Fusion Plasma Theory II. 2019

Week 1

Fusion Plasma Theory II . 2019

Unit 4 : Waves in Fluid Plasma

Ch.15: Basic Concepts of Waves

Ch.16: Waves in unmagnetized Plasma

Ch.17: High-frequency waves in magnetized plasma

Ch.18: Low-frequency waves "

– MHD description of waves

Unit 5 : Instabilities in Fluid Plasma

Ch.19. Rayleigh-Taylor Instabilities

Ch.20. Resistive Tearing Instability

Fusion Plasma Theory II, 2019

Unit 5 : Instabilities in Fluid Plasma

Ch 21 : Drift Waves and Instabilities

Unit 6 : Kinetic Theory of Plasmas

Ch 22 : The Vlasov – Maxwell Equations

Ch 23 – 24 : Kinetic Effects on Plasma Waves

Ch ~~24~~ 25 : Velocity-space Instabilities and Nonlinear Theory

Ch 26 : Drift-Kinetic Equations and Kinetic Drift Waves

Ch. 15: Basic Concepts of small-amplitude Waves

* Relevant governing equations for plasma dynamics are nonlinear.

* Assume amplitudes of waves are small enough (infinitesimally small).

  $\Rightarrow$ Equations can be linearized.

  $\Rightarrow$ Fourier analysis becomes useful.

* 0-th order: equilibrium of the system.

* 1-st order: Solve for frequency "ω" $= \omega(\vec{k})$, and relative amplitudes and phases of various oscillating quantities.

* Exponential Notation:

For example, density perturbation:

$$\delta n = \underline{\delta \bar{n}} \exp\left[i \left(\underline{\vec{k} \cdot \vec{x} - \omega t} + \underline{\delta_n} \right) \right]$$

amplitude (real number); spatial, temporal phase dependences.

(cf: sometimes, $\delta \bar{n}\, e^{i\delta_n}$ is combined into a complex amplitude $\delta \tilde{n}$)

So, understanding the notations depending on the context of discussion is useful, rather than being trapped in irrelevant details.

Of course,

$$e^{iA} = \cos A + i \sin A$$

is a useful identity.

Physical Quantities of Interests

* $\vec{k}$: wave vector

$\lambda = 2\pi/k = 2\pi/|\vec{k}|$: wavelength.

* ω : frequency

$$\omega = \omega(\vec{k})$$

Dispersion Relation

* **Phase Velocity :**

" an observer travelling with the phase velocity, $\vec{V}_p$, stays at a constant wave phase. "

⇒ Demand $\frac{d}{dt}$ (total phase factor) $= 0$

(in $\exp[i(\vec{k}\cdot\vec{x} - \omega t + \delta_n)]$)

* Phase Velocity :

From $\frac{d}{dt}\left(\vec{k}\cdot\vec{x} - \omega t\right) = \vec{k}\cdot\frac{d\vec{x}}{dt} - \omega = 0$,

$$\boxed{\vec{V}_p = \frac{\vec{k}}{|\vec{k}|^2}\,\omega} = \frac{\omega k_x}{k^2}\hat{x} + \frac{\omega k_y}{k^2}\hat{y} + \frac{\omega k_z}{k^2}\hat{z}$$

in Cartesian coordinates

not $\frac{\omega}{k_x}\hat{x} + \frac{\omega}{k_y}\hat{y} + \frac{\omega}{k_z}\hat{z}$.

* If we consider a wave-packet consisting of many components with different $\vec{k}$'s (and ω's), $\vec{V}_p$ is the velocity at which individual crests within the packet travel.

