Fusion Plasma Theory II. 2019

Week 2

Ch. 16. Waves in an unmagnetized plasma

16.1. Langmuir Waves

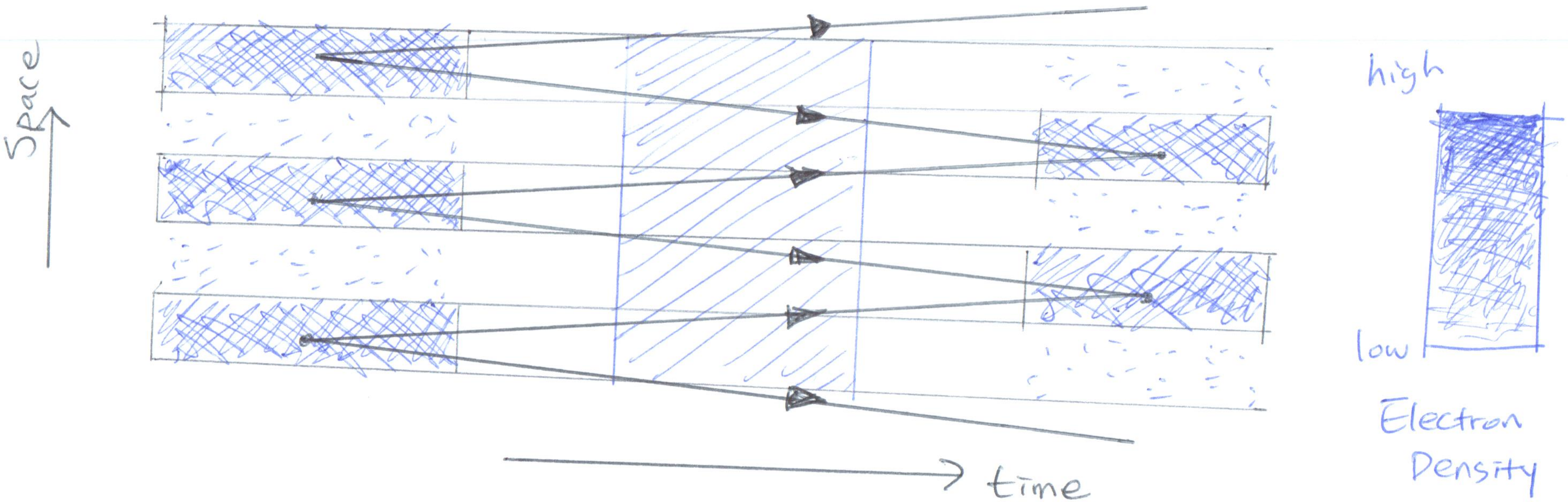


Fig 16.1.

* Initial displacements of electrons from equilibrium position
→ charge separation → electric field generation
→ pulling electrons back ~~to~~ → overshoot → oscillations.

* This oscillation happens in very Short time scale

$\Rightarrow$ ions with heavy mass cannot respond, and can be considered as stationary.

* **Electron Dynamics :**

- $m n_e [\dot{\vec{u}}_e + (\vec{u}_e \cdot \vec{\nabla}) \vec{u}_e] = -e n_e \vec{E} - \vec{\nabla} p_e$ (16.1)
- $\dot{n}_e + \vec{\nabla} \cdot (n_e \vec{u}_e) = 0$ (16.2)
- $\epsilon_0 \vec{\nabla} \cdot \vec{E} = e(n_i - n_e)$ (16.3)

* Consider electrons moving in x direction and waves propagating " .

$\Rightarrow$ perturbed quantities vary as "$\exp[i(kx - \omega t)]$"

$\boxed{\frac{\partial}{\partial x} \rightarrow ik}$ and $\boxed{\frac{\partial}{\partial t} = -i\omega}$

* Linearize Eqs. (16.1)–(16.3) i.e., keep only the 1st order quantities:

In addition, relate p_1 to n_1 by using the equation of state for 1-dimensional compression faster than the thermal conduction, i.e., $p \propto n^{\gamma}$, where $\gamma = \frac{2+N}{N} = \frac{2+1}{1} = 3$.

$$\Rightarrow \begin{cases} i\omega m n_0 u_1 = e n_0 E_1 + 3ikT n_1 & (16.6) \\ -i\omega n_1 + ikn_0 u_1 = 0. & (16.7) \\ ik\epsilon_0 E_1 = -e n_1 & (16.8) \end{cases}$$

3 equations for 3 perturbed quantities, u_1, n_1 and E_1.

