. (8.3) and (8.4) can be written as

$$\vec{u}_i \approx \vec{u} + \frac{m}{M} \frac{\vec{j}}{ne}, \quad \vec{u}_e \approx \vec{u} - \frac{\vec{j}}{ne} \quad (8.5)$$

- \* From  $\frac{\partial n_i}{\partial t} + \vec{\nabla} \cdot (n_i \vec{u}_i) = 0$ , Continuity eqn for ions,  
 and  $\frac{\partial n_e}{\partial t} + \vec{\nabla} \cdot (n_e \vec{u}_e) = 0$ . " electrons

- \* By multiplying with  $M$  and  $m$  respectively, and adding

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0} \quad \text{Mass Continuity Equation (8.7)}$$

- \* By subtracting one from another,

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0} \quad \text{Charge Continuity Equation (8.8)}$$

- \* From

$$M n_i \frac{d\vec{u}_i}{dt} = e n_i (\vec{E} + \vec{u}_i \times \vec{B}) - \vec{\nabla} p_i + \vec{R}_{ie} \quad \text{ion momentum balance}$$

$$m n_e \frac{d\vec{u}_e}{dt} = -e n_e (\vec{E} + \vec{u}_e \times \vec{B}) - \vec{\nabla} p_e + \vec{R}_{ei} \quad \text{electron " (8.9)}$$

\* By adding two momentum balance equations, we obtain

$$\rho \frac{d}{dt} \vec{u} = \rho \left( \frac{\partial}{\partial t} \vec{u} + (\vec{u} \cdot \vec{\nabla}) \vec{u} \right) = \sigma \vec{E} + \vec{j} \times \vec{B} - \vec{\nabla} p \quad (8.10)$$

cf. - Here, the pressure is defined w.r.t. the mass velocity  $\vec{u}$ ,  
unlike  $P_i$  and  $P_e$  w.r.t.  $\vec{u}_i$  and  $\vec{u}_e$  respectively.

- Typically,  $\vec{u} \approx \vec{u}_i + \left(\frac{m}{M}\right) \vec{u}_e \approx \vec{u}_i$

\* By ignoring electron inertia, and using  $\vec{R}_{ei} = m n \langle v_{ei} \rangle (\vec{u}_e - \vec{u}_i) = \eta n e \vec{j}$ ,  
the electron momentum balance equation can be written as

$$\vec{E} + \vec{u}_e \times \vec{B} = \eta \vec{j} - \frac{\vec{\nabla} P_e}{ne} \quad (8.12)$$

or

$$\vec{E} + \vec{u} \times \vec{B} = \eta \vec{j} + \frac{\vec{j} \times \vec{B} - \vec{\nabla} P_e}{ne} \quad (8.12)$$

The generalized Ohm's law.

\* The equation of state :

- often adiabatic law  $\boxed{\frac{d}{dt} \left( \frac{P}{\rho^\gamma} \right) = 0}$  is used (8.14).

- sometimes isothermal law with  $T_e, T_i = \text{constant}$  is used

\* The system of equations is closed by including the four Maxwell equations,

### 8.2. The quasi-neutrality Approximation

From  $\rho \frac{d\vec{u}}{dt} = \rho \left( \frac{\partial}{\partial t} \vec{u} + \underbrace{(\vec{u} \cdot \vec{\nabla}) \vec{u}}_{\textcircled{1}} \right) = \underbrace{\sigma \vec{E}}_{\textcircled{2}} + \vec{j} \times \vec{B} - \vec{\nabla} P,$

$$\textcircled{2}/\textcircled{1} \sim \frac{\sigma E}{\rho (\vec{u} \cdot \vec{\nabla}) \vec{u}} \sim \frac{\epsilon_0 E^2 / L}{\rho u^2 / L} \sim \frac{\epsilon_0 E^2}{\rho u^2} \sim \frac{\epsilon_0 B^2}{\rho} \ll 1 \quad (8.19)$$

using Poisson's eqn  $\vec{\nabla} \cdot (\epsilon_0 \vec{E}) = \sigma$ , and  $u \sim E/B$ ,

now  $\epsilon_0 B^2 / \rho$  is a very small quantity ( $\sim 10^{-2} - 10^{-3}$ )  $\therefore \textcircled{2}$  is negligible.  
for almost all plasmas of interest.

