Fusion Plasma Theory II. 2019

Week 10

where c_1 is a coefficient on the order of one which depends on many assumptions.
We'll assume n and T_e profiles will behave in the same way.
When magnetic island size is large enough, very fast $\|$ conduction along $\mathbf{B}$ flattens T_e profile.

$$\Rightarrow \frac{\partial}{\partial x}(T_{eo} + \delta T_e) \simeq 0 \quad \left(\Rightarrow \frac{\partial}{\partial x}\delta p_e = -\frac{\partial p_e}{\partial r}\right)$$

where $x \equiv r - r_s$.
δj_b in the Ohm's law is

$$\delta j_b = -j_b! \quad \left(\because \delta j_b = -c_1 \frac{\epsilon^{1/2}}{B_\theta}\frac{\partial}{\partial x}\delta p_e\right)$$

i.e. a deficiency of Bootstrap current in the island region.

$$\therefore \frac{\partial \psi}{\partial t} = \eta \left(\frac{1}{\mu}\frac{\partial^2}{\partial r^2}\psi - c_1 \frac{\epsilon^{1/2}}{B_\theta}\frac{\partial p}{\partial r}\right)$$

$$\therefore \frac{\partial \psi}{\partial t} = \eta \left(\frac{1}{\mu} \frac{\partial^2}{\partial r^2} \psi - c_1 \frac{\epsilon^{1/2}}{B_\theta} \frac{\partial p}{\partial r} \right)$$

As we did before for a classical tearing mode, we'll integrate this equation over the width "w" of a magnetic island in radial direction.

$$\int_{r_s - w/2}^{r_s + w/2} dr \Rightarrow \int_{-w/2}^{+w/2} dx$$

$$\Rightarrow \text{LHS} = w \frac{\partial \psi}{\partial t}, \quad \text{RHS} = \frac{\eta}{\mu_0} \Delta' \psi - c_1 \eta \frac{\epsilon^{1/2}}{B_\theta} \frac{\partial p}{\partial r} \cdot w.$$

Divide each side by 2ψ and recall $w \propto \psi^{1/2}$.

$$\Rightarrow \text{LHS} = \frac{w}{2\psi} \frac{\partial}{\partial t} \psi = \frac{\partial}{\partial t} w, \quad \text{RHS} = \frac{\eta}{2\mu_0} \Delta' - c_1 \frac{\eta}{2} \frac{\epsilon^{1/2}}{B_\theta} \frac{\partial p}{\partial r} \frac{w}{\psi}.$$

Using the relation between "w" and "ψ"

$$w = 4 \left(\frac{q\psi}{q' B_\theta} \right)^{1/2}$$

and defining

$$\alpha \equiv -\frac{8p'q}{pq'} \text{ and } \beta_p = \frac{2\mu_0 p}{B_\theta^2}$$

we can obtain a modified Rutherford equation for NTM.

$$\frac{dw}{dt} = \frac{\eta}{2\mu_0}\left(\Delta'(w) + c_1\frac{\alpha\epsilon^{1/2}\beta_p}{w}\right)$$

where

$$\Delta'(w) \equiv \frac{\left.\frac{\partial\psi}{\partial x}\right|_{w/2} - \left.\frac{\partial\psi}{\partial x}\right|_{-w/2}}{\psi}, \quad \alpha = \frac{8r}{\hat{s}L_p}, \quad \hat{s} \equiv \frac{rq'}{q}.$$

This equation appears to be singular (i.e., the 2nd term on the RHS $\nearrow \infty$ as $w \to 0$) but by multiplying both sides with w, we obtain

$$\frac{d}{dt}w^2 = \frac{\eta}{4\mu_0}\left(\Delta'(w)w + c_1\alpha\epsilon^{1/2}\beta_p\right).$$

Note that $\alpha > 0$ for $\hat{s} > 0$, $p' < 0$ and $1/L_p > 0$.