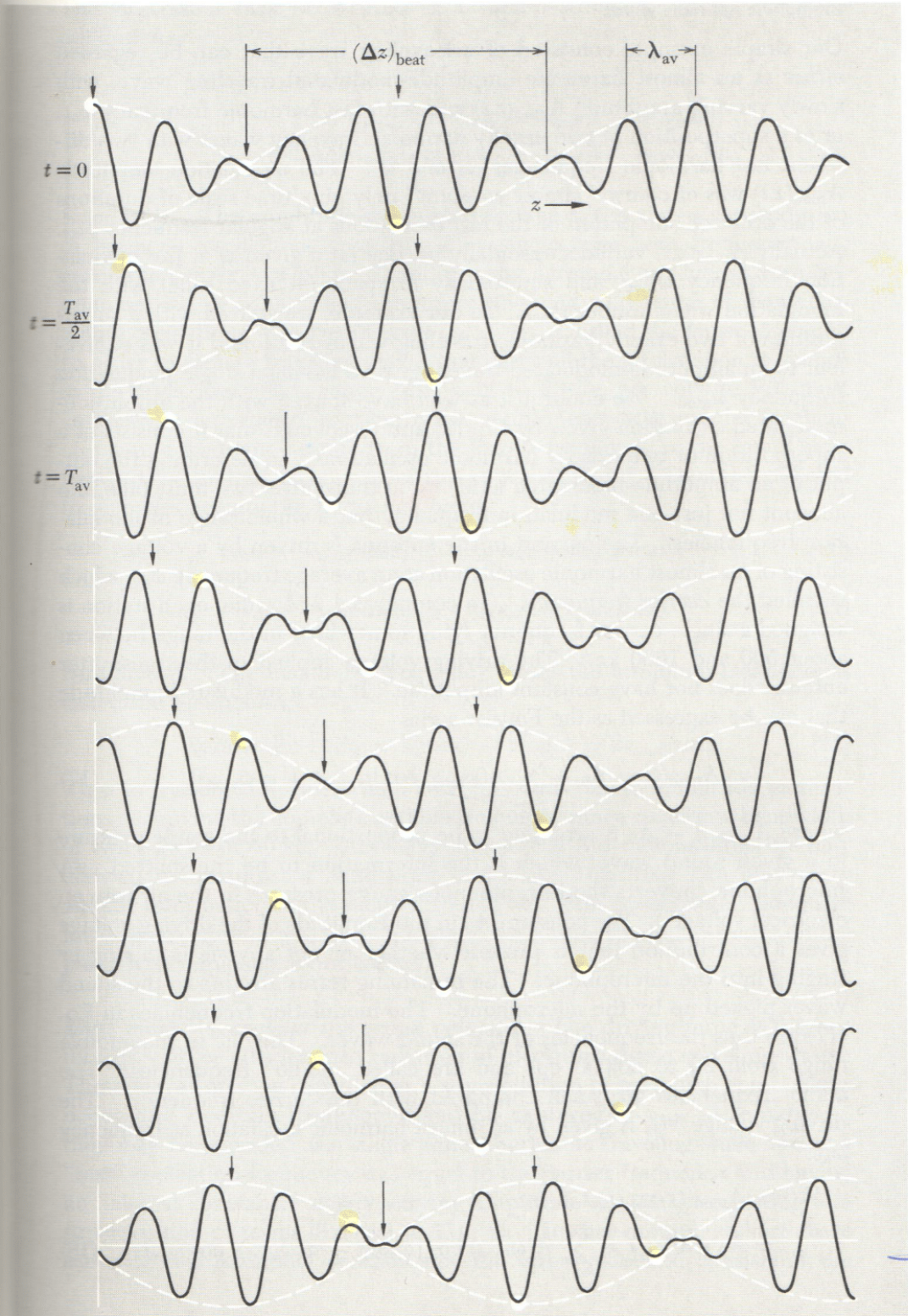


Fig. 6.1 Group velocity. The arrows follow the beats, which travel at the group velocity v_g. The white circles follow individual wave crests, which travel at the average phase velocity v_{av}.

draw thru ob 'envelope'

I.-7.

of crests & troughs

$\frac{1}{4}$ wl.

thru packet prop. by

$\frac{1}{4}$ wl.

