

Fusion Plasma Theory I. 2019

Week 7

11.2. Electron and Ion Collision Frequencies

(*) So far, we discuss collision frequency of each ptl, i.e,

$\nu_{ei} \propto v^{-3}$. Now, we derive collision frequencies averaged over distribution of particles.

(*) Let's evaluate the frictional force on a distribution of electrons drifting through stationary ions,

$$\vec{F} = -n_{em} \underbrace{\langle \nu_{ei} \vec{v} \rangle}_{\text{avg. over distribution of electrons.}} \quad (11.20)$$

Consider drifting Maxwellian:

$$f_e(\vec{v}) = \frac{n_e}{(2\pi)^{3/2} v_{Te}^{3/2}} \exp \left[-\frac{(\vec{v} - \vec{u})^2}{2v_{Te}^2} \right] \approx \left(1 + \frac{u_z v_z}{v_{Te}^2} \right) f_{eo}(\vec{v}),$$

where $f_{eo}(\vec{v})$ is the "unshifted" Maxwellian. $u_z \ll v_{Te}$ is assumed, and $\vec{u} = u_z \hat{z}$ is the non-zero mean velocity.

$$\textcircled{+} \quad F_z = -m \int v_{ei} v_z f_{fe} d^3v = -mu_z \int \frac{v_z^2}{v_{Te}^2} v_{ei} f_{fe} d^3v \\ = -\frac{mu_z}{3} \int \frac{v^2}{v_{Te}^2} v_{ei} f_{fe} d^3v.$$

Here, we used the spherical symmetry of $f_{fe}(v^2)$.

The velocity space integral is (because $v_{ei} \propto 1/v^3$):

$$\int \frac{f_{fe}(v)}{v} d^3v = 4\pi \int_0^\infty f_{fe}(v) v^2 dv = \left(\frac{2}{\pi}\right)^{1/2} \frac{n_e}{v_{Te}} \quad (11.21)$$

" $4\pi v^2 dv$ "

\textcircled{+} The frictional force becomes: $F_z = -nem \langle v_{ei} \rangle u_z$

where

$$\langle v_{ei} \rangle = \frac{2^{1/2} n_i Z^2 e^4 \ln \Lambda}{12 \pi^{3/2} \epsilon_0^2 m^{1/2} T_e^{3/2}} \quad (11.22)$$

- Note that this is independent of ion mass $\therefore \frac{m_e}{M_i} \ll 1$ and ions are assumed to be stationary during scattering.

④

In a plasma containing many different ions,

" $n_i Z^2$ " in Eq (11.22) is replaced by " $n_e Z_{\text{eff}}$ ",

where

$$Z_{\text{eff}} = \sum_i n_i Z_i^2 / n_e$$

: measure of impureness
of a plasma.

⑤

For electron-electron or ion-ion collisions, we should consider the frictional force ~~on~~ on a population of electrons (ions) drifting through another population of " ^(*) " ~~the same species~~.

In this situation, the Coulomb (Rutherford) scattering with a fixed target particle can be repeated if we ~~calculation~~

consider a scattering event at the center-of-mass frame.

For a consideration of relative motion of two pts,

"Now, the incoming ptl suffering a collisional scattering

(^{effective})

has a reduced mass $\mu = \frac{M_1 M_2}{M_1 + M_2}$ "

* Reduced mass for electron-electron collision : $m_e/2$

" ion-ion " : $M_i/2$

(In fact "m" in Eq. (11.22) should have been $\mu = \frac{m_e M_i}{m_e + M_i} (\approx m_e)$, to be more precise)

* Noting that Frictional force $F_z \propto m^{1/2}$, and identifying the origins of various parametric dependences of Eq. (11.22), we can deduce

$$\boxed{\langle v_{ee} \rangle = \frac{n_e e^4 \ln \Lambda}{12\pi^{3/2} \epsilon_0^2 m^{1/2} T_e^{3/2}}} \quad (11.23)$$

and

$$\boxed{\langle v_{ii} \rangle = \frac{n_i e^4 \ln \Lambda}{12\pi^{3/2} \epsilon_0^2 M^{1/2} T_i^{3/2}}} \quad (11.24)$$

(*)

Momentum exchange for ions through electrons is generally not very important, since the momentum gained or lost by the ion in ~~a~~ such a collision is relatively small.