\* Comparing the two terms appearing in the charge continuity equation,

$$\frac{\partial \sigma / \partial t}{\vec{\nabla} \cdot \vec{j}} \sim \frac{\epsilon_0 E / L \tau}{j / L} \sim \frac{\epsilon_0 \omega B / \tau}{\rho \omega / B \tau} \sim \frac{\epsilon_0 B^2}{\rho} \ll 1 \quad (8.20)$$

where  $\vec{j} \times \vec{B} \sim \rho \frac{\partial}{\partial t} \vec{u}$  has been used to estimate  $j$ .

∞

$\sigma \vec{E}$  in momentum equation and  $\frac{\partial \sigma}{\partial t}$  in charge continuity equation can be neglected. → "Quasi-neutrality approximation."

\* But quasi-neutrality equation does NOT mean that

$\sigma$  can be neglected in the Poisson equation  $\vec{\nabla} \cdot (\epsilon_0 \vec{E}) = \sigma$  !

\* " $\sigma$ " does not appear elsewhere in MHD equations,

and Poisson equation can be dropped from the system of MHD equations.

### 8.3 The 'Small Larmor Radius' Approximation

\* Compare the last terms of the generalized Ohm's law, to  $\vec{u} \times \vec{B}$  term

$$\frac{\vec{\nabla} p_e / ne}{\vec{u} \times \vec{B}} \sim \frac{I}{euBL} \sim \frac{Mv_{Ti}^2}{euBL} \sim \frac{Mv_{Ti}}{eBL} \sim \frac{v_{Ti}}{\omega_c L} \sim \frac{r_{Li}}{L} \quad (8.22)$$

where  $u \sim v_{Ti}$  has been used.

Also,  $\frac{\vec{j} \times \vec{B} / ne}{\vec{u} \times \vec{B}} \sim \dots \sim \frac{r_{Li}}{L}$ , since  $\vec{\nabla} p \sim \vec{j} \times \vec{B}$ .

\* For  $\frac{r_{Li}}{L} \ll 1$ , the Ohm's can be simplified.

$$\underline{* \vec{E} + \vec{u} \times \vec{B} = \eta \vec{j}} \quad (8.23)$$

the effective electric field seen by a fluid element moving with velocity  $\vec{u}$  across  $\vec{B}$

(for  $|\vec{u}| \ll c$ ).

"MHD model" consists of the set of following equations.

$$\bullet \quad \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0$$

$$\bullet \quad \vec{\nabla} \cdot \vec{j} = 0$$

(8.24)

$$\bullet \quad \rho \frac{d}{dt} \vec{u} = -\vec{\nabla} \rho + \vec{j} \times \vec{B}$$

$$\bullet \quad \vec{E} + \vec{u} \times \vec{B} = \eta \vec{j}$$

(8.23)

together with the required versions of the Maxwell eqns.

$$\bullet \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$$

$$\bullet \quad \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{B}$$

(8.25)

$$\bullet \quad \vec{\nabla} \cdot \vec{B} = 0$$

## 8.4. The Approximation of "Infinite Conductivity"

$\eta$  is very small in magnitude in a high-temperature plasma

→ Some dynamical phenomena at large scale can often be described with an approximation of  $\eta = 0$

or  
infinite conductivity

→ Ideal MHD

The relation

$$\vec{E} + \vec{u} \times \vec{B} = 0 \quad \text{leads to strong constraints on}$$

the evolution of  $\vec{u}$  and  $\vec{B}$

"Plasma is tied to the magnetic field lines."

"Any two elements of an ideal plasma that lie initially on a given field line will still lie on the same field line after an arbitrary motion of the plasma."

