Fusion Plasma Theory II. 2019

Week 13

Ch. 23. Kinetic Effects on Plasma Waves

$$\frac{\partial}{\partial t} f + \vec{v} \cdot \vec{\nabla} f + \frac{q}{m} (\vec{E} + \vec{v} \times \vec{B}) \cdot \frac{\partial}{\partial \vec{v}} f = 0 \qquad (22.22)$$

— Vlasov Eqn.

- Unlike Drift-kinetic Equation (which can be derived from the Vlasov equation), it contains the Lorentz force explicitly and it follows charged particles' trajectory in phase space. which is 6D.

Linearize with $f(x, v, t) = f_0(v) + \delta f(x, v, t)$ (23.1)

and $\delta E(x, t) = \delta\hat{E} \exp(-i\omega t + ikx)$ (23.2).

for one dimensional (in x) ~~a~~ uniform back ground, with $\vec{B}_0 = 0$.

⊛ $$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\delta f - \frac{e}{m}\delta E\frac{\partial}{\partial v}f_0 = 0 \qquad (23.5).$$

for electrons.

⊛ For electron plasma waves,

Poisson equation:

$$\epsilon_0 \vec{\nabla}\cdot\delta\vec{E} = \sigma = -e\int \delta f\, d^3v \qquad (23.6).$$

∴ In 1d, $$\epsilon_0 \frac{\partial \delta E}{\partial x} = -e\int_{-\infty}^{\infty} \delta f\, dv. \qquad (23.7).$$

Taking $$\delta f(x,v,t) = \hat{\delta f}(v)\exp(-i\omega t + ikx) \qquad (23.8)$$,

$$\hat{\delta f} = \frac{ie\hat{\delta E}}{m}\,\frac{\frac{\partial f_0}{\partial v}}{\omega - kv}. \qquad (23.10).$$

Substituting to Poisson equation, we get

$$i k \epsilon_0 \delta\hat{E} = -e\int_{-\infty}^{\infty} \delta\hat{f} dv = -\frac{i e^2 \delta\hat{E}}{m}\int_{-\infty}^{\infty} \frac{\frac{\partial f_0}{\partial v}}{\omega - kv} dv \qquad (23.11)$$

$$\Downarrow$$

$$\boxed{D(k,\omega) \equiv 1 + \frac{e^2}{m k \epsilon_0}\int_{-\infty}^{\infty} \frac{\frac{\partial f_0}{\partial v}}{\omega - kv} dv = 0} \qquad (23.12)$$

$D(k,\omega)$: Plasma Dispersion Function.

(sometimes, plasma dielectric function).

$D(k,\omega) = 0 \Rightarrow$ dispersion relation

— for the case of electron plasma waves in an unmagnetized plasma.

While this is a solution, in principle, the exact integration is rarely done analytically.

23.3. Thermal Effects on Electron Plasma Waves.

(*) · Assume $\frac{\omega}{k} \gg v$, and expand

$$\frac{1}{\omega - kv} = \frac{1}{\omega} + \frac{kv}{\omega^2} + \frac{k^2v^2}{\omega^3} + \cdots$$

Then, integrals in Eq (22.8) can be carried out,

$$\int_{-\infty}^{\infty} \frac{\partial f_0}{\partial v} dv = 0, \quad \int_{-\infty}^{\infty} \frac{\partial f_0}{\partial v} v\, dv = -n,$$

$$\int_{-\infty}^{\infty} \frac{\partial f_0}{\partial v} v^2 dv = 0, \quad \int_{-\infty}^{\infty} \frac{\partial f_0}{\partial v} v^3 dv = -3n v_{Te}^2, \quad (23.14)$$

$\cdots$

yielding

$$D(k,\omega) \simeq 1 - \frac{\omega_{pe}^2}{\omega^2}\left(1 + \frac{3k^2 v_{Te}^2}{\omega^2} + \cdots\right) = 0 \quad (23.15)$$

where

$$\omega_{pe}^2 \equiv \frac{n_e e^2}{m_e \epsilon_0}$$

Eq. (23.15) can be solved to the 1st order in

$\frac{k^2 v_{Te}^2}{\omega^2}$:

$$\omega^2 \approx \underbrace{\omega_{pe}^2}_{\text{electron plasma wave}} + \underbrace{3k^2 v_{Te}^2}_{\text{thermal corrections to it}} \qquad (23.16)$$

