Fusion Plasma Theory lI. 2019

Week 3

Ch. 17. High-Frequency Waves in Magnetized Plasmas

⊛ In the presence of a strong equilibrium magnetic field, $\vec{B}_0$, the medium is anisotropic.

17.1. High-Frequency EM waves propagating ⊥ to $\vec{B}_0$

⊛ $\vec{K} \perp \vec{B}_0$:

(1). O-waves : $\vec{E}_1 \parallel \vec{B}_0$ → $\vec{B}_0$ plays no role in wave dynamics.
"ordinary" —

⇒ we can apply the results for unmagnetized plasma with $\vec{u}_1 \parallel \vec{E}_1$.

$\vec{u}_1 \times \vec{B}_0 = 0$ and O-mode (O-wave) never notices $\vec{B}_0$.

⇒ Recall that EM waves in unmagnetized plasma is "transverse".
" $\vec{E}_1 \perp \vec{K}$ ", With ⊥ propagation ($\vec{K} \perp \vec{B}_0$) :

⊛ $\vec{k} \perp \vec{B}_o$:

(2). X-waves : $\vec{E}_1 \perp \vec{B}_o$

extra-ordinary ...

- In general, $\vec{E}_1$ of this X-wave has a component along $\vec{k}$ ($\perp$ to $\vec{B}_o$) and also a component $\perp$ to both $\vec{k}$ and $\vec{B}_o$.

⊛ For $\vec{B}_o = B_o \hat{z}$ and $\vec{k} = k \hat{x}$,

$$\vec{E}_1 = E_{1x} \hat{x} + E_{1y} \hat{y} .$$

- For this hi-freq. wave, we will take the ions to be stationary.

- In addition, we will neglect the electron pressure assuming a cold plasma with $T_i \approx T_e \approx 0$.

(*) Recall the wave equation, (16.27):

$$k^2 \vec{E}_1 - \vec{k}(\vec{k}\cdot\vec{E}_1) = \left(\frac{\omega}{c}\right)^2 \left(\vec{E}_1 + i\vec{j}_1/\epsilon_0\omega\right), \qquad (17.4)$$

where $\vec{j}_1 = -n_0 e \vec{u}_1$ carried by electrons.

* $\vec{u}_1$ can be obtained from the linearized electron fluid eqn of motion including the Lorentz force, $-e(\vec{u}_1 \times \vec{B}_0)$,

$$-i\omega m\, u_{x1} = -e(E_{x1} + u_{y1} B_0) \qquad (17.1)$$
$$-i\omega m\, u_{y1} = -e(E_{y1} - u_{x1} B_0)$$

* This can be solved for u_{x1} and u_{y1}, which will be used in Eq. (17.4).

$$u_{x1} = \frac{\left(\frac{e}{m}\right)(i\omega E_{x1} + \omega_c E_{y1})}{\omega_c^2 - \omega^2} \qquad (17.3)$$

$$u_{y1} = \frac{\left(\frac{e}{m}\right)(i\omega E_{y1} - \omega_c E_{x1})}{\omega_c^2 - \omega^2},$$

where $\omega_c = \frac{eB_0}{m}$ is the electron cyclotron freq.

⊛ From this procedure, we can obtain two coupled eqns for E_{x1} and E_{y1}. Determinant of 2x2 matrix (of coefficients) $= 0$

⇒ after some arrange-ments ;

⊥ prop^n, X-mode

$$\frac{c^2k^2}{\omega^2} = \frac{c^2}{v_p^2} = 1 - \frac{\omega_p^2(\omega^2 - \omega_p^2)}{\omega^2(\omega^2 - \omega_h^2)} \quad (17.12)$$

where $\omega_h^2 \equiv \omega_p^2 + \omega_c^2$, "Upper-hybrid" frequency

⊛ Resonance :

- At $\omega = \omega_h$, $k \to \infty$ $(\lambda \to 0)$, $v_p \to 0$ and wave fronts pile up.

