

# Fusion Plasma Theory I. 2019

## Week 6

# UNIT 3 : Collisional Processes in Plasmas

III-1.

## Ch 11. Collisions in fully ionized Plasmas.

### 11.1 Coulomb Collisions

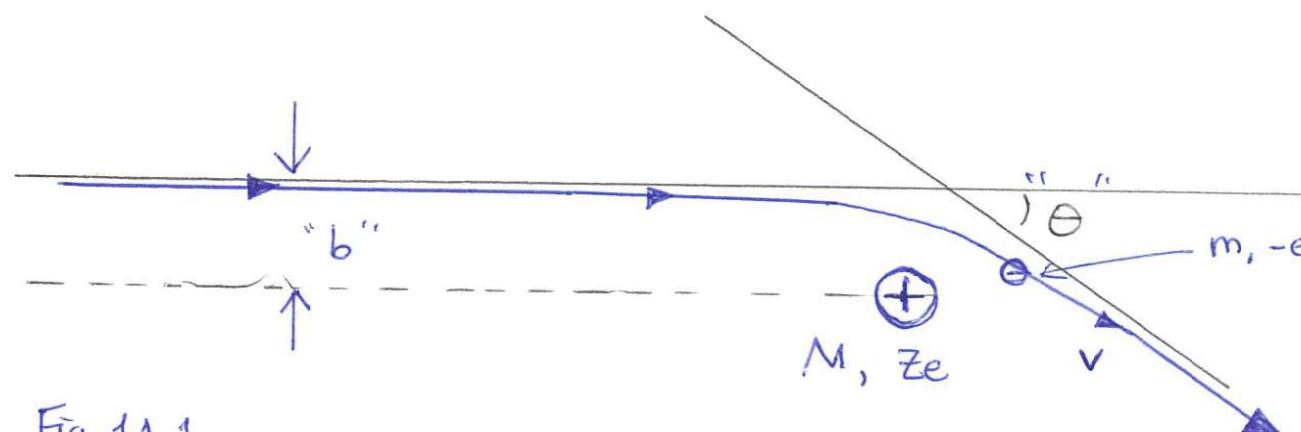


Fig 11.1.

- ④ Consider an electron with velocity  $v$  approaching a fixed ion of charge " $Ze$ ." The angle of deflection " $\theta$ " is related to "the impact parameter"  $b$ , and is due to Coulomb interaction with an inverse-square-law force.

$$\tan \frac{\theta}{2} = \frac{Ze^2}{4\pi\epsilon_0 m v^2 b}$$

(11.2)

"Rutherford Scattering"

- ⊗ For scattering through  $90^\circ$  (i.e.,  $\theta/2 = 45^\circ$ ,  $\tan \frac{\theta}{2} = 1$ ), the impact parameter "b" is given by