(*)

Comparing electron and ion collision frequencies in a plasma with $T_e \sim T_i$, we have

$$\nu_{ei} \sim \nu_{ee} \gg \nu_{ii} \gg \nu_{ie}$$

and

$$\frac{\nu_{ei}}{\nu_{ii}} \sim \left(\frac{M_i}{m_e}\right)^{1/2}, \quad \frac{\nu_{ii}}{\nu_{ie}} \sim \left(\frac{M_i}{m_e}\right)^{1/2}$$

(71.25)

(*)

$$\langle \nu_{ei} \rangle \sim \langle \nu_{ee} \rangle \sim 5 \times 10^{11} n / T_e^{3/2} \text{ (s}^{-1}\text{)} \quad (\text{temp in eV})$$

$$\langle \nu_{ii} \rangle \sim 10^{12} n / T_i^{3/2} \text{ (s}^{-1}\text{)}.$$

Note :

$$\sigma_{\text{Coulomb}} \propto T^{-2}$$

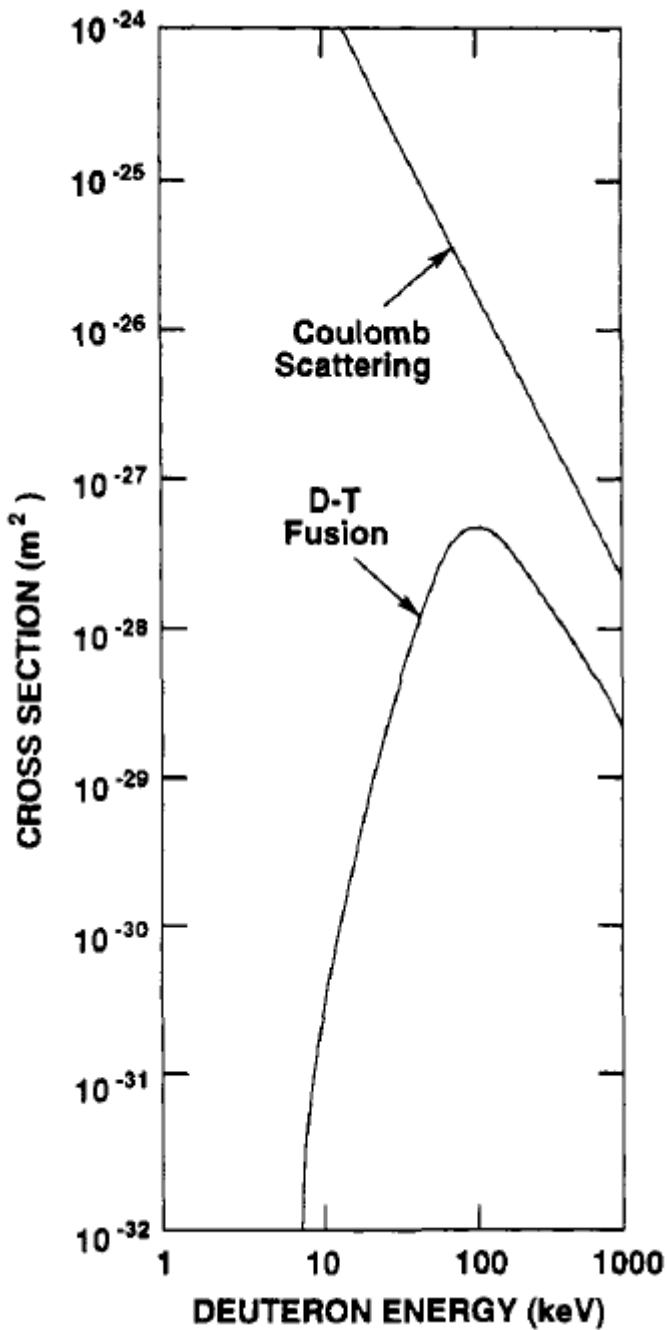


Figure 11.3. Coulomb and fusion cross sections compared for a deuterium ion (deuteron) in a deuterium-tritium plasma.

11.3. Plasma Resistivity

III-16

- Ohm's Law :

$$\vec{E} = \eta \vec{j} \quad (11.26)$$

- * Its origin is the electron momentum equation

$$m_e n_e \frac{d\vec{u}_e}{dt} = -e n_e \vec{E} + \vec{R}_{ei}, \quad (11.27)$$

where

$$\vec{R}_{ei} = -m_e n_e \langle v_{ei} \rangle (\vec{u}_e - \vec{u}_i) \quad (11.28)$$

- * Since $\vec{j} = -n_e e (\vec{u}_e - \vec{u}_i)$,

$$\boxed{\eta = \frac{m_e \langle v_{ei} \rangle}{n_e e^2} = \frac{2^{1/2} m_e^{1/2} Z e^2 \ln \Lambda}{12 \pi^{3/2} \epsilon_0^2 T_e^{3/2}}} \quad (11.30)$$

- This expression based on the shifted Maxwellian of f_e is an overestimation by a factor of "2".
- In fact, more accurate calculation taking into account of the distortions of f_e from shifted Maxwellian has been performed.

$$\textcircled{*} \quad n \propto T_e^{-3/2}$$

→ The "freezing" of plasma to \vec{B} field lines.

(a property of ideal MHD), works better
for high-temperature plasma.

→ Ohmic heating : $P = \vec{j} \cdot \vec{E} = \gamma j^2$

so efficiency drops sharply as $T_e \uparrow$ at fixed j .

not sufficient for fusion purpose.

→ Auxiliary heating methods are necessary.