Note that

$$\frac{\text{2nd term on RHS}}{\text{1st term} \quad ''} \sim \frac{k^2 v_{Te}^2}{\omega^2} \sim \frac{k^2 v_{Te}^2}{\omega_{pe}^2}$$

$\sim k^2 \lambda_{De}^2$, (λ_{De}: the Debye length.)

∴ This approximation is valid for long wavelength modes

with $k^2 \lambda_{De}^2 \ll 1$.

23.4. The Two-Stream Instability

⊛ So far, we have considered various instabilities in plasmas. Almost all of them were driven by free energy associated with $\vec{\nabla} P$ or $\vec{\nabla} J_0$.

⊛ Now, we consider instabilities which are driven by ~~[illegible]~~ gradients in velocity space for non-Maxwellian f_0.

⊛ The 1st example (which is an idealization) is the two-stream instability.

⊛ Take

$$f_0(v) = \frac{1}{2} n \left[\delta(v - v_0) + \delta(v + v_0) \right] \qquad (23.17)$$

delta functions.

This is a crude model for counter-propagating one dimensional cold beams.

(∵ no thermal spread)

⊛ Delta functions appearing in the integral with $\frac{\partial f_0}{\partial v}$ can be treated ~~by~~ with an integration by parts.

$$\int_{-\infty}^{\infty} \frac{\frac{\partial f_0}{\partial v}}{\omega - kv} dv = -\int_{-\infty}^{\infty} f_0 \frac{\partial}{\partial v}\left(\frac{1}{\omega - kv}\right) + \left[\frac{f_0}{\omega - kv}\right]_{-\infty}^{\infty}$$

$$= -k \int_{-\infty}^{\infty} \frac{f_0}{(\omega - kv)^2} dv = -\frac{kn}{2}\left(\frac{1}{(\omega - kv_0)^2} + \frac{1}{(\omega + kv_0)^2}\right) \qquad (23.18)$$

from the property of δ ftn.

$$\Rightarrow \quad D(k,\omega) \equiv 1 - \frac{1}{2}\left\{\frac{\omega_{pe}^2}{(\omega - kv_0)^2} + \frac{\omega_{pe}^2}{(\omega + kv_0)^2}\right\} = 0 \qquad (23.19).$$

This is a quartic (the 4th order) algebraic eqn for "ω".

∴ There must be 4 roots.

i) For $k^2v_0^2 > \omega_{pe}^2$, there are 4 real roots and no instabilities.

ii) For $k^2v_0^2 < \omega_{pe}^2$, there are 2 real roots and 2 complex (with non-zero imaginary part) roots which are complex conjugates ~~of~~ of one another.

"two-stream instability."

$\Rightarrow$ one of those is an instability.

23-6.
23-9.