Now, we can discuss nonlinear evolution of NTM magnetic island.

i) If "w" is very small (thin island),

the 2nd term dominates on the RHS.

$$\therefore \frac{d}{dt} w^2 = \frac{c_1 \eta \alpha}{4\mu_o} \epsilon^{1/2} \beta_p \Rightarrow w(t)^2 = w(0)^2 + c_1 \frac{\eta\alpha}{4\mu_o} \epsilon^{1/2} \beta_p t$$

$$\Rightarrow w(t) = w(0) \sqrt{1 + c_1 \frac{\eta\alpha}{4\mu_0} \frac{\epsilon^{1/2}\beta}{w(0)}} \cdot t \propto t^{1/2}$$

For an initial size of island $w(0) \equiv w(t)|_{t=0}$

Note that this possible even for $\Delta'(0) < 0$! (i.e., when the classical tearing mode is stable)

As w increases, the 1st term on the RHS becomes non-negligible and we should consider its effect.

ii) If $\Delta'(0) > 0$

the Neoclassical term $\propto p'$ simply adds to the drive due to current gradient characterized by Δ'.

$\Rightarrow$ Island will eventually saturate($w \to w(t \to \infty) < \infty$, due to $\Delta'(w) < 0$) as w gets larger.

iii) If $\Delta'(0) < 0$,

the island will saturate when

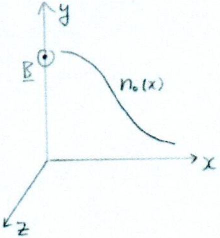
$$w_{\text{sat}} \simeq \frac{c_1 \alpha \epsilon^{1/2} \beta_p}{|\Delta'|}, \quad \text{if } \Delta'(w) \simeq \Delta'(0).$$

This model predicts every rational surface in tokamak will be unstable to NTM! Fortunately, only a few relatively low (low or moderate) mode number modes have been typically observed from tokamak experiments.

2. Examples of Basic Microinstabilities

Consider a uniform magnetic field $\mathbf{B} = B_0\hat{z}$, nonuniform density profile $n_0 = n_0(x)$, periodicity in y and z.(topologically a flat torus!) Let's consider uniform temperatures for simplicity.

In this simple geometry, any perturbed quantities can be Fourier-decomposed in



y and z directions. E.g.,

$$\delta\phi(x,y,z) = \sum_{\mathbf{k},\omega} \delta\phi_{\mathbf{k},\omega}(x) \exp\left[i(k_y y + k_z z - \omega t)\right]$$

$$\delta n(x,y,z) = \sum_{\mathbf{k},\omega} \delta n_{\mathbf{k},\omega}(x) \exp\left[i(k_y y + k_z z - \omega t)\right]$$

We'll pursue a <u>local</u> theory at first (at one point in x).

2.1. Electron Drift Wave

Electron drift wave has been seriously considered in theoretical community since early days. It can be driven unstable only in the presence of density gradient. Let's search for an "<u>electrostatic</u>" (i.e. $\delta B = 0 \Rightarrow \nabla \times \mathbf{E} = 0 \Rightarrow \mathbf{E} = -\nabla\phi$) wave with a phase velocity satisfying $v_{\text{th},i} \ll \omega/k_z \ll v_{\text{th},e}$. Here $v_{\text{th},e} = \sqrt{T_e/m_e}$, $v_{\text{th},i} = \sqrt{T_i/M_i}$, $k_z = k_\parallel \equiv \mathbf{B}\cdot\mathbf{k}/\left|\mathbf{B}\right|$.

A.Electron Response

Since electrons move fast (can cover the system size during one wave period! $k_{\parallel} v_{\mathrm{th},e} \gg \omega$), we can consider them in a thermal equilibrium in the presence of electrostatic fluctuation $\delta\phi$.

<u>Maxwell-Boltzmann Statistics</u>
$\Rightarrow f_e(E) \propto \exp\left(-E/T_e\right) = \exp\left[-\left(\frac{1}{2}m_e v^2 - |e|\,\delta\phi\right)/T_e\right]$
$\Rightarrow n_e = \int d^3v f_e(E) = n_{e0} \exp\left(|e|\,\delta\phi/T_e\right)$: Boltzmann relation.