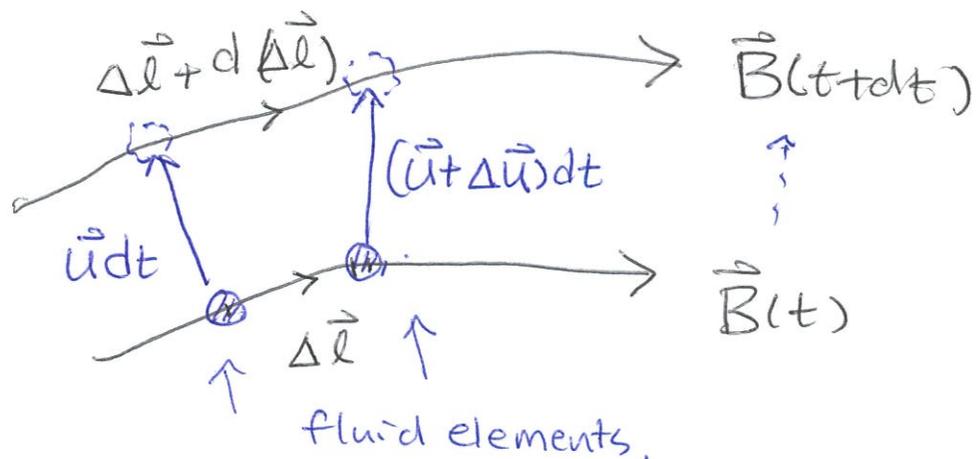


Fig 8.1,

\* Let's consider two plasma fluid elements separated by  $\Delta \vec{l}$  on the same magnetic field line  $\vec{B}(t)$ .

\* After "dt", the two fluid elts move  $\vec{u} dt$  and  $(\vec{u} + \Delta \vec{u}) dt$  respectively, while  $\vec{B}(t)$  moves to  $\vec{B}(t+dt)$ .

\* Let's calculate "d( $\Delta \vec{l}$ )". From Fig 8.1, the vectors should add up.

$$\Delta \vec{l} + d(\Delta \vec{l}) = \Delta \vec{l} + (\vec{u} + \Delta \vec{u}) dt - \vec{u} dt = \Delta \vec{l} + \Delta \vec{u} dt \quad (8.28)$$

on top

$$\rho_0 \quad \frac{d \Delta \vec{l}}{dt} = \Delta \vec{u} = (\Delta \vec{l} \cdot \vec{\nabla}) \vec{u} \quad \text{from Taylor expansion} \quad (8.27)$$

\* On the other hand, starting from the Faraday's law,

$$\frac{\partial}{\partial t} \vec{B} = -\vec{\nabla} \times \vec{E} \stackrel{\text{Ohm's law}}{=} \vec{\nabla} \times (\vec{u} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla}) \vec{u} - (\vec{u} \cdot \vec{\nabla}) \vec{B} - \vec{B} (\vec{\nabla} \cdot \vec{u})$$

(8.30)

$$\therefore \vec{\nabla} \cdot \vec{B} = 0$$

\* The total derivative of  $\vec{B}$ , following the motion of the plasma is

$$\frac{d}{dt} \vec{B} = \left( \frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla} \right) \vec{B} = (\vec{B} \cdot \vec{\nabla}) \vec{u} - \vec{B} (\vec{\nabla} \cdot \vec{u}). \quad (8.31)$$

\* Let's evaluate

$$\frac{d}{dt} (\Delta \vec{l} \times \vec{B}) = \frac{d}{dt} \Delta \vec{l} \times \vec{B} + \Delta \vec{l} \times \frac{d}{dt} \vec{B} = \underbrace{\left[ (\Delta \vec{l} \cdot \vec{\nabla}) \vec{u} \right]}_{\text{wavy}} \times \vec{B} + \underbrace{\Delta \vec{l} \times \left[ (\vec{B} \cdot \vec{\nabla}) \vec{u} - \vec{B} (\vec{\nabla} \cdot \vec{u}) \right]}_{\text{wavy}} \quad (8.32)$$

= 0 because  $\Delta \vec{l} \times \vec{B} = 0$  initially

- Two terms underlined by wavy cancel, if  $\Delta \vec{l} \parallel \vec{B}$  (the same expression if  $\Delta \vec{l}$  and  $\vec{B}$  are interchanged).