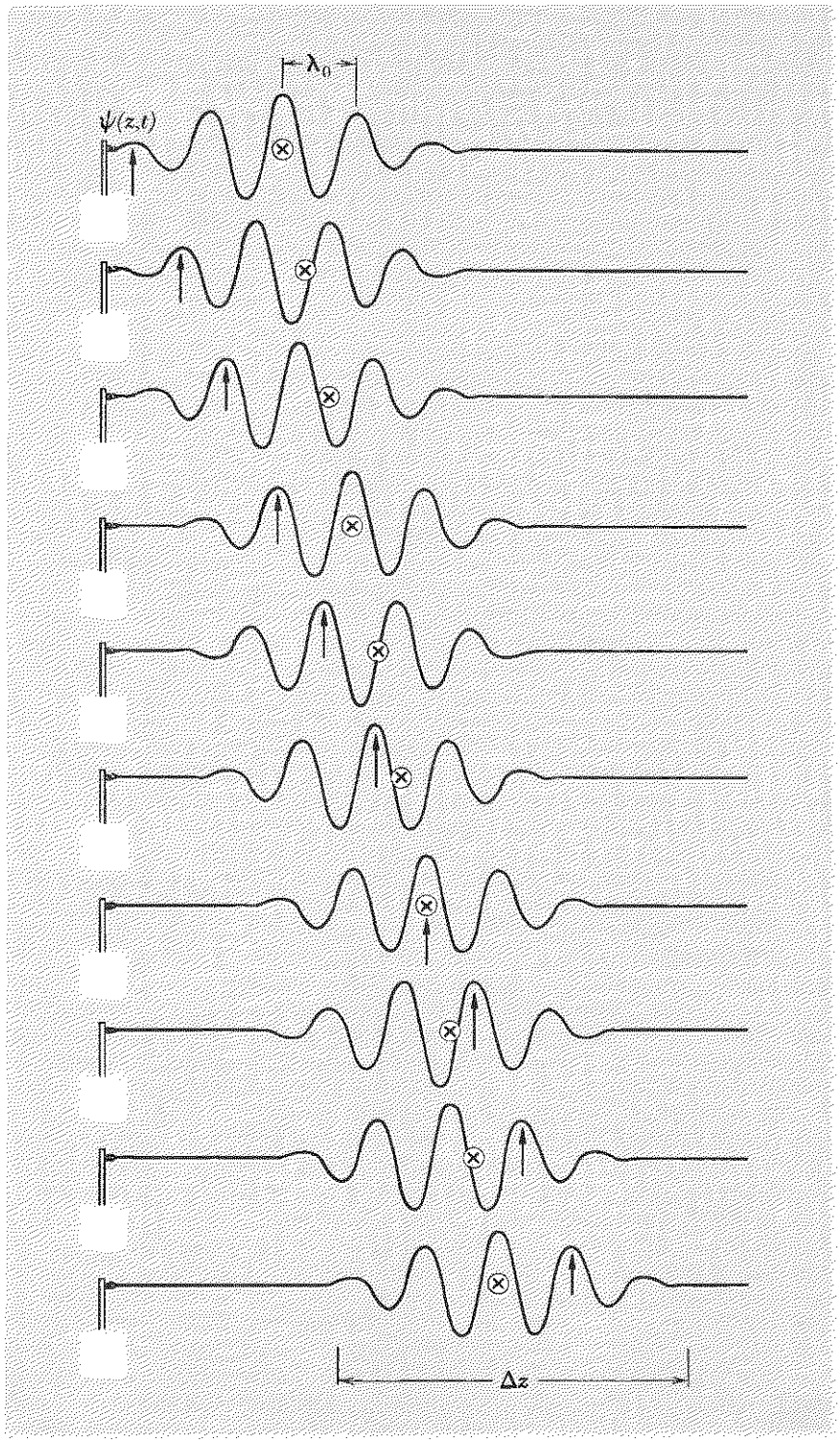
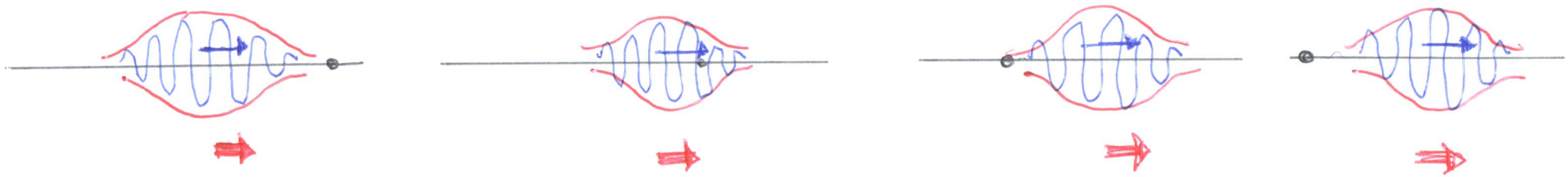