The perturbed part is given by $\delta n_e = n_e - n_{e0} = n_{e0}\Big[1 + \dfrac{|e|\,\delta\phi}{T_e} + \mathcal{O}\Big(\Big(\dfrac{|e|\,\delta\phi}{T_e}\Big)^2\Big) - 1\Big] = n_{e0}\dfrac{|e|\,\delta\phi}{T_e}$

$\boxed{\delta n_e/n_{e0} = |e|\,\delta\phi/T_e}$: Electrons obey Boltzmann response.
This Boltzmann response is also called the adiabatic response.
"Adiabatic" here refers to a slow time variation of a wave.

It is also instructive to recover this from a fluid description.
The fluid momentum equation (of motion) for electrons is given by

$$m_e n_e \frac{d}{dt}\mathbf{u}_e = -n_e\,|e|\left(\mathbf{E} + \frac{1}{c}\mathbf{u}_e \times \mathbf{B}\right) - \nabla p_e \tag{1}$$

Linearize (assuming $\mathbf{u}_{e0} = 0$) to obtain,

$$m_e n_0 \frac{\partial}{\partial t}\delta\mathbf{u}_e = -n_0\,|e|\left(\delta\mathbf{E} + \frac{1}{c}\delta\mathbf{u}_e \times \mathbf{B}\right) - \nabla \delta p_e \tag{2}$$

Take $\mathbf{b}\cdot$ and ignore electron inertia; $m_e \to 0$ (recall $v_{\mathrm{th},e} = \sqrt{T_e/m_e}$ is very fast)

$$\Rightarrow |e|\, n_0 \nabla_{\parallel} \delta\phi - T_e \nabla_{\parallel} \delta n_e = 0$$

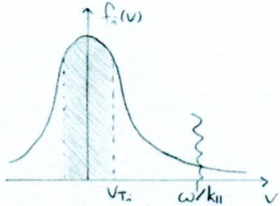
$$\delta n_e / n_{e0} = |e|\, \delta\phi / T_e \tag{3}$$

(Here, $\nabla_{\parallel} = \mathbf{b}\cdot\nabla = \partial/\partial z$, we assumed isothermal plasma with $\delta T_e = 0 \Rightarrow \delta p_e = T_e \delta n_e$)

B. Ion Response

Ions satisfy $\omega/k_{\parallel} \gg v_{\mathrm{th},i}$, we further assume "cold ions",
i.e. $k_\perp \rho_i \ll 1$ (ignore FLR effect), but $T_i \ll T_e \Rightarrow \boxed{k_\perp \rho_s \sim 1}$.
Here $\rho_i = v_{\mathrm{th},i}/\Omega_{ci}$, $\Omega_{ci} \equiv |e|\, B_0 / M_i c$, $\rho_s = C_s/\Omega_{ci} = \sqrt{T_e/T_i}\rho_i$, $C_s = \sqrt{T_e/M_i}$.
In tokamak, Ohmically heated or ECRH heated plasma satisfy $T_i \ll T_e$. But here just for algebraic simplicity.
In this situation, most of ions move slowly enough. From the wave's point of view, they more or less move together like a fluid.

Their equation of motion is



$$M_i n_i \frac{d}{dt}\mathbf{u}_i = n_i\,|e|\left(\mathbf{E} + \frac{1}{c}\mathbf{u}_i \times \mathbf{B}\right) - \nabla p_i \tag{4}$$

Linearize and drop ∇p_i (cold ions!) (with $\mathbf{u}_{i0} = 0$)

$$\Rightarrow M_i n_i \frac{\partial}{\partial t}\delta\mathbf{u}_i = n_i\,|e|\left(\delta\mathbf{E} + \frac{1}{c}\delta\mathbf{u}_i \times \mathbf{B}\right) \tag{5}$$

Along $\mathbf{B} \Rightarrow$

$$M_i n_0 \frac{\partial}{\partial t}\delta u_{i\parallel} = n_0\,|e|\,\delta E_\parallel = -n_0\,|e|\,\nabla_\parallel \delta\phi \tag{6}$$

Across $\mathbf{B} \Rightarrow$
We can solve Equation (5) via iteration knowing (or assuming) $\omega/\Omega_{ci} \ll 1$.
(We are dealing with "low frequency" microinstabilities.)
$(\omega/\Omega_{ci} \sim \omega/\frac{|e|B}{M_i c} \propto |M_i/e| \ll 1)$