∴  $\frac{d}{dt} (\Delta \vec{l} \times \vec{B}) = 0$  and

$\Delta \vec{l}$  moves so as to remain parallel to  $\vec{B}$ .

## 8.5. Conservation of Magnetic Flux

II.- 21

⊗ "The magnetic flux through any closed contour that moves with the plasma is constant."

$$\Phi = \int \vec{B} \cdot d\vec{S}$$

Magnetic Flux, closed contour      differential element of area.

$$\frac{d}{dt} \Phi = \int \vec{\nabla} \times (\vec{u} \times \vec{B}) \cdot d\vec{S} + \int \vec{B} \cdot \frac{d}{dt} (\Delta \vec{S})$$

due to  $\left(\frac{\partial \vec{B}}{\partial t}\right)$       due to movement of area.

(8.36)

$$\frac{d}{dt}(\Delta \vec{S}) = \vec{u} \times \Delta \vec{\ell} \quad \leftarrow \quad (8.37)$$

$$d \Delta \vec{S} = \vec{u} dt \times \Delta \vec{\ell}$$

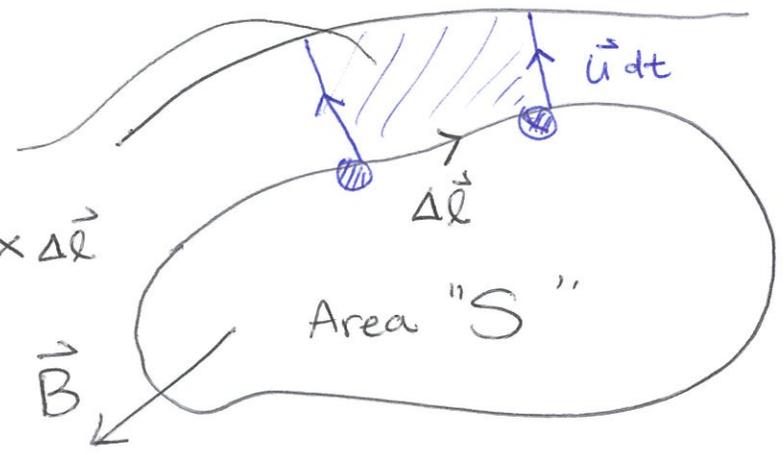


Figure 8.2.

From Eq. (8.36)

$$\begin{aligned} \frac{d}{dt} \Phi &= \int \vec{\nabla} \times (\vec{u} \times \vec{B}) \cdot d\vec{S} + \int \vec{B} \cdot \frac{d}{dt} \Delta \vec{S} \\ &\quad \downarrow \text{Stokes' Theorem} \qquad \qquad \qquad \downarrow \text{Eq. (8.37)} \\ &= \int (\vec{u} \times \vec{B}) \cdot \Delta \vec{\ell} + \int \vec{B} \cdot (\vec{u} \times \Delta \vec{\ell}) = 0 ! \\ &\qquad \qquad \text{from vector identity.} \end{aligned}$$

" The magnetic flux through an area bounded by any closed contour 'painted' on the plasma is unchanged in any motion of the plasma. "

## 8.6. Conservation of Energy

II-23.

⊗ The total energy of a system described by ideal MHD;

$$W = \int \left( \frac{\rho}{2} |\vec{u}|^2 + \frac{P}{\gamma-1} + \frac{\epsilon_0}{2} |\vec{E}|^2 + \frac{1}{2\mu_0} |\vec{B}|^2 \right) d^3x \quad (8.40)$$

Kinetic energy density      thermal energy density      electric field energy density      magnetic field energy density

~ directed motion      ~ random motion

$$\text{" } \frac{dW}{dt} = 0 \text{ "}$$

Recall:  $\rho \left( \frac{\partial}{\partial t} \vec{u} + \vec{u} \cdot \vec{\nabla} \vec{u} \right) = \sigma \vec{E} + \vec{j} \times \vec{B} - \vec{\nabla} P \quad (8.10)$

"Single-fluid equation of motion"