- We can also check that $E_{y1}/E_{x1} \to 0$ at $\omega = \omega_h$, $\vec{E}_1 = E_{x1}\hat{x} \parallel \vec{k} = k\hat{x}$ at resonance, and wave is electrostatic at resonance.

(*) Cutoffs ; where $k \to 0$ $(\lambda \to \infty)$.

From Eq. (17.12), numerator on RHS $= 0$ for $k = 0$,

$\Rightarrow$

$$\omega = \left[(\omega_c^2 + 4\omega_p^2)^{1/2} \pm \omega_c \right] \equiv \begin{cases} \omega_R \\ \omega_L \end{cases} \qquad (17.19)$$

(*) Examination of Eq. (17.12); $\Rightarrow$

Waves can propagate for $\omega > \omega_R$ and $\omega_h > \omega > \omega_L$,

but cannot " $\omega_R > \omega > \omega_L$ and $\omega_L > \omega$.

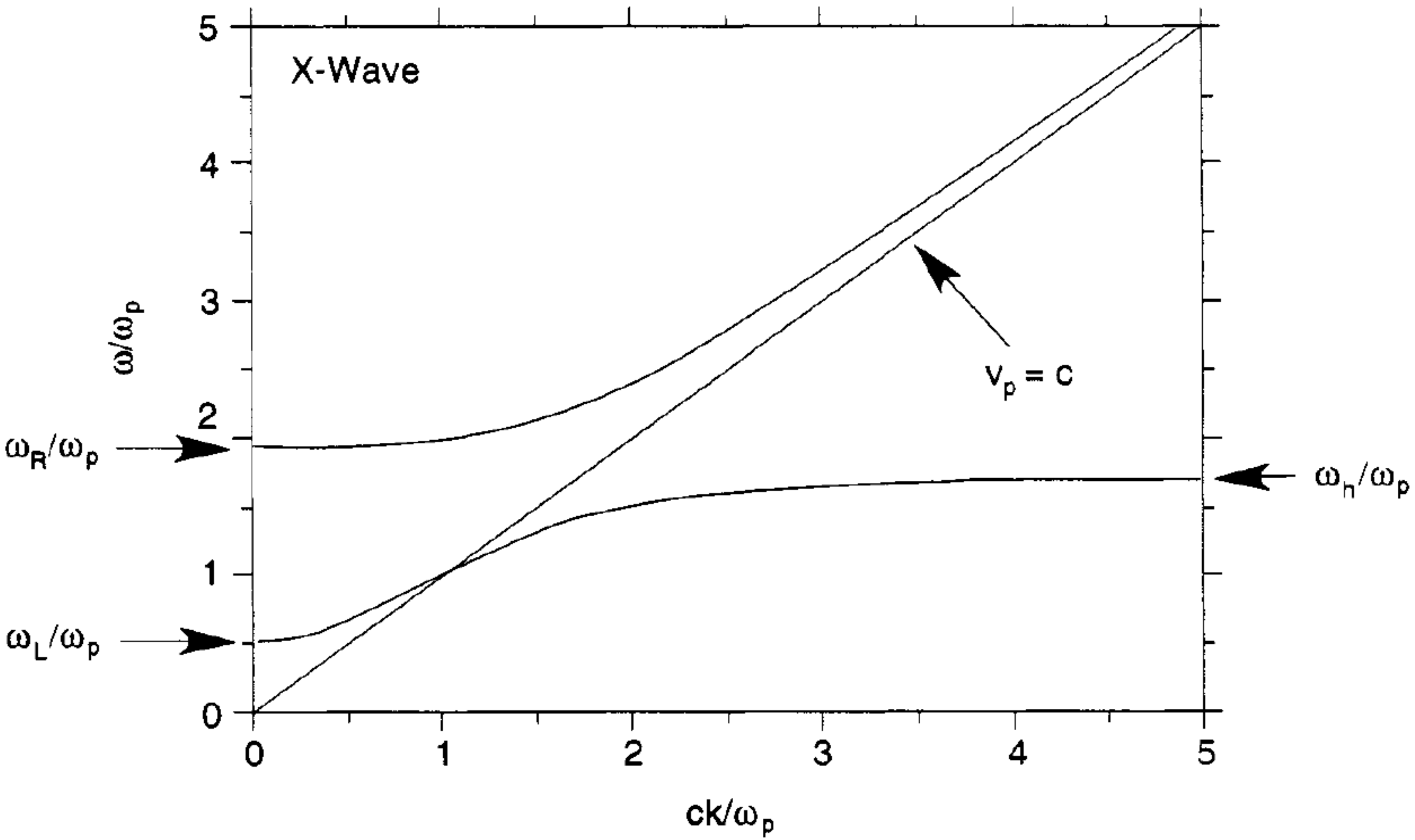

Figure 17.1. Dispersion relation for the extraordinary wave propagating perpendicular to **B** in a magnetized plasma, with ω_c^2 chosen to be equal to $2\omega_p^2$.

(*) Discussion related to Experiments:

- In general, a fixed-frequency wave is driven by a generator.

- It is much easier to arrange a wave propagating up a density gradient, since the RF source is usually located outside of the plasma.

- It is possible to arrange a wave propagating down a $\vec{B}_0$ gradient (high-field side launch in tokamak).

Recall that $\omega_p^2 \propto n_e$ and $\omega_c \propto |\vec{B}_0|$.

$\Rightarrow$ It's easy to arrange a wave to propagate and reach ω_R-cutoff.

$\Rightarrow$ To make a wave propagate and reach ω_h-resonance, one should rely on $|\vec{B}_0|$ variation in space.