1st order : RHS=0.

$$\delta\mathbf{E} + \frac{1}{c}\delta\mathbf{u}_i^{(1)} \times \mathbf{B} = 0$$
$$\Rightarrow \delta\mathbf{u}_\perp^{(1)} = \delta\mathbf{u}_E = \frac{c\mathbf{b} \times \nabla\delta\phi}{B} \tag{7}$$

2nd order :

$$M_i n_0 \frac{\partial}{\partial t}\delta\mathbf{u}_\mathbf{E} = n_0 \left|e\right| \frac{1}{c}\delta\mathbf{u}_\perp^{(2)} \times \mathbf{B} \tag{8}$$
$$\delta\mathbf{u}_\perp^{(2)} = \delta\mathbf{u}_{\text{polarization drift}} = \frac{M_i c^2}{\left|e\right| B^2}\frac{\partial}{\partial t}\mathbf{E}_\perp = -\frac{M_i c^2}{\left|e\right| B^2}\frac{\partial}{\partial t}\nabla_\perp\delta\phi \tag{9}$$

(1) $\Rightarrow$

$$\frac{\delta n_e}{n_0} = \frac{\left|e\right|\delta\phi}{T_e} \tag{10}$$

(6), (7), (9) with a continuity equation, $\frac{\partial}{\partial t}n_i + \nabla\cdot\left(n_i\mathbf{u}_i\right) = 0$, we can derive a dispersion relation. Linearize to get

$$\frac{\partial}{\partial t}\delta n_i + \delta\mathbf{u}_E\cdot\nabla n_0 + n_0\nabla\cdot\delta\mathbf{u}_{\text{pol}} + n_0\nabla_\parallel\delta u_{i\parallel} = 0 \tag{11}$$

$$\frac{\partial}{\partial t}\delta n_i + \delta \mathbf{u}_E \cdot \nabla n_0 + n_0 \nabla \cdot \delta \mathbf{u}_{\text{pol}} + n_0 \nabla_{\parallel} \delta u_{i\parallel} = 0 \tag{11}$$

Here you can check $\nabla \cdot \delta \mathbf{u}_E$ and other contributions are even smaller or vanish. Fourier decompose, i.e. $\sim \exp\left[i(k_y y + k_z z - \omega t)\right] \Rightarrow$

$$\frac{\partial}{\partial t} \to -i\omega,\ \nabla_{\parallel} \to ik_{\parallel},\ \nabla_{\perp} \to i\mathbf{k}_{\perp}, \text{but } \nabla n_o = -\hat{x}\frac{n_o}{L_n}$$

Here $L_n k_{\perp} \gg 1$, i.e. (system size) $\gg$ ($\perp$ wavelength), and therefore $\delta \mathbf{u}_{\text{pol}} \cdot \nabla n_0$ term is dropped in the linearized continuity equation.

$$\Rightarrow\ \underline{\frac{\delta n_i}{n_0} = \left(\frac{\omega_{*e}}{\omega} + \frac{k_{\parallel}^2 c_s^2}{\omega^2} - k_{\perp}^2 \rho_s^2\right)\frac{|e|\,\delta\phi}{T_e}} \tag{12}$$

Here the 1st term on RHS is from $\delta \mathbf{u}_E \cdot \nabla n_0$, and the 2nd term and the 3rd term are from $\nabla_{\parallel} \delta u_{i\parallel}$ and $\nabla \cdot \delta \mathbf{u}_{\text{pol}}$, respectively.
The Poisson equation in a normalized form is

$$-\lambda_{De}^2 \nabla^2 \frac{|e|\delta\phi}{T_e} = \frac{\delta n_i - \delta n_e}{n_0} \tag{13}$$

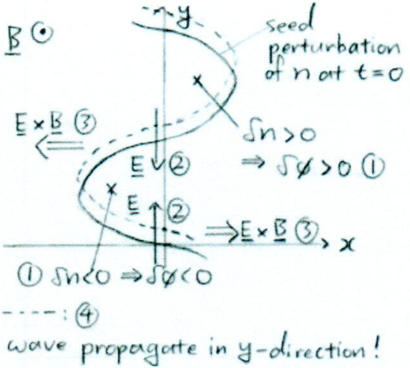
For long enough wavelength $\lambda \gg \lambda_{De}$. Therefore, it can be approximated by the quasi-neutrality equation : $\delta n_e = \delta n_i$
Finally, we obtain the linear dispersion relation for the electron drift wave.