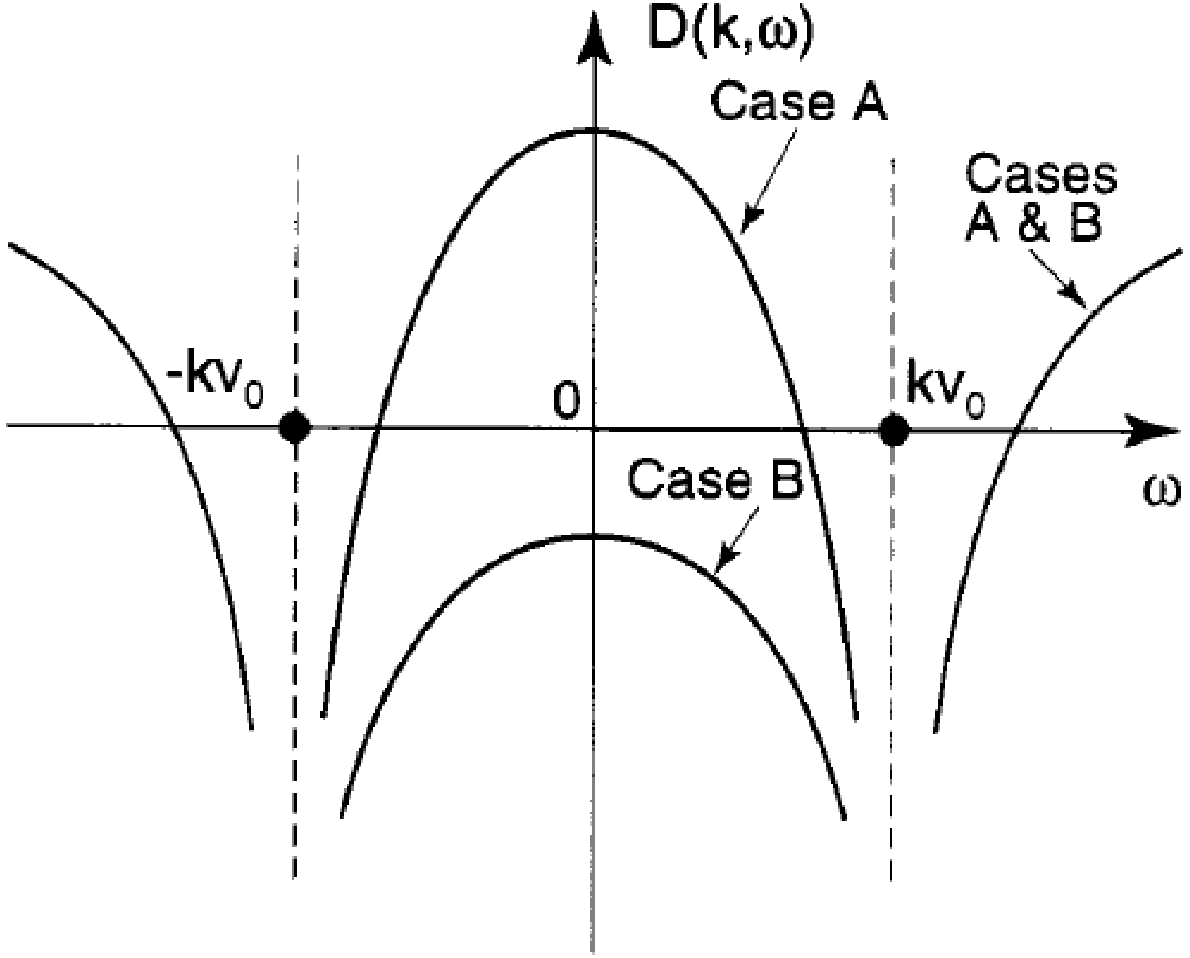

Figure 23.1. The dispersion function $D(k, \omega)$ for the two-stream instability plotted against ω for the Cases A (four real roots ω of $D(k, \omega) = 0$) and **B** (two real roots ω of $D(k, \omega) = 0$).

23.5. Ion Acoustic Waves

⊛ Look for 'electrostatic wave' for which both ions and electrons are allowed to oscillate. Taking both species' contribution to Poisson eqn into account, we have

$$D(k,\omega) \equiv 1 + \sum_\sigma \frac{e_\sigma^2}{m_\sigma k \epsilon_0} \int_{-\infty}^{\infty} \frac{\frac{\partial f_{0\sigma}}{\partial v}}{\omega - kv} dv = 0 \quad (23.20)$$

Solve for

$$v_{Ti} \ll \frac{\omega}{k} \ll v_{Te} \quad (23.21).$$

⊛ For ions,

$$\frac{1}{\omega - kv} \simeq \frac{1}{\omega} + \frac{kv}{\omega^2} + \cdots \quad (23.22)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\frac{\partial f_0}{\partial v}}{\omega - kv} dv \cong -\frac{nk}{\omega^2} + \cdots \quad (23.23)$$