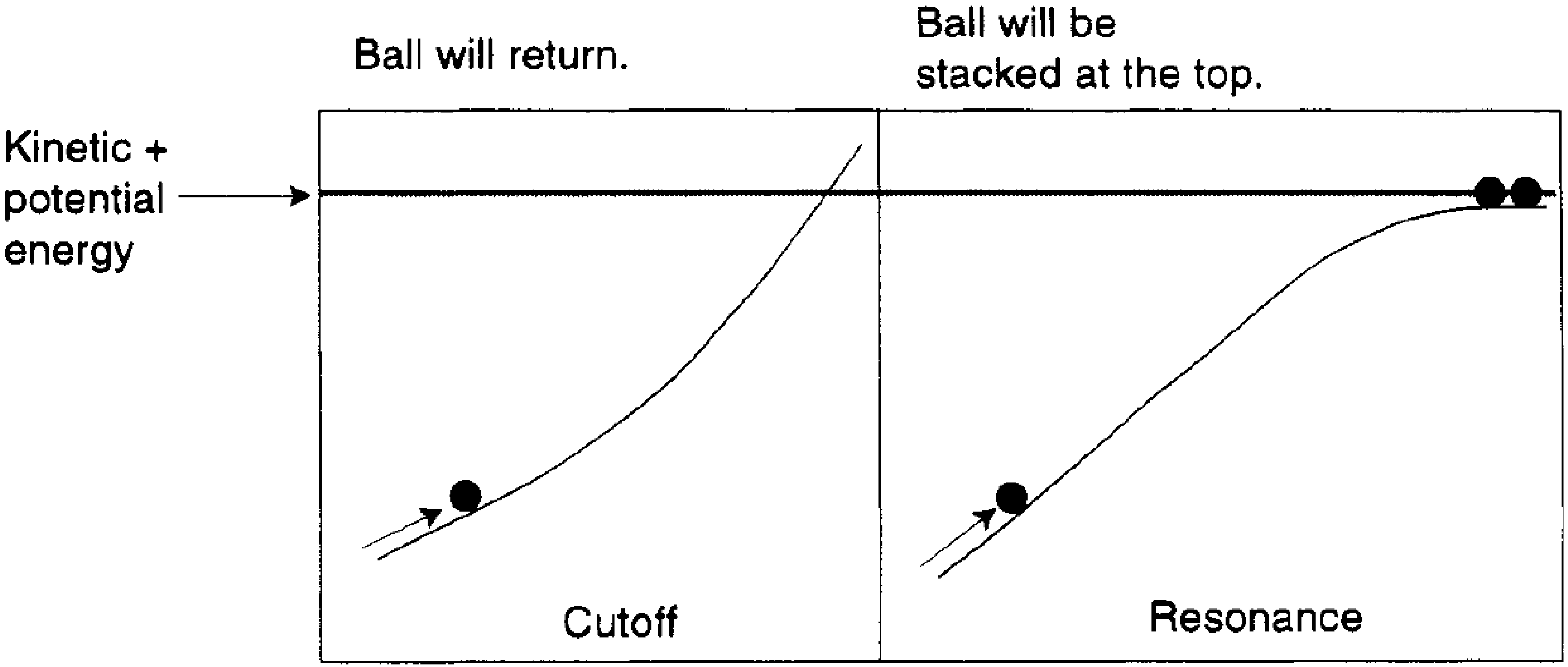

Figure 17.2. Mechanical analog to wave cutoffs and resonances.

(*) Within the context of ⊥ propagation of EM waves based on linear cold-plasma theory, the wave amplitude must grow steadily at the upper-hybrid resonance layer when we pump energy from the outside.

(*) On the other hand, it can be shown that the wave energy is reflected at the cut-off (accelerating the wave back out of the plasma) at $\omega = \omega_p$ in an unmagnetized plasma.
(by using an EM wave example)

Homework : Problem 17.2 on page 277

* 17.2. Hi-Freq. EM waves propagating // $\vec{B}_0$

(*) Consider $\vec{k} \parallel \vec{B}_0$ and high frequency limit such that ions can be considered to be stationary.

Once again, we use the wave eqn.

$$k^2 \vec{E}_1 - \vec{k}(\vec{k}\cdot\vec{E}_1) = \left(\frac{\omega}{c}\right)^2 \left[\vec{E}_1 + i\,\vec{j}_1/\epsilon_0\omega\right]. \quad (17.27)$$