Fig. 6.7 *Wave packet with phase velocity twice the group velocity. The arrow travels at the phase velocity, following a point of constant phase for the dominant wavelength. The cross travels at the group velocity with the packet as a whole.*

15.2. Group Velocities



* $\vec{v}_p$: phase velocity of individual crests within the packet.

  (cf. it can be in oppositive direction to ⇒ of packet)

* $\vec{v}_g$: group velocity of the wave packet

  (~ envelope containing many wave crests).

* In general, $\vec{v}_g \neq \vec{v}_p$.

* Energy and Information are carried at $\vec{v}_g$!

* Packet of oscillations with Gaussian envelope;

- $A(x) = \mathrm{Re}\left[\exp\left(-\frac{x^2}{2\sigma^2}\right)\exp(ik_0x)\right]$ (15.8)

See Fig 15.1 in G&R ; $k_0\sigma >> 1$ (typo in Fig. caption)

It can be shown that,

- $A(x) = \mathrm{Re}\left(\frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\ \exp(ikx)\exp\left[-\frac{\sigma^2(k-k_0)^2}{2}\right]\right)$ (15.9)

*** Home work: Problem 15.2. on page 253

This can be viewed as $t=0$ snap shot (free-frames) of a set of propagating waves.

* Ensuing time-evolution of this system is then,

$$A(x,t) = \mathrm{Re}\left(\frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, \exp[i(kx - \omega(k)t)] e^{-\frac{\sigma^2(k-k_0)^2}{2}} \right) \qquad (15.10)$$

Note:

- Dependences of Gaussian envelope on σ:
  - large σ → narrow (localized) packet in k-space
    → wide packet in x-space.
    (and vice versa for small σ)

∿ uncertainty principle in quantum mechanics.

approximate; $\omega(k) = \omega(k_0) + \left(\frac{\partial \omega}{\partial k}\right)_{k_0} (k - k_0) + \ldots$

$$A(x,t) = \mathrm{Re}\left[\exp\left[i\left\{ k_0 \left(\frac{\partial \omega}{\partial k}\right)_{k_0} - \omega(k_0)\right\} t \right] \right.$$

$$\left. \times \text{"} \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, \exp\left\{ i\left[kx - k\left(\frac{\partial \omega}{\partial k}\right)_{k_0} t \right] \right\} e^{-\frac{\sigma^2 (k-k_0)^2}{2}} \right] \text{"} \qquad (15.11)$$

Comparing this expression to $A(x, t=0)$ in Eq (15.9),

the second line " $\sim$ " is exactly " $A\left(x - \left(\frac{\partial \omega}{\partial k}\right)_{k_0} t, 0\right)$ "

i.e., a translation of the original $t=0$ freeze-frame

at velocity $\left(\frac{\partial \omega}{\partial k}\right)_{k_0}$!

15.3. Ray-tracing Equations

* Consider an inhomogeneous plasma in 3d.

Without loss of generality, take $\vec{k_0} = k_0 \hat{x}$.