$$b_0 = \frac{Ze^2}{4\pi\epsilon_0 m v^2} \quad (11.3)$$

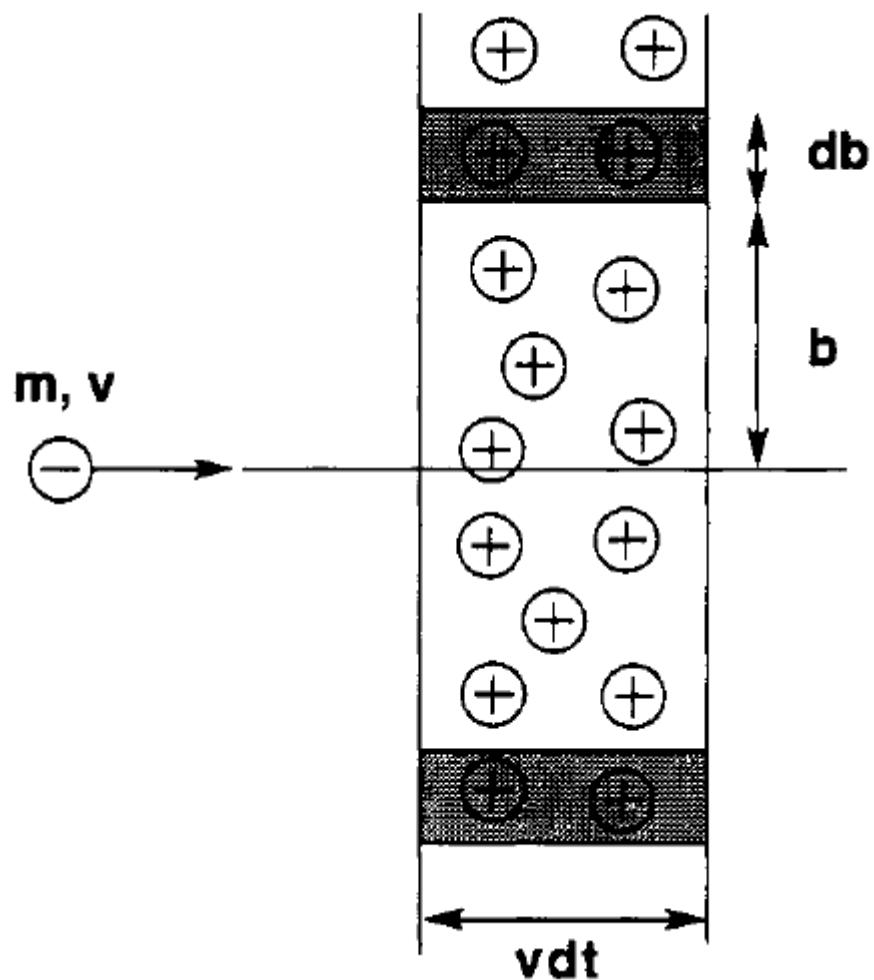
$$\rightarrow \tan \left( \frac{\theta}{2} \right) = b_0/b,$$

- ⊗ "Cross section" of the ion for  $90^\circ$ -scattering:

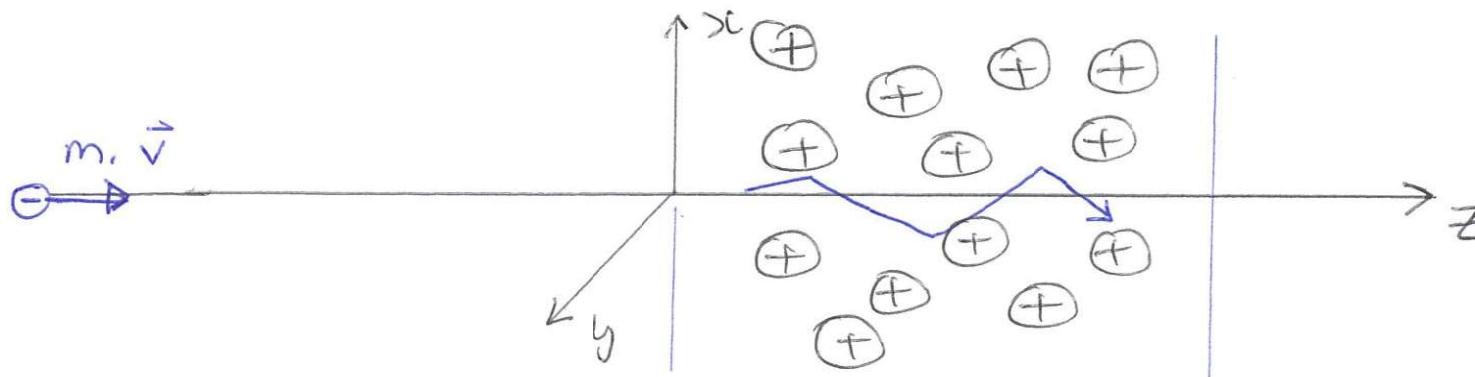
$$\sigma_{i,90^\circ} = \pi b_0^2 = \frac{\pi Z^2 e^4}{(4\pi\epsilon_0)^2 m^2 v^4} \quad (11.4)$$

- ⊗ But for most situations in plasmas, the cumulative effect of many small-angle deflections turns out to be larger than the effect of relatively fewer large-angle deflections.

⇒ "Effective cross section for Coulomb scattering is considerably larger than Eq.(11.4)."