(*) Since a longitudinal mode ($\vec{E}_1 \parallel \vec{k}$) corresponds to the electrostatic Langmuir wave, we consider a new EM wave with $\vec{k}\cdot\vec{E}_1 = 0$ (transverse wave).

(*) Once again, we can solve the linearized electron fluid eqn of motion, to get expressions for u_{x1} and u_{y1} in terms of E_{x1} and E_{y1} and substitute to $\vec{j}_1 = -n_e e\,\vec{u}_1$.

(*) Once again, we get a coupled set of eqns for E_{x1} and E_{y1},

$$\begin{bmatrix} \text{matrix} \end{bmatrix} \begin{bmatrix} E_{x1} \\ E_{y1} \end{bmatrix} = 0 . \quad \text{Det} \begin{bmatrix} \text{matrix} \end{bmatrix} = 0 \ \text{yields},$$

$$\tilde{n}^2 \equiv \frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_c)} \qquad (17.35)$$

$\tilde{n}$: the index of refraction

- "+" and "−" signs correspond to "L-wave" and "R-wave" respectively.
- Both correpond to "circularly polarized waves."
- E_{x1} and E_{y1} are $\pi/2$ out of phase.

[L / R wave's $\vec{E}_1$ vector rotates according to a left-hand / right-hand rule.