Contribution from LHS of Eq. (8.10); from taking  $\vec{u} \cdot (8.10)$  and integrating over all space,

$$\begin{aligned}
 \otimes \int \rho \vec{u} \cdot \left( \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) d^3x &= \frac{1}{2} \int \left( \rho \frac{\partial}{\partial t} |\vec{u}|^2 + \rho \vec{u} \cdot \nabla |\vec{u}|^2 \right) d^3x \\
 &= \frac{1}{2} \int \left( \rho \frac{\partial}{\partial t} |\vec{u}|^2 - |\vec{u}|^2 \nabla \cdot (\rho \vec{u}) \right) d^3x = \frac{1}{2} \int \left( \rho \frac{\partial}{\partial t} |\vec{u}|^2 + |\vec{u}|^2 \frac{\partial \rho}{\partial t} \right) d^3x \\
 &\quad \xrightarrow{\text{from integration by part}} \qquad \qquad \qquad \xrightarrow{\text{Continuity equation}} \\
 &= \frac{\partial}{\partial t} \int \frac{\rho}{2} |\vec{u}|^2 d^3x
 \end{aligned}$$

Contribution from the 3rd term on RHS of Eq. (8.10);

$$\otimes \int \vec{u} \cdot \nabla p d^3x = - \int \rho \nabla \cdot \vec{u} d^3x = \frac{1}{\gamma-1} \int \rho \frac{d}{dt} \left( \frac{P}{\rho} \right) d^3x$$

IBP from Eq. (8.42) below:

$$0 = \frac{d}{dt} \left( \frac{P}{\rho^\gamma} \right) = \frac{1}{\rho^{\gamma-1}} \frac{d}{dt} \left( \frac{P}{\rho} \right) - \frac{(\gamma-1)P}{\rho^{\gamma+1}} \frac{d\rho}{dt} = \frac{1}{\rho^{\gamma-1}} \frac{d}{dt} \left( \frac{P}{\rho} \right) + \frac{(\gamma-1)P}{\rho^\gamma} \nabla \cdot \vec{u} \tag{8.42}$$

adiabatic eqn of state

# 8.7. Magnetic Reynolds Number

- ⊛ Ideal MHD assumes infinite electrical conductivity (zero resistivity).  
 "What is the consequence of non-zero resistivity?"

$$\textcircled{*} \quad \frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E} = \underbrace{\vec{\nabla} \times (\vec{u} \times \vec{B})}_{\text{Ohm's law}} - \underbrace{\vec{\nabla} \times (\eta \vec{j})}_{\text{Ampère's law}} = \vec{\nabla} \times (\vec{u} \times \vec{B}) + \frac{\eta}{\mu_0} \nabla^2 \vec{B} \quad (8.44)$$

$$= \underbrace{-(\vec{u} \cdot \vec{\nabla}) \vec{B}}_{\text{Convection of } \vec{B} \text{ with plasma}} - \underbrace{\vec{B} (\vec{\nabla} \cdot \vec{u})}_{\text{Plasma Compression}} + \underbrace{(\vec{B} \cdot \vec{\nabla}) \vec{u}}_{\text{" " } \perp \text{ to } \vec{B} \text{ "}} + \frac{\eta}{\mu_0} \nabla^2 \vec{B} \quad (8.45)$$

"Resistive Diffusion" of ~~the~~  $\vec{B}$  across the plasma.

- ⊛ For motion with characteristic scale length "L" and " plasma speed "u",

$$\frac{\text{"Convection"}}{\text{"diffusion"}} \sim \frac{\mu_0 u L}{\eta} \equiv R_M. \quad \begin{array}{l} \text{- Magnetic Reynolds \#} \\ \text{- Lundquist \#} \end{array} > 10^8! \quad \text{for fusion-relevant plasmas.}$$

# Homework

II-25 a)

Problem 8.2: on page 127

## Ch. 9. MHD Equilibrium

9.1. Assuming  $\eta = 0$ ,  $\vec{u} = 0$ , isotropic pressure, a steady-state solution of MHD equations satisfies;

$$\boxed{\vec{\nabla} p = \vec{j} \times \vec{B}}, \quad \vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{j} \quad (9.1)$$

$$\rightarrow \underline{\vec{j}_{\perp} = \frac{\vec{B} \times \vec{\nabla} p}{B^2}} \quad \text{; diamagnetic current} \quad (9.2)$$