Then, 3d-extension of Eq. (15.8) is:

$$\mathcal{A}(x) = \mathrm{Re}\left[\exp\left[-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{z^2}{2\sigma_z^2} \right] \exp(i k_0 x) \right]$$

which can be re-expressed through the Fourier transform,

(15.12)

$$\mathcal{A}(x) = \mathrm{Re}\left[\frac{\sigma_x \sigma_y \sigma_z}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3\vec{k} \exp\left(i \vec{k} \cdot \vec{x} \right) \right.$$

$$\left. \exp\left[-\sigma_x^2 (k_x - k_0)^2/2 - \sigma_y^2 k_y^2/2 - \frac{\sigma_z^2 k_z^2}{2} \right] \right]$$

(15.13)

* As done in 1d before, expand

$$\omega(\vec{k}) \simeq \omega(\vec{k}_0) + (\vec{k}-\vec{k}_0)\cdot \vec{\nabla}_{\vec{k}}\,\omega\Big|_{\vec{k}_0} + \cdots \qquad (15.14)$$

where $$\vec{\nabla}_k \equiv \hat{x}\frac{\partial}{\partial k_x} + \hat{y}\frac{\partial}{\partial k_y} + \hat{z}\frac{\partial}{\partial k_z}\,. \qquad (15.15)$$

* A similar analysis as Eqs (15.9) – (15.11) in 3d will lead to our 'freeze-frame' $\mathbb{A}(\vec{x})$ translating at a vector group velocity given by $(\mathbb{A}(\vec{x}-\vec{v}_g t))$,

$$\boxed{\vec{v}_g = \frac{\partial \omega}{\partial \vec{k}} \quad \left(= \nabla_{\vec{k}}\,\omega\right)}$$

* Note that $\vec{v}_g$ may not only have a different magnitude from $\vec{v}_p$, but even a different direction,

* As a packet propagates in an inhomogeneous plasma, the peak of $\vec{k}$ spectrum, $\vec{k}_0$ can change.

* On the other hand, if the medium (background) is by hypothesis linear and time-independent, $\omega(\vec{k}_0)$ should be constant.

⇒ Total derivative of ω, moving with the wave-packet must vanish. i.e,

$$\frac{d}{dt}\,\omega(\vec{k},\vec{x}) = 0$$

along the ray.

* Using the chain-rule,

$$\frac{d}{dt}\omega(\vec{k},\vec{x}) = \frac{d\vec{x}_0}{dt}\cdot\frac{\partial}{\partial\vec{x}}\Big|_{\vec{k}}\omega + \frac{d\vec{k}_0}{dt}\cdot\frac{\partial}{\partial\vec{k}}\Big|_{\vec{x}}\omega = 0 \qquad (15.17)$$

∴ we can identify the relation;

$$\bullet\ \frac{d\vec{k}_0}{dt} = -\frac{\partial\omega}{\partial\vec{x}}\Big|_{\vec{k}}$$

$$\bullet\ \frac{d\vec{x}_0}{dt} = \frac{\partial\omega}{\partial\vec{k}}\Big|_{\vec{x}} \qquad (15.18)$$

* Ray-tracing equations for wave-packet.

* As the wave-packet propagates it maintains the peak of its frequency spectrum, while $\vec{k}$-spectrum transforms.

* Ray-tracing Equation for Wave-Packet : ← Wave-Particle duality → Hamiltonian Mechanics for Particle

$$\frac{d\vec{k}_0}{dt} = -\frac{\partial}{\partial \vec{x}}\omega \qquad \frac{d\vec{x}_0}{dt} = \frac{\partial}{\partial \vec{k}}\omega$$

$\hbar\omega = E$

$\hbar\vec{k} = \vec{p}$

in Q.M.

$$\frac{d\vec{p}}{dt} = -\frac{\partial}{\partial \vec{q}} H(\vec{p},\vec{q}) \qquad \frac{d\vec{q}}{dt} = \frac{\partial}{\partial \vec{p}} H(\vec{p},\vec{q})$$

* $\vec{V_g}$ of de Broglie wave ~ $\vec{V}$ particle.

* Non conservation of $\vec{k}_0$ in an inhomogeneous medium is related to the non-conservation of momentum

* Conservation of ω in time-independent medium is related to the conservation of energy.