**Figure 11.2.** Electron Coulomb scattering by ions in an annular element of volume with impact parameters between  $b$  and  $b + db$  as the electron moves a distance  $vdt$ .



⊗ Taking averages over many small-angle scattering events,

$\langle \Delta v_x \rangle = \langle \Delta v_y \rangle = 0$ , because there's no preferred direction for scattering.

However,

$$\langle \Delta v_x^2 \rangle = \langle \Delta v_y^2 \rangle = \frac{1}{2} \langle \Delta v_z^2 \rangle \neq 0.$$

⊗ From  $\tan(\frac{\theta}{2}) = b_0/b$ , we obtain ( $\because \sin\theta = \frac{2\tan(\theta/2)}{1+\tan^2(\theta/2)}$ )

$$\boxed{\sin\theta = \frac{2b/b_0}{1+(b/b_0)^2}} \quad (11.8)$$

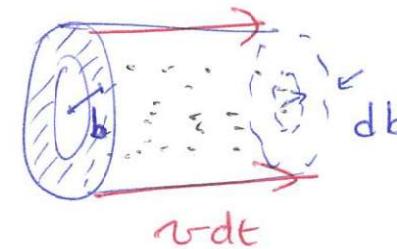
∴ For a single scattering event, we have

$$\boxed{\langle \Delta v_z^2 \rangle = v^2 \sin^2\theta = \frac{4v^2 (b/b_0)^2}{[1 + (b/b_0)^2]^2}} \quad (11.9)$$

(\*) According to Fig 11.2,

the number of ions in a scattering element between "b" and "b+db" is given by

$$\frac{n_i}{\text{ion number density}} \frac{2\pi b db}{\text{area}} \frac{v dt}{\text{distance}}$$



(\*) Integrating over impact parameters and differentiating w.r.t. time,

$$\frac{d}{dt} \langle (\Delta v_{\perp})^2 \rangle = 4\pi n_i v \int (\Delta v_{\perp})^2 b db = 8\pi n_i v^3 \int_0^{b_{\max}} \frac{(b/b_0)^2 b db}{[1 + (b/b_0)^2]^2}$$

The integral is well-defined at  $b=0$ , but diverges for

$b_{\max} \rightarrow \infty$ . We will avoid this problem by physics consideration.

$$\begin{aligned} \frac{d}{dt} \langle (\Delta v_{\perp})^2 \rangle &= \boxed{4\pi n_i v^3 b_0^2 \left[ \ln \left[ 1 + \left( \frac{b_{\max}}{b_0} \right)^2 \right] + \text{"smaller terms for"} \right]} \\ &= 8\pi n_i v^3 b_0^2 \cdot \ln \Lambda \quad \boxed{\frac{b_{\max}/b_0}{1} \gg 1} = \frac{n_i Z^2 e^4}{2\pi \epsilon_0^2 m^2 v} \ln \Lambda \end{aligned} \quad (11.11)$$

- ④ Since the electron energy is conserved in the collision for  $\frac{m_e}{M_i} \rightarrow 0$ ,

$$(v + \Delta v_{\parallel})^2 + \Delta v_{\perp}^2 = v_{\parallel}^2 \Rightarrow v - \Delta v_{\parallel} + \frac{1}{2} (\Delta v_{\perp})^2 = 0.$$

$$\therefore \frac{d}{dt} \Delta v_{\parallel} = -4\pi n_i v^2 b_0^2 \ln \Lambda = -\frac{n_i Z^2 e^4}{4\pi \epsilon_0^2 m^2 v^2} \ln \Lambda$$

$$\equiv -\nu_{ei} v$$

- ⑤ The collision frequency (for loss of electron momentum) :

$$\boxed{\nu_{ei} = 4\pi n_i v b_0^2 \ln \Lambda = \frac{n_i Z^2 e^4}{4\pi \epsilon_0^2 m^2} \frac{\ln \Lambda}{v^3}} \quad (11.16)$$

Here,

$$-\Lambda \equiv b_{\max.}/b_0 \gg 1 \quad \text{and}$$

$\ln \Lambda$  is called the Coulomb logarithm

- ⑥ In a quasi-neutral plasma, a charged ptl will interact weakly  
 with ptls further removed from it than the "Debye length  $\gamma_D$ ".  
 (much weaker than " $1/r^2$ -law" implies).