* [Determinant] = 0 of 3x3 matrix. $\Rightarrow$ Dispersion Relation (between "ω" and "k")

* $$\boxed{\omega^2 = \omega_{pe}^2 + 3k^2 v_{Te}^2}$$

where $\omega_{pe}^2 \equiv n_e e^2/\epsilon_0 m_e$

electron plasma frequency

$v_{Te}^2 = (T_e/m_e)$.

electron thermal velocity.

Bohm-Gross dispersion relation [PR 1949].

* Fig 16.2 illustrates the behavior at different k regimes.

* At long wavelength (low k); $\omega^2 \approx \omega_{pe}^2$

~ non-propagating "plasma oscillation".

(Langmuir)

* At short wavelength (high k); $\omega^2 \approx 3k^2 v_{Te}^2$

~ propagating "electron sound wave."

16.2. Ion Sound Waves

* Take a look at another electrostatic longitudinal wave ($\vec{K} \parallel \vec{E}_1$) which oscillates in longer time scale so that ions dynamics plays an essential role.

→ Electrons can move fast enough to establish nearly exact force balance (between pressure gradient and electric field), i.e., a Boltzmann response.

* Follow a similar derivation as the one for Langmuir wave.

- Ion Fluid momentum eqn. (16.16)
- Poisson eqn. (16.19)
- Ion Continuity eqn (16.21)

* A crucial difference (from the Langmuir wave where $n_{i1} = 0$) is that

$$n_{e1} = n_e - n_{e0} = n_{e0}\left(\exp(e\phi_1/T_e) - 1\right)$$
$$\simeq n_{e0}\, e\phi_1/T_e \qquad (16.18)$$

i.e. Boltzmann response (or adiabatic response),

* Linearization → Solve ~~for~~ coupled eqns for n_{i1}, ϕ_1, and u_{i1} → to get:

$$\boxed{\left(\omega/k\right)^2 = \frac{T_e/M}{1 + k^2\lambda_D^2} + \gamma_i T_i/M} \qquad (16.23)$$

$$\lambda_D^2 \equiv \epsilon_0 T_e/n_e e^2 = v_{Te}^2/\omega_p^2.$$

Debye length.

* In the long wavelength limit (~~sm~~ low k) ;

$$\boxed{\left(\omega/k\right)^2 \approx \frac{T_e + \gamma_i T_i}{M}} \quad (\star)$$

Sound wave

- Both electron and ion pressure contribute, but only ions provide "mass".

- In collisionless plasma, Eq. ☆ is valid for $T_e \gg T_i$ because one needs to consider wave-particle resonant ~~et~~ interaction (Landau damping) for $\omega/k \sim v_{T_i}$ using kinetic theory.

* In the short wavelength limit (high k) ;

$$\omega^2 \approx \frac{T_e/M}{\lambda_D^2} \approx \left(\frac{m}{M}\right)\omega_p^2 \equiv \Omega_p^2 \; : \text{ ion plasma frequency.}$$

non-propagating wave.

16.3. High-Frequency Electromagnetic Waves in an Unmagnetized plasma.

* Now, we consider magnetic field perturbation $\vec{B}_1$ in addition to $\vec{E}_1$. Then, we ~~ea~~ need to consider Ampère's law and Faraday's law. With $\vec{\nabla} \Rightarrow i\vec{k}$,

- $i\vec{k} \times \vec{B}_1 = \mu_0 \vec{j}_1 - i\omega \vec{E}_1 / c^2$ (16.24)
- $i\vec{k} \times \vec{E}_1 = i\omega \vec{B}_1$ (16.25)

⊛ Taking $\vec{k} \times$ Eq. (16.25) and expanding the triple vector product,

$$\boxed{k^2 \vec{E}_1 - \vec{k}(\vec{k} \cdot \vec{E}_1) = \left(\frac{\omega}{c}\right)^2 \left(\vec{E}_1 + i\vec{j}_1/\epsilon_0\omega\right)} \qquad (16.27)$$

* For longitudinal waves with $\vec{k} \parallel \vec{E}_1$, LHS $= 0$ and we have an electrostatic wave in which Poisson eqn plays an important role.