From quasi-neutrality,  $\vec{\nabla} \cdot \vec{j} = 0$ ; and with  $\vec{\nabla} \cdot \vec{B} = 0$ ,

$$\Rightarrow \underline{(\vec{B} \cdot \vec{\nabla}) \left( \frac{j_{\parallel}}{B} \right) + \vec{\nabla} \cdot \vec{j}_{\perp} = 0} \quad (9.5)$$

## 9.2. Magnetic Pressure

II-27

$$\textcircled{*} \quad \vec{\nabla} P = \vec{j} \times \vec{B} = \frac{1}{\mu_0} (\vec{\nabla} \times \vec{B}) \times \vec{B} = \frac{1}{\mu_0} \left\{ (\vec{B} \cdot \vec{\nabla}) \vec{B} - \vec{\nabla} \left( \frac{B^2}{2} \right) \right\} \quad (9.6)$$

Ampère's law

$$\Rightarrow \quad \vec{\nabla} \left( P + \frac{B^2}{2\mu_0} \right) = \frac{1}{\mu_0} (\vec{B} \cdot \vec{\nabla}) \vec{B} \quad (9.7)$$

"Pressure-balance condition"

where

$$(\vec{B} \cdot \vec{\nabla}) \vec{B} = B^2 (\hat{b} \cdot \vec{\nabla}) \hat{b} + \hat{b} (\hat{b} \cdot \vec{\nabla}) \frac{B^2}{2}$$

bending force

( $\perp \vec{B}$ )

parallel compression  
of field lines.

In some cases, the field lines are approximately straight and parallel,  $\Rightarrow$  RHS of (9.7)  $\approx 0$ .

$$\Rightarrow \quad \text{" } P + \frac{B^2}{2\mu_0} = \text{constant} \text{ "}$$

$\downarrow$  plasma pressure                       $\downarrow$  magnetic-field pressure.

$$\beta = 2\mu_0 \frac{P}{B^2} \quad (9.9)$$

" a measure of the degree to which the magnetic field is holding a ~~non~~ non-uniform plasma in equilibrium."

⊛ To achieve magnetic fusion efficiently, we need plasmas with higher  $\beta$  value.

\* Conventional tokamak plasmas :  $\beta < 10\%$ ,

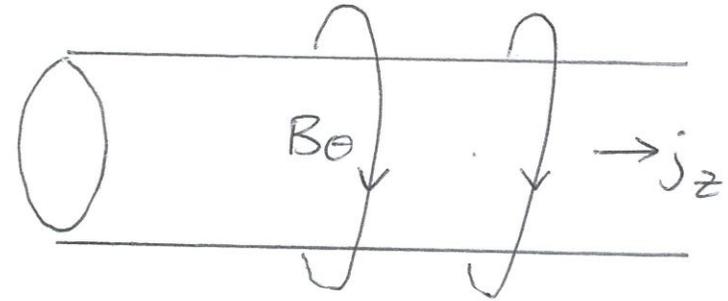
\* Relatively innovative configurations  
such as spherical torus :  $\beta < 1$ ,

\* Astrophysical plasmas :  $\beta \geq 1$

## 9.3. Cylindrical Pinch

II-29.

⊛ Cylindrical plasma with  $B_\theta$  only and  $\hat{j}_z$  only.



Since  $\frac{\partial}{\partial \theta} \hat{\theta} = -\hat{r}$ , Eq. (9.7) becomes

Fig 9.2.

$$\boxed{\frac{\partial}{\partial r} \left( p + \frac{B_\theta^2}{2\mu_0} \right) = -\frac{B_\theta^2}{\mu_0 r}} \quad (9.10)$$

From integration from 0 to  $r$ , we obtain

$$p(r) = p(0) - \frac{B_\theta^2(r)}{2\mu_0} - \frac{1}{\mu_0} \int_0^r \frac{B_\theta^2}{r} dr \quad (9.11)$$

There exist infinitely many possible equilibria of this kind.