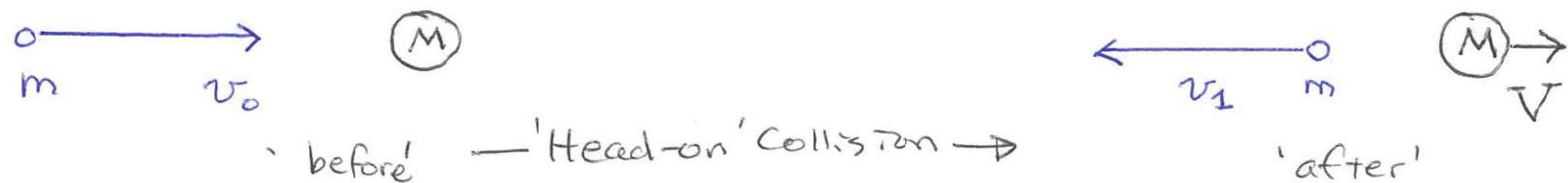
Neutral Beam Heating, Ion Cyclotron Resonance Heating,

Electron "

etc.,

11.4. Energy Transfer

- (*) Consider energy transfer between electrons with T_e and ions with T_i via collisional process, for $T_e \gg T_i$.
- (*) This process will take the "equilibration time" τ_{eq} which is much longer than times for electrons and ions separately to come to thermodynamic equilibrium among themselves, which are τ_{ee}^{-1} and τ_{ii}^{-1} respectively.



$$* \text{ Momentum Conservation: } m v_0 + M v_i = M V \quad (11.32)$$

$$* \text{ Energy Conservation: } \frac{1}{2} m v_0^2 + \frac{1}{2} M v_i^2 = \frac{1}{2} M V^2 \quad (11.33)$$

$$\Rightarrow \frac{1}{2} M V^2 \approx \underbrace{\left(\frac{4m}{M}\right)}_{\text{for } v_i \approx v_0} \frac{m v_0^2}{2} \quad \text{for } v_i \approx v_0. \quad (11.34)$$

Energy Transfer

- ⊕ Consider Coulomb collisions btwn electrons and ions.

From $m \Delta \vec{v} = -M \Delta \vec{V}$, we obtain

$$\frac{1}{2} M |\Delta \vec{V}|^2 = \frac{m^2}{2M} |\Delta \vec{v}|^2. \quad (11.36)$$

Recall $\Delta \vec{v}$ is mostly in \perp direction on avg (Fig 11.1).

$$\therefore \frac{1}{2} M |\Delta \vec{V}|^2 = \frac{m^2}{2M} \langle (\Delta v_i)^2 \rangle \quad (11.37)$$

- ⊕ For the case where many electrons colliding with many ions, from (11.1), we have

$$\frac{d}{dt} \langle (\Delta v_i)^2 \rangle = \frac{n_i Z^2 e^4 \ln \Lambda}{2\pi \epsilon_0^2 m^2 v} \quad (11.38)$$

- ⊕ Total rate of energy loss from electrons by collisional transfer to ions,

$$\frac{d}{dt} \bar{W}_e = -\frac{m^2}{2M} \int \frac{d}{dt} \langle (\Delta v_i)^2 \rangle f_e(v) d^3 v \quad (11.40)$$

(*) From $W_i = \frac{3}{2} n_i T_i$ and $\frac{d}{dt} \bar{W}_i = -\frac{d}{dt} W_e$, (11.41)

we get

$$\frac{d \bar{T}_i}{dt} = \frac{m^2}{3n_i M} \int \frac{d}{dt} \langle (\bar{v}_i)^2 \rangle f_e(v) d^3 v \quad (11.42)$$

$$= \frac{Z^2 e^4 \ln \Lambda}{6\pi \epsilon_0^2 M} \int \frac{f_e(v)}{v} d^3 v \quad (11.43)$$

For a Maxwellian f_e ,

$$\int \frac{f_e(v)}{v} d^3 v = \left(\frac{2}{\pi}\right)^{1/2} \frac{n_e m_e^{1/2}}{T_e^{1/2}} \quad (11.44)$$

$\Rightarrow \frac{d}{dt} \bar{T}_i = \frac{T_e}{\tau_{eq}}$ (11.45)

where

$$\boxed{\tau_{eq}^{-1} = \frac{n_e Z^2 e^4 m_e^{1/2} \ln \Lambda}{3\pi (2\pi)^{1/2} \epsilon_0^2 M T_e^{3/2}}} \quad (11.46)$$

'Temperature equilibration rate'

Note that

$$\tau_{eq}^{-1} \approx 2 \frac{m}{M} \langle v_i \rangle \quad (11.47)$$

* This can be generalized to the case with finite T_e ,

$$\frac{d}{dt} T_i = \frac{T_e - T_i}{T_{eq}}$$

$$\left(\rightarrow \frac{d}{dt} T_e = \frac{n_i}{n_e} \frac{T_i - T_e}{T_{eq}} \right)$$

from energy conservation

(11.48),

Homework

* Problem 11.4. on page 180