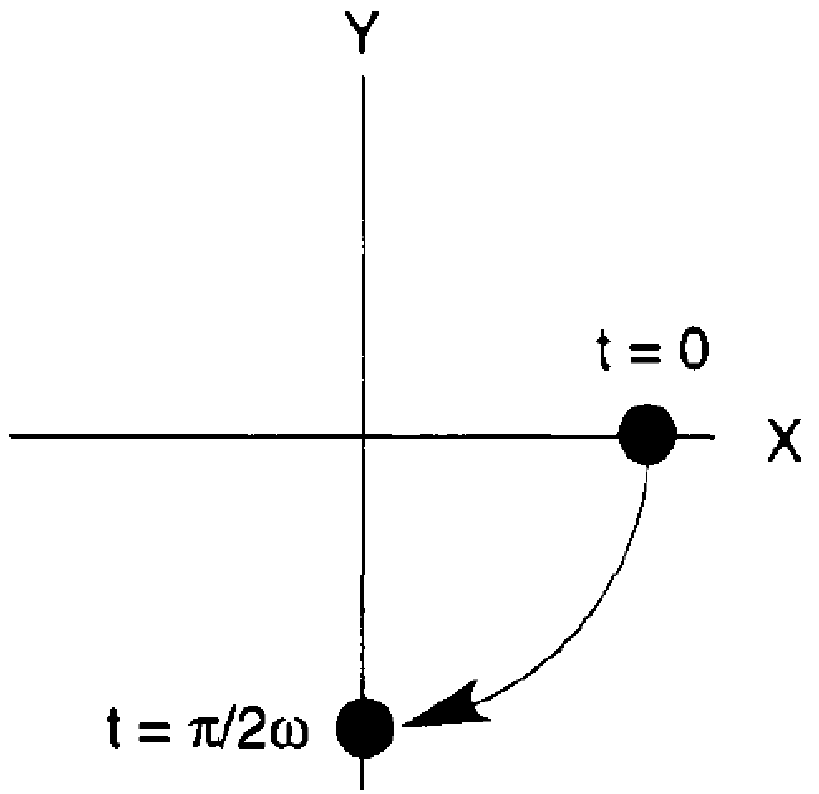

Figure 17.3. Time progression of the **E**-field vector for a left-hand circularly polarized wave with $\mathbf{B}_0$ along the z direction, out of the page.

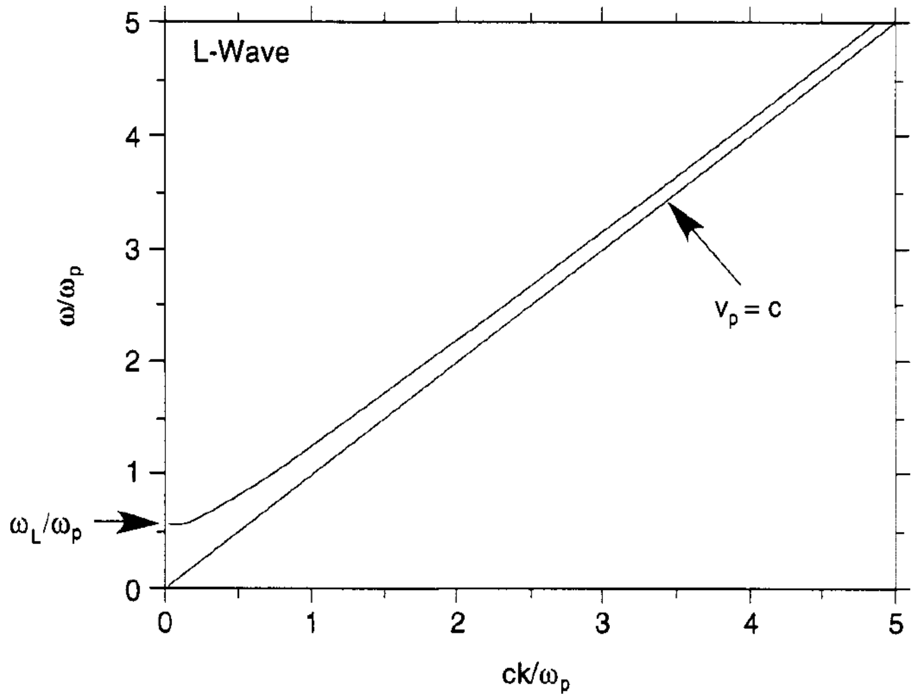

Figure 17.4. Left-hand circularly polarized electromagnetic wave propagating parallel to $\mathbf{B}_0$ in a magnetized plasma, with ω_c^2 chosen to equal $2\omega_p^2$.