## 1.7 Debye Shielding

- ③ Consider a plasma in which electrons and ions are in thermal equilibrium among themselves with  $T_e$  and  $T_i$  respectively.
- Suppose  $T_e$  and  $T_i$  are uniform and  $n_e = Zn_i = n_\infty$  at infinity.

$$\begin{aligned} n_e(x) &= n_\infty \exp [e\phi(x)/T_e] \\ Zn_i(x) &= n_\infty \exp [-eZ\phi(x)/T_i] \end{aligned} \quad (1.31)$$

- ④ The Poisson equation in 1-d is ,

$$\epsilon_0 \frac{d^2}{dx^2} \phi(x) = e[n_e - Zn_i] = e n_\infty \left[ e^{\frac{e\phi(x)}{T_e}} - e^{-\frac{eZ\phi(x)}{T_i}} \right]$$

$$\approx e n_\infty \left[ \frac{e\phi(x)}{T_e} + \frac{eZ\phi(x)}{T_i} \right], \quad (1.32-33)$$

for  $\frac{Ze\phi}{T_e} \ll 1$ .

$$\Rightarrow \phi \propto \exp [-x/\lambda_D] \quad (1.35)$$

④ Debye Length .

$$\lambda_D = \left[ \frac{\epsilon_0 T_e}{n_e e^2 (1 + Z T_e / T_i)} \right]^{1/2} \quad (1.36)$$

$$(\lambda_{De} = (\epsilon_0 T_e / n_e e^2)^{1/2})$$

For  $T_e = 3 \text{ eV}$ ,  $n_e = 10^{19} \text{ m}^{-3}$ ,  $\lambda_{De} \approx 3 \times 10^{-6} \text{ m}$ ,

⑤ Characteristics of Plasmas :

① System size  $\gg \lambda_D$

② Number of pts in Debye sphere :  $n \left( \frac{4\pi}{3} \lambda_D^3 \right) \gg 1$ .

Both are easily satisfied for systems we consider.

⑥ In 3-d, the Debye shielding results in an exponential decrease  
too in the electric potential for  $r > \lambda_D$ .

" $\phi(r) \propto \exp(-r/\lambda_D)/r$ "

[ Problem 1.3. ]

- Back to

## Coulomb logarithm

$$\Lambda = b_{\max.}/b_0 \sim \lambda_D/b_0,$$

$$\text{where } b_0 \sim Ze^2/(2\pi\epsilon_0 T) \sim \left(\frac{Z}{12\pi}\right)(n\lambda_D^2)^{-1}$$

<sup>so</sup>  $\Lambda \sim \left(\frac{12\pi}{Z}\right)n\lambda_D^3$  is typically a very large number.

(Recall:  $n\lambda_D^3 \gg 1$  was one of definitions of plasma).

## Coulomb Cross Section

From

$$\sigma_{ei} = n_i \sigma_{ei} v \quad (11.18)$$

We obtain,

$$\boxed{\sigma_{ei} = \frac{Z^2 e^4 \ln \Lambda}{4\pi \epsilon_0^2 m_e^2 v^4}} \quad \left( = \frac{4 \ln \Lambda}{\pi} \sigma_{90^\circ} \text{ of Eq(11.4)} \right) \quad (11.19)$$

$\sim 70$  (a consequence of  $1/r^2$ -law).

**Table 11.1.** Values of  $\ln\Lambda$  for naturally occurring and laboratory plasmas.

	$n(\text{m}^{-3})$	$T(\text{eV})$	$\ln\Lambda$
Solar wind	$10^7$	10	26
Van Allen belts	$10^9$	$10^2$	26
Earth's ionosphere	$10^{11}$	$10^{-1}$	14
Solar corona	$10^{13}$	$10^2$	21
Gas discharge	$10^{16}$	$10^0$	12
Process plasma	$10^{18}$	$10^2$	15
Fusion experiment	$10^{19}$	$10^3$	17
Fusion reactor	$10^{20}$	$10^4$	18