* We consider here a transverse wave with $\vec{k} \perp \vec{E}_1$ (i.e., $\vec{k} \cdot \vec{E}_1 = 0$).

$\Rightarrow$ no need to consider Poisson eqn or density perturbation or continuity eqn.

* $$\boxed{\vec{j}_1 = -n_0 e \vec{u}_1} \quad (16.28)$$

(assume stationary ions for high frequency waves as before).

where

$$\boxed{-i\omega m \vec{u}_1 = -e \vec{E}_1} \quad (16.29)$$

$\Rightarrow$ Eq. (16.27) becomes,

$$(c^2k^2 - \omega^2)\vec{E}_1 = i\omega \vec{j}_1 / \epsilon_0 = -\left(\frac{n_0 e^2}{m \epsilon_0}\right)\vec{E}_1 \quad (16.31)$$

$$\boxed{\omega^2 = c^2 k^2 + \omega_p^2} \qquad (16.32)$$

Dispersion Relation for "EM wave" in unmagnetized plasma.

⊛ Note that: $v_p \equiv \dfrac{\omega}{k} = c\left(1 + \dfrac{\omega_p^2}{c^2 k^2}\right)^{1/2} \underline{> c}$.

while

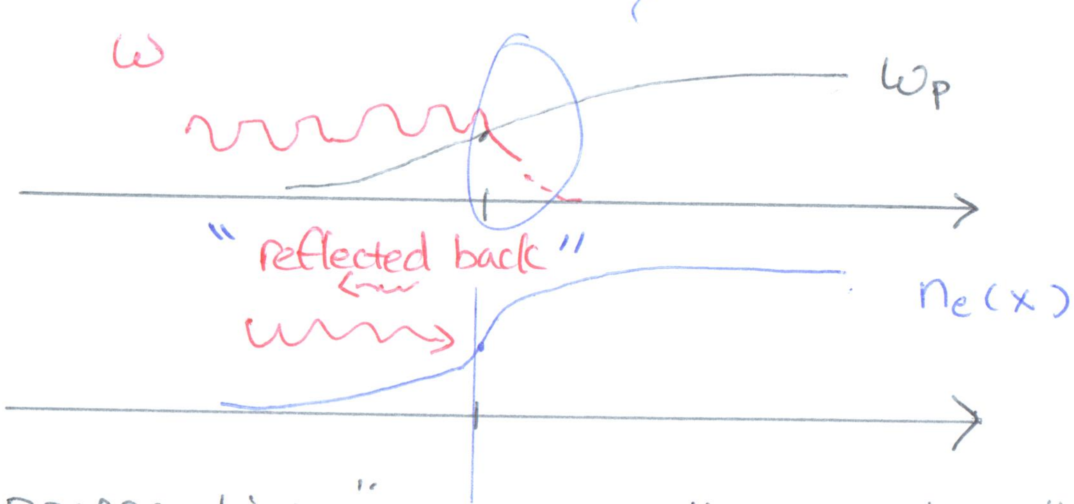
$$v_g \equiv \frac{\partial \omega}{\partial k} = \frac{c^2}{v_p} < c.$$

This is NOT a violation of the special relativity because the information and energy propagate with v_g not v_p.

- low k ; ⇝ constant freq. non-propagating ~~osc~~ oscillation
- hi k ; ⇝ EM waves in vacuum propagating with "c".

⊛ Homework Problem 16.2 on page 266.

(*) EM waves cannot propagate in a plasma with $\omega_p > \omega$.



"propagation" with $\omega > \omega_p$

Cut-off layer

"propagation" not possible

with cut-off density

$$n_c = m_e \epsilon_0 \omega^2 / e^2 .$$

For $\omega_p > \omega$, D.R. admits "k" which is imaginary, i.e.,

$$k = (\omega^2 - \omega_p^2)^{1/2} / c$$

$$= \pm i \, (\omega_p^2 - \omega^2)^{1/2} / c$$

(16.37)

- $\exp(ikx) = \exp\left(-x \frac{(\omega_p^2 - \omega^2)^{1/2}}{c}\right)$

  evanescent solution

- Collisionless Skin depth: $\equiv$ "c/ω_p"