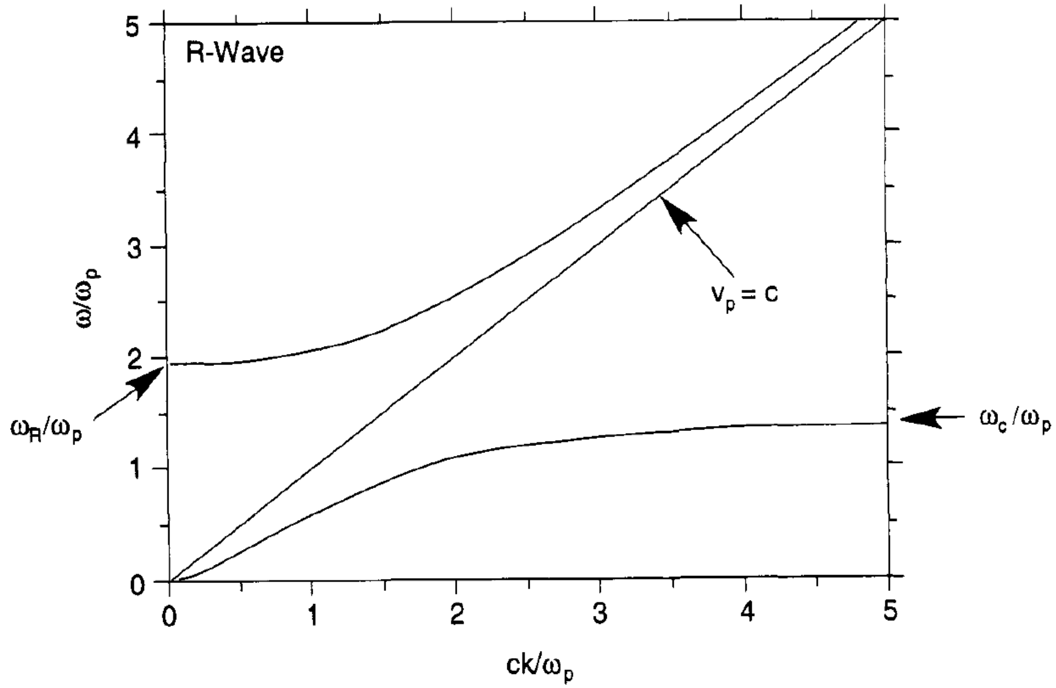

Figure 17.5. Right-hand circularly polarized electromagnetic wave propagating parallel to $\mathbf{B}_0$ in a magnetized plasma, with ω_c^2 chosen to equal $2\omega_p^2$.

⊛ Whistler wave:

⊛ R-wave in the region of $\omega < \omega_c$.

(lower frequency range)

⊛ One can show that $v_g = \frac{\partial \omega}{\partial k} \nearrow$ as $\omega \nearrow$.

→ White radio noise generated in a burst in the ionosphere due to lightning flashes, and propagating as a whistler wave will travel faster at high frequencies than a low.

⇒ Ground-based receiver will then hear a "whistle" going from hi-freq. to low due to lighting flashes.

⊛ Faraday Rotation :

⊛ For the same freq. "ω", the upper band of the R-wave ($\omega > \omega_R$) has a higher phase velocity v_p than the corresponding L-wave at the same "ω".

⇒ When a linearly polarized EM wave propagates // to $\vec{B}_0$, the angle of polarization of the wave rotates as it travels.

⊛ This is illustrated on page 283 of G and R.

⊛ Since the resulting difference in the phase depends on ω_c and ω_p, one can determine the magnetic field in plasma if density of plasma is known from other means.

For details, try Problem 17.1 and 17.4.