$$1 + k_{\perp}^2 \rho_s^2 - \frac{\omega_{*e}}{\omega} - \frac{k_{\parallel}^2 c_s^2}{\omega^2} = 0 \tag{14}$$

Here $\omega_{*e} = k_y \frac{\rho_s}{L_n} c_s = k_y v_{*e}$ is electron diamagnetic frequency, where v_{*e} is electron diamagnetic drift velocity.

(cf. Another possible notation v_{de} could be confused with ∇B or curvature drift.)

The figure below is an illustration of drift wave propagation. We consider a seed perturbation in the presence of equilibrium density gradient.



2.2. Electron Drift Wave in Uniform Magnetic Field

Linear dispersion relation:

$$1 + k_\perp^2 \rho_s^2 - \frac{\omega_{*e}}{\omega} - \frac{k_\parallel^2 c_s^2}{\omega^2} = 0 \tag{15}$$

Here, $k_\perp^2 = k_x^2 + k_y^2$, $\mathbf{B} = B_0 \hat{z}$ and the diamagnetic drift frequency is $\omega_{*e} \equiv (k_y \rho_s / L_n)\, c_s = k_y v_{*e}$. We assumed

$$\delta\phi\,(x, y, z) = \sum_{\mathbf{k},\omega} \delta\phi_{\mathbf{k},\omega}\,(x) \exp i\,(k_y y + k_z z - \omega t) \tag{16}$$

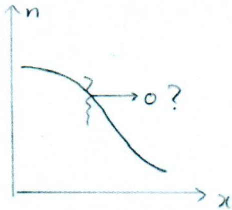
and used the WKB approximation,

$$\delta\phi_{\mathbf{k},\omega}\,(x) = \delta\hat{\phi}\,(x) \exp\left[i \int^x k_x\,(x)\,dx\right], \tag{17}$$

which is valid for $k_x L_n \gg 1$. Here, the eikonal actor $\exp\left(i \int^x k_x\,(x)\,dx\right)$ captures the fast variation in x and $\delta\hat{\phi}\,(x)$ and $k_x\,(x)$ are slowly varying in x. To the lowest order in the $1/k_x L_n$ expansion, there is only a local value in k_x in the linear dispersion relation.
Let's calculate the particle flux in the x direction carried by a drift wave.

$$\Gamma_{\text{ptl}} = \langle \delta n_e \delta v_x \rangle \tag{18}$$

Here, $\langle \ldots \rangle$ is an ensemble average, or a long time average. Practically, it's replaced by an average over ignorable coordinate(s) (i.e., direction of symmetry).



In this simple slab geometry, both y and z are ignorable coordinates. (In tokamak geometry, only the toroidal angle "ζ" is an ignorable coordinate.) The $\mathbf{E} \times \mathbf{B}$ drift is

$$\delta v_x = \frac{c\delta E_y}{B^2} B_z = -\frac{c}{B}\frac{\partial}{\partial y}\delta\phi \tag{19}$$

and the density fluctuation is

$$\delta n = \frac{|e|\,\delta\phi}{T_e} n_0 \tag{20}$$

Therefore,

$$\langle \delta n \delta v_x \rangle = -\frac{c\,|e|}{BT_e}\left\langle \delta\phi \frac{\partial}{\partial y}\delta\phi \right\rangle = -\frac{c}{2B}\frac{|e|}{T_e} n_0 \left\langle \frac{\partial}{\partial y}\delta\phi^2 \right\rangle = 0, \tag{21}$$

because the last expression is a perfect derivative, and

$$\Gamma_{ptl} = Re\,\langle \delta n \delta v_x \rangle = 0. \tag{22}$$

with

$$\langle \ldots \rangle = \frac{1}{L_y}\oint_0^{L_y} dy\,(\ldots) \Leftarrow \frac{1}{2\pi}\oint d\theta\,(\ldots) \quad r d\theta = dy. \tag{23}$$