For illustration, let's consider 
$$\begin{cases} \hat{j}_z(r) = \hat{j}_{z0} & , r \leq a \\ \hat{j}_z(r) = 0 & , r > a \end{cases} \quad (9.12)$$

For that current profile, we obtain  $I = \pi a^2 j_{z0}$  and,

$$B_\theta(r) = \frac{B_{\theta a} r}{a}, \text{ and } p(r) = p_0 - \frac{B_{\theta a}^2 r^2}{\mu_0 a^2} \quad (9.13)$$

Since  $p(a) = 0$ , i.e. Parabolic pressure profile, we obtain " $p(0) = \frac{B_{\theta a}^2}{\mu_0} = \frac{\mu_0 I^2}{4\pi^2 a^2}$ " (9.14).

### Pinch Condition

- In this configuration,  $j_z$  is entirely from the plasma diamagnetic current, and this provides the entire magnetic field.
- Unfortunately, this configuration is strongly unstable against MHD instability.

## 9.4. Force-free Equilibria: 'Cylindrical' Tokamak

II-31.

⊗ Consider a low- $\beta$  cylindrical plasma,

$$\Rightarrow \boxed{0 = \vec{j} \times \vec{B}} \quad (9.15)$$

Force-free Equilibria

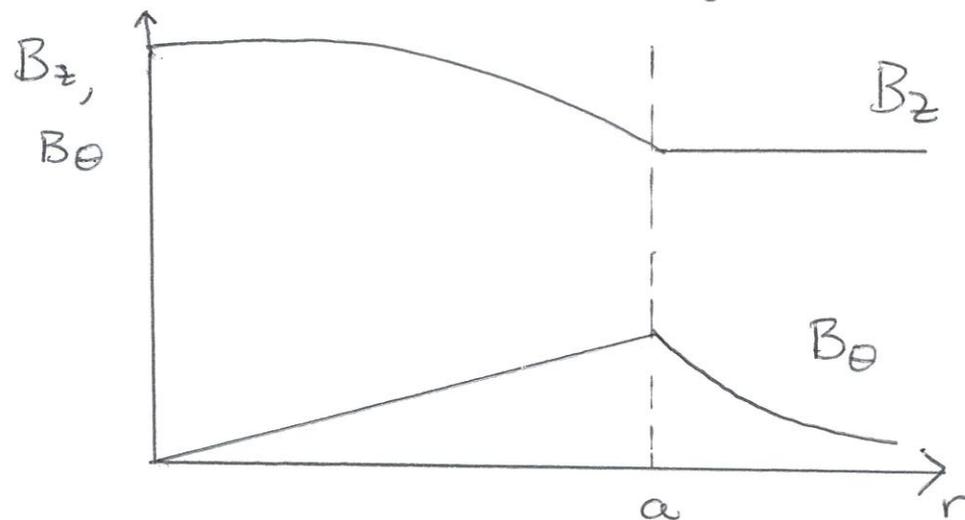
$$\rightarrow \frac{\partial}{\partial r} \left( \frac{B_\theta^2}{2} + \frac{B_z^2}{2} \right) = - \frac{B_\theta^2}{r} \quad (9.16)$$

Then consider the current density  $j_z(r)$  which is uniformly distributed within the plasma so that  $B_\theta(r) = B_{\theta a} \left( \frac{r}{a} \right)$ .

Integrating Eq. (9.16)  $\Rightarrow$

$$B_z(r)^2 = B_z(0)^2 - B_\theta(r)^2 - 2 \int_0^r \frac{B_\theta^2}{r} dr = B_z(0)^2 - 2B_\theta(r)^2 \quad (9.17)$$

Eq (9.17) is illustrated in Fig. 9.4.



← "Paramagnetic"  
plasma

$$B_z(0) > B_z(a)$$

But, when a moderate amount of plasma pressure  $P > B_0^2 / 2\mu_0$  is added, the plasma becomes "diamagnetic".

i.e.,  $B_z(0) < B_z(a)$ .

## Home work

Problem 9.